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Research Article

Common Fixed Point Theorems for Weakly Compatible Maps Satisfying a General Contractive Condition

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We introduce a new generalized contractive condition for four mappings in the framework of metric space. We give some common fixed point results for these mappings and we deduce a fixed point result for weakly compatible mappings satisfying a contractive condition of integral type.

1. Introduction and preliminaries

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research during recent years. The most general of the common fixed point theorems pertaining to four mappings, A, B, S, and T of a metric space (X,d), uses either a Banach-type contractive condition of the form

$$d(Ax,By) \le kM(x,y), \quad 0 \le k < 1, \tag{1.1}$$

where

$$M(x,y) = \max\left\{d(Sx,Ty),\, d(Ax,Sx),\, d(By,Ty),\, \frac{[d(Sx,By)+d(Ax,Ty)]}{2}\right\}, \tag{1.2}$$

or a Meir-Keeler-type (ε,δ)-contractive condition, that is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \le M(x,y) < \varepsilon + \delta \Longrightarrow d(Ax,By) < \varepsilon, \tag{1.3}$$

or a ψ-contractive condition of the form

$$d(Ax, By) \leq \psi(M(x, y)), \tag{1.4}$$

involving a contractive gauge function $\psi : [0, +\infty[\rightarrow [0, +\infty[$ such that $\psi(t) < t$ for each $t > 0$. Clearly, Banach-type contractive condition is a special case of both conditions Meir-Keeler-type (ε, δ)-contractive and ψ-contractive. A ψ-contractive condition does not guarantee the existence of a fixed point unless some additional condition is assumed. Moreover, a ψ-contractive condition, in general, does not imply the Meir-Keeler-type (ε, δ)-contractive condition [1, Example 1.1].

Recently, some fixed point results for mappings satisfying an integral-type contractive condition are obtained in [2–5]. Suzuki [6] showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. Zhang [7] introduced a generalized contractive-type condition for a pair of mappings in metric space and proved common fixed point theorems that extend results in [3–5]. In this paper, we give a new generalized contractive-type condition for four mappings in metric space and prove some common fixed point results for these mappings. The results obtained extend well-known comparable results in [2–5, 7].

Lemma 1.1 (see [8]). *For every function $\psi : [0, +\infty[\rightarrow [0, +\infty[$, let ψ^n be the nth iterate of ψ. Then the following hold:*

(i) *if ψ is nondecreasing, then for each $t > 0$, $\lim_{n \to +\infty} \psi^n(t) = 0$ implies $\psi(t) < t$;*

(ii) *if ψ is right continuous with $\psi(t) < t$ for $t > 0$, then $\lim_{n \to +\infty} \psi^n(t) = 0$.*

2. Common fixed points

In this section, we give our main result. Two self-mappings A and S of a metric space (X, d) are called weakly compatible if they commute at their coincidence points. Let A, B, S, and T be self mappings of a metric space (X, d). In the sequel, we set

$$M(x, y) = \max\left\{d(Sx, Ty), \, d(Ax, Sx), \, d(By, Ty), \, \frac{[d(Sx, By) + d(Ax, Ty)]}{2}\right\}. \tag{2.1}$$

Lemma 2.1. *Let A, B, S, and T be self-mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume that there exist $F, \psi : [0, +\infty[\rightarrow [0, +\infty[$ such that*

(i) *F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;*

(ii) *ψ is nondecreasing, right continuous, and $\psi(t) < t$ for every $t > 0$.*

If for all $x, y \in X$,

$$F(d(Ax, By)) \leq \psi(F(M(x, y))), \tag{2.2}$$

then for each $x_0 \in X$, the sequence (y_n) of points of X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n-1} = Bx_{2n-1} = Sx_{2n} \tag{2.3}$$

is a Cauchy sequence.

Proof. We have

$$
\begin{aligned}
M(x_{2n}, x_{2n+1}) \\
&= \max\left\{ d(y_{2n-1}, y_{2n}),\ d(y_{2n}, y_{2n-1}),\ d(y_{2n+1}, y_{2n}),\ \frac{[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]}{2} \right\} \\
&= \max\left\{ d(y_{2n}, y_{2n-1}),\ d(y_{2n}, y_{2n+1}),\ \frac{d(y_{2n-1}, y_{2n+1})}{2} \right\} \\
&= \max\left\{ d(y_{2n}, y_{2n-1}),\ d(y_{2n}, y_{2n+1}) \right\}.
\end{aligned} \tag{2.4}
$$

Similarly

$$M(x_{2n}, x_{2n-1}) = \max\left\{ d(y_{2n}, y_{2n-1}),\ d(y_{2n-1}, y_{2n-2}) \right\}. \tag{2.5}$$

If for some n we have either $y_{2n} = y_{2n-1}$ or $y_{2n} = y_{2n+1}$, then by condition (2.2) we obtain that the sequence (y_n) is definitely constant and thus is a Cauchy sequence. Suppose $y_n \neq y_{n-1}$ for each n.

From

$$
\begin{aligned}
F(d(y_{2n}, y_{2n+1})) &= F(d(Ax_{2n}, Bx_{2n+1})) \le \psi(F(M(x_{2n}, x_{2n+1}))) \\
&= \psi(F(d(y_{2n}, y_{2n-1}))) < F(d(y_{2n}, y_{2n-1})), \\
F(d(y_{2n}, y_{2n-1})) &= F(d(Ax_{2n}, Bx_{2n-1})) \le \psi(F(M(x_{2n}, x_{2n-1}))) \\
&= \psi(F(d(y_{2n-1}, y_{2n-2}))) < F(d(y_{2n-1}, y_{2n-2})),
\end{aligned} \tag{2.6}
$$

we deduce

$$F(d(y_{n+1}, y_n)) < F(d(y_n, y_{n-1})), \tag{2.7}$$

for all $n \in \mathbb{N}$. Now, from

$$F(d(y_{n+1}, y_n)) \le \psi(F(d(y_n, y_{n-1}))) \le \cdots \le \psi^n(F(d(y_0, y_1))) \tag{2.8}$$

and (ii) of Lemma 1.1, we obtain $\lim_{n\to+\infty} F(d(y_{n+1}, y_n)) = 0$, which implies

$$\lim_{n\to+\infty} d(y_{n+1}, y_n) = 0. \tag{2.9}$$

We prove that (y_n) is a Cauchy sequence. Suppose not, then there exists $\varepsilon > 0$ such that $d(y_n, y_m) \ge 2\varepsilon$ for infinite values of m and n with $m < n$. This assures that there exist two sequences (m_k), (n_k) of natural numbers, with $m_k < n_k$, such that

$$d(y_{2m_k}, y_{2n_k+1}) > \varepsilon \quad \forall k. \tag{2.10}$$

It is not restrictive to suppose that n_k is the least positive integer exceeding m_k and satisfying (2.10). We have

$$\begin{aligned}\varepsilon &< d(y_{2m_k}, y_{2n_k+1})\\ &\leq d(y_{2m_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1})\\ &\leq \varepsilon + d(y_{2n_k-1}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1}).\end{aligned} \tag{2.11}$$

Then $d(y_{2m_k}, y_{2n_k+1}) \to \varepsilon$. We note

$$\begin{aligned}&d(y_{2m_k}, y_{2n_k+1}) - d(y_{2m_k}, y_{2m_k+1}) - d(y_{2n_k+2}, y_{2n_k+1})\\ &\quad\leq d(y_{2m_k+1}, y_{2n_k+2})\\ &\quad\leq d(y_{2m_k}, y_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+1}) + d(y_{2n_k+2}, y_{2n_k+1}),\end{aligned} \tag{2.12}$$

and thus $d(y_{2m_k+1}, y_{2n_k+2}) \to \varepsilon$ as $k \to +\infty$. We have

$$\begin{aligned}&M(x_{2n_k+2}, x_{2m_k+1})\\ &= \max\left\{d(y_{2m_k}, y_{2n_k+1}), d(y_{2n_k+1}, y_{2n_k+2}), d(y_{2m_k}, y_{2m_k+1}), \frac{d(y_{2m_k+1}, y_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+2})}{2}\right\}\\ &= d(y_{2m_k}, y_{2n_k+1}) + d_k,\end{aligned} \tag{2.13}$$

where $d_k \to 0$ as $k \to +\infty$ and $d_k \geq 0$ for all k. Then from

$$\begin{aligned}F(d(y_{2m_k+1}, y_{2n_k+2})) = F(d(Ax_{2n_k+2}, Bx_{2m_k+1})) &\leq \psi(F(M(x_{2n_k+2}, x_{2m_k+1})))\\ &= \psi(F(d(y_{2m_k}, y_{2n_k+1}) + d_k)),\end{aligned} \tag{2.14}$$

as $k \to +\infty$, F being continuous and ψ right continuous, we get

$$F(\varepsilon) \leq \psi(F(\varepsilon)) < F(\varepsilon). \tag{2.15}$$

This is a contradiction. Therefore (y_n) is a Cauchy sequence. □

Lemma 2.2. *Let (X, d) be a metric space and let A, B, S, T, F, and ψ be as in Lemma 2.1. If one of AX, TX, BX, and SX is a complete subspace of X, then the following hold:*

(i) *A and S have a coincidence point;*

(ii) *T and B have a coincidence point.*

Proof. Fix $x_0 \in X$ and let (y_n) be the sequence defined in Lemma 2.1. If $y_{2n} = y_{2n-1}$ for some n, then $Ax_{2n} = Tx_{2n+1} = Bx_{2n-1} = Sx_{2n}$, and A and S have a coincidence point. If $y_{2n} = y_{2n+1}$

for some n, then $Ax_{2n} = Tx_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, and T and B have a coincidence point. Assume that $y_n \neq y_{n+1}$ for every n and TX is complete. By Lemma 2.1, the sequence (y_n) is Cauchy; as $(y_{2n}) \subset TX$, there exists $u \in TX$ such that $y_n \to u$. Let $v \in X$ be such that $Tv = u$. To prove that $Bv = u$. We have

$$M(x_{2n}, v) = \max\left\{d(y_{2n-1}, u),\ d(y_{2n}, y_{2n-1}),\ d(Bv, u),\ \frac{[d(y_{2n-1}, Bv) + d(y_{2n}, u)]}{2}\right\}. \tag{2.16}$$

If $Bv \neq u$, then $M(x_{2n}, v) = d(u, Bv)$ definitely and consequently for large n,

$$F(d(Ax_{2n}, Bv)) \leq \psi(F(M(x_{2n}, v))) = \psi(F(d(u, Bv))). \tag{2.17}$$

F being continuous, as $n \to +\infty$, we obtain

$$F(d(u, Bv)) \leq \psi(F(d(u, Bv))) < F(d(u, Bv)). \tag{2.18}$$

This is a contradiction, therefore $Bv = u$ and v is a coincidence point for T and B. From $BX \subset SX$, which gives $u \in SX$, we deduce that there exists $w \in X$ such that $Sw = u$. To prove that $Aw = u$. We have

$$M(w, v) = \max\left\{d(u, u),\ d(Aw, u),\ d(u, u),\ \frac{[d(u, u) + d(Aw, u)]}{2}\right\} = d(Aw, u), \tag{2.19}$$

and hence

$$F(d(Aw, Bv)) \leq \psi(F(M(w, u))) = \psi(F(d(Aw, u))) < F(d(Aw, u)), \tag{2.20}$$

which gives $Aw = u$.

The same result holds if we suppose that one of SX, AX, BX is complete. □

Theorem 2.3. *Let A, B, S, and T be self-mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume that there exist $F, \psi : [0, +\infty[\to [0, +\infty[$ such that*

(i) *F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;*

(ii) *ψ is nondecreasing, right continuous, and $\psi(t) < t$ for every $t > 0$;*

(iii) *$F(d(Ax, By)) \leq \psi(F(M(x, y)))$ for all $x, y \in X$.*

If one of AX, TX, BX, and SX is a complete subspace of X, then the following hold:

(iv) *A and S have a coincidence point;*

(v) *T and B have a coincidence point.*

Further, if A and S as well as B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Proof. Fix $x_0 \in X$ and let (y_n) be the sequence defined in Lemma 2.1. Assume that TX is complete and let u, v, and w be as in Lemma 2.2. If A and S are weakly compatible, then

$$Au = ASw = SAw = Su, \tag{2.21}$$

therefore u is a coincidence point of A and S. To prove that $d(Au, u) = 0$. Suppose that $d(Au, u) \neq 0$. We have

$$\begin{gathered} M(u,v) = \max\left\{ d(Su,u),\, d(Au,Su),\, d(u,u),\, \frac{[d(Su,u) + d(Au,u)]}{2} \right\} = d(Au,u) \\ F(d(Au,Bv)) = F(d(Au,u)) \leq \psi(F(M(u,v))) = \psi(F(d(Au,u))) < F(d(Au,u)). \end{gathered} \tag{2.22}$$

This is a contradiction, and thus $Au = u$. Since $Au = Su = u$, we obtain that u is a common fixed point for A and S.

Similarly, if B and T are weakly compatible, we deduce that u is a common fixed point for B and T. Now if A and S as well as B and T are weakly compatible, then u is a common fixed point for A, B, S, and T. If $z \in X$ is also a common fixed point for A, B, S, and T with $u \neq z$, then

$$F(d(Au,Bz)) \leq \psi(F(M(u,z))) = \psi(F(d(Au,Bv))) < F(d(Au,Bv)), \tag{2.23}$$

which gives $u = z$. □

Let $\varphi : [0,+\infty[\to [0,+\infty[$ be a Lebesgue integrable function which is nonnegative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \quad \text{for every } \varepsilon > 0. \tag{2.24}$$

The function $F : [0,+\infty[\to [0,+\infty[$, with $F(s) = \int_0^s \varphi(t)dt$ satisfies condition (i) of Lemma 2.1 and from Theorem 2.3 we deduce the following theorem.

Theorem 2.4 (see [2, Theorem 2.1]). *Let A, B, S, and T be self-mappings of a metric space (X, d) such that $AX \subset TX$, $BX \subset SX$. Assume that there exists a nondecreasing right continuous function $\psi : [0,+\infty[\to [0,+\infty[$, with $\psi(t) < t$ for all $t > 0$, such that*

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{2.25}$$

where $\varphi : [0,+\infty[\to [0,+\infty[$ is a Lebesgue integrable function which is nonnegative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \quad \textit{for every } \varepsilon > 0. \tag{2.26}$$

If one of AX, TX, BX, *and* SX *is a complete subspace of* X, *then the following hold:*

(i) A *and* S *have a coincidence point;*

(ii) T *and* B *have a coincidence point.*

Further, if A *and* S *as well as* B *and* T *are weakly compatible, then* A, B, S, *and* T *have a unique common fixed point.*

Remark 2.5. Theorem 2.4 is a generalization of the main theorem in [3], of [4, Theorem 2], and of [5, Theorem 2].

If in Theorem 2.3, we assume $S = T = I_X$, where I_X is the identity map on X, we obtain the following theorem.

Theorem 2.6. *Let* A *and* B *be self-mappings of a metric space* (X, d). *Assume that there exist* $F, \psi : [0, +\infty[\to [0, +\infty[$ *such that*

(i) F *is nondecreasing, continuous, and* $F(0) = 0 < F(t)$ *for every* $t > 0$*;*

(ii) ψ *is nondecreasing, right continuous, and* $\psi(t) < t$ *for every* $t > 0$*;*

(iii) $F(d(Ax, By)) \leq \psi(F(m(x, y)))$ *for all* $x, y \in X$,

where

$$m(x, y) = \max\left\{d(x, y),\ d(Ax, y),\ d(By, y),\ \frac{[d(Ax, y) + d(x, By)]}{2}\right\}. \tag{2.27}$$

If one of AX *and* BX *is a complete subspace of* X, *then* A *and* S *have a unique common fixed point. Moreover, for each* $x_0 \in X$, *the iterated sequence* (x_n) *with* $x_{2n+1} = Ax_{2n}$ *and* $x_{2n+2} = Bx_{2n+1}$ *converges to the common fixed point of* A *and* B.

Theorem 2.6 includes [7, Theorem 1].

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