Asymptotic Stress Field of a Mode III Crack Growing Along an Elastic/Elastic Power-Law Creeping Bimaterial Interface

C.-Y. Hui

M. T. A. Saif

Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853

The asymptotic stress field near the tip of a crack subjected to antiplane shear loading is analysed. The crack is growing quasi-statically along an elastic/elastic power-law creeping bimaterial interface. We find there is a separable solution with the following characteristics: for $n < 3$*, where* n *is the power-law creeping exponent, the asymptotic stress field is dominated by the elastic strain rates and has an inverse square root singularity,* $r^{-1/2}$*, where* r *is the distance from the current crack tip. For* $n \geq 3$*, the near-tip stress and strain fields has a singularity of the form* $r^{-1/(n-1)}$*. The strength of this field is completely specified by the current crack growth rate, besides material properties, and is otherwise independent of the applied load and of the prior crack growth history.*

Introduction

Recent interest in composite materials, such as ceramic-ceramic and metal matrix systems has renewed interest in the studies of the growth of interface cracks along bimaterial systems. The stress field near the tip of a crack between two linear elastic homogeneous materials is well understood (Williams, 1959; Rice and Sih, 1965; England, 1965; Rice, 1988; Shih, 1991). Progress has been made by Shih and Asaro (1988, 1989, 1991), who carried out detail numerical studies to compute the crack-tip fields for a stationary plane-strain crack along the interface between a linearly elastic and a rate-independent plastic material. Analytic study of the crack-tip field has been performed by Wang (1990) and Sharma and Aravas (1991, 1993) for a crack lying along the interface of an elastic and power-law hardening material under plane-strain condition. Champion and Atkinson (1990, 1991) derived asymptotic fields of a crack lying along the interface of two power-law hardening materials. Drugan and Chen (1989) and Chen and Drugan (1991) investigated the stress and deformation fields near the tip of a quasi-statically growing crack through a homogeneous, isotropic, incompressible elastic ideally plastic Prandtl-Reuss-Mises material. Drugan (1991) also studied the fields for a crack quasi-statically growing along the interface of a Prandtl-Reuss-Mises material and a rigid substrate. A recent article by Bose and Castañeda (1992) reports an elegant study of near-tip fields of a quasi-statically growing crack along the interface of two elastoplastic materials characterized by J_2 flow theory of plasticity. The paper also contains a comprehensive literature review on the subject.

Experimental study of the deformation field near a quasi-statically and dynamically growing crack along PMMA/Aluminium interface has been reported by Tippur and Rosakis (1990) who indicate that the experimentally measured stress intensity factors are in good agreement with the numerically evaluated values. Experimental investigations reported by O'Dowd et al. (1992) indicates that the fracture toughness of the interface depends strongly on the load mixity for Aluminium/Niobium interface.

Also, fracture toughness predicted, based on full-field finite element calculations and hoop stress criterion, captures the trend of experimental results.

There is considerably less literature on the problem of the growth of interface cracks along the interface of bimaterial systems of which one of the materials is rate dependent. Such problems may be important for composite material systems like metal matrix composites operating under sufficiently high temperature.

The purpose of this paper is to study analytically the nature of crack-tip fields for a crack lying along the interface of two different materials under antiplane shear (mode III) loading. Mode III loading is chosen for its mathematical simplicity and to provide insight on the crack-tip fields. The crack is growing quasi-statically with velocity $\dot{a}$ along the interface of two homogeneous materials one of which is linearly elastic and the other one is rate dependent. The latter under uniaxial tension deforms as

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the ASME Journal of Applied Mechanics.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME Journal of Applied Mechanics.

Manuscript received by the ASME Applied Mechanics Division, Apr. 15, 1992; final revision, Feb. 19, 1993. Associate Technical Editor: C. F. Shih.

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + B\sigma^n. \tag{1}$$

The first term represents the elastic strain rate where E is Young's modulus, and the second term describes power-law secondary creep where B is the creep coefficient and n is the creep exponent. Asymptotic stress field is solved exactly in this case. We show that for $n \geq 3$, the stress and strain singularities are of the form $\sigma, \epsilon \propto r^{-(1/n-1)}$ where r is radial distance from the moving crack tip. This asymptotic field has the remarkable property to be uniquely determined by the current crack growth rate, $\dot{a}$ (besides material properties), independent of the remotely applied load.

Problem Definition

Figure 1 shows the asymptotic view of the crack with a reference frame $(x, y)/(r, \theta)$ moving with the tip with velocity $\dot{a} = da/dt$ along the positive x direction, which coincides with the interface. The z-axis lies along the crack front. Antiplane (mode III) is characterized by the only nonzero out of plane (along z direction) displacement component, w, and the two nonzero stresses and strains $\tau_i = \sigma_{zi}$, and $\gamma_i = 2\epsilon_{zi}$, $i = (x, y)/(r, \theta)$. The magnitude of the resultant shear stress is $\tau_e = (\tau_x^2 + \tau_y^2)^{1/2} = (\tau_\theta^2 + \tau_r^2)^{1/2}$. At any time t, the stresses in (r, θ) coordinates satisfy the equilibrium equation,

$$\frac{\partial \tau_\theta}{r\partial\theta} + \frac{\partial \tau_r}{\partial r} + \frac{\tau_r}{r} = 0, \tag{2}$$

and the strains satisfy the compatibility condition,

$$\frac{\partial \gamma_r}{\partial \theta} - \frac{\partial (r\gamma_\theta)}{\partial r} = 0. \tag{3}$$

The equilibrium equation is identically satisfied by the stress function ψ if

$$\tau_\theta = \frac{\partial \psi}{\partial r}$$

$$\tau_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}. \tag{4}$$

The material law for the upper half-plane (UHP), generalized to antiplane shear from Eq. (1), is

$$\dot{\gamma}_i^+ = \frac{\dot{\tau}_i^+}{G^+} + \bar{B}\tau_e^{+\,n-1}\tau_i^+ \tag{5}$$

and those for the linear elastic lower half-plane (LHP) are

$$\gamma_i^- = \frac{\tau_i^-}{G^-} \tag{6}$$

where G is the shear modulus, $\bar{B} = (\sqrt{3})^{n+1} B$, and $i = (x, y)/(r, \theta)$. The superscripts $^+$ and $^-$ represent the upper and lower half-planes.

For the time being, consider steady-state crack growth for which

$$\frac{d(\cdot)}{dt} = \frac{d(\cdot)}{dx}\frac{d(x)}{dt} = -\dot{a}\frac{d(\cdot)}{dx} \tag{7}$$

where $(\cdot)$ is a differentiable function of $(x, y)/(r, \theta)$. Using Eqs. (7), (4), and (5) in (3), we get the governing differential equation for ψ^+, which is

$$-\frac{\dot{a}}{G^+}\nabla^2\frac{\partial \psi^+}{\partial x} + \bar{B}\nabla\cdot\{\tau_e^{+\,n-1}\nabla\psi^+\} = 0 \tag{8}$$

where ∇ is the gradient operator. Similarly, the governing differential equation for ψ^-,

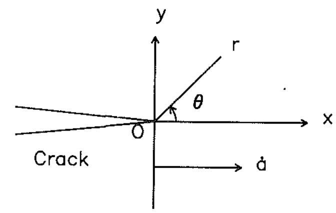

Fig. 1 Crack tip moving with velocity $\dot{a}$ under antiplane shear load (Mode III)

$$\nabla^2\psi^- = 0, \tag{9}$$

is obtained by using Eqs. (7), (4) and (6) in Eq. (3). The boundary conditions for Eqs. (8) and (9) are

(1) traction-free crack surface

$$\frac{\partial \psi^+}{\partial r}(\theta = \pi) = 0$$

$$\frac{\partial \psi^-}{\partial r}(\theta = -\pi) = 0, \tag{10}$$

(2) continuity of traction across the interface

$$\tau_\theta^+(r, \theta = 0) = \tau_\theta^-(r, \theta = 0), \tag{11}$$

(3) continuity of displacement across the interface

$$w(r, \theta = 0^+) = w(r, \theta = 0^-),$$

$$\text{or}\quad \gamma_{r,r}^+(r, \theta = 0) = \gamma_{r,r}^-(r, \theta = 0). \tag{12}$$

Earlier work by Hui and Riedel (1981) shows that the crack-tip stress field of a quasi-statically growing crack under antiplane shear through a *homogeneous* material satisfying Eq. (5) with $n \geq 3$ is given by

$$\tau_i = \left(\frac{\dot{a}}{G^+\bar{B}r}\right)^{\frac{1}{n-1}} f_i(\theta, n) \tag{13}$$

where $f_i(\theta, n)$ is a function of θ and n, and $i = r, \theta$. Note that the stress field given by Eq. (13) satisfies the boundary condition $w = 0$ along $\theta = 0$ and traction-free boundary condition for $\theta = \pi$, which are also the required boundary conditions for the special case where the LHP is rigid. The fact that the elastic strains and creep strains have the same asymptotic behavior as $r \to 0$ when the material is homogeneous, as implied by Eq. (13), leads us to anticipate that if the rigid LHP is replaced by a linearly elastic material, then the nature of the field will be similar to Eq. (13), i.e., separable, and the singularity in the upper and lower half-plane should be the same and equal to $r^{-1/(n-1)}$. We will show that this is indeed the case in the following analysis.

Derivation of ψ^- in the LHP. We will first consider the case of $n \geq 3$. We look for a separable solution, i.e., let

$$\psi^- = r^s\phi^-(\theta) \tag{14}$$

where s is an unknown constant and $\phi^-(\phi)$, a function of θ, is independent of r. Substituting Eq. (14) in Eq. (9), we get

$$\frac{d^2\phi^-(\theta)}{d\theta^2} + s^2\phi^-(\theta) = 0 \tag{15}$$

with the boundary condition that

$$\phi^-(\theta = -\pi) = 0$$

from stress-free crack surface. The general solution of Eq. (15) is

$$\phi^-(\theta)=A^-\frac{\sin(s(\theta+\pi))}{\cos(s\pi)} \quad (16)$$

and

$$\psi^-(r,\theta)=\frac{A^-}{\cos(s\pi)}r^s\sin\{(s\theta+\pi)\}$$

where A^- is the unknown constant of integration. The stresses are

$$\tau_\theta^-(r,\theta)=\frac{\partial\psi^-}{\partial r}=\frac{A^-}{\cos(s\pi)}sr^{s-1}\sin\{s(\theta+\pi)\} \quad (17)$$

$$\tau_r^-(r,\theta)=-\frac{1}{r}\frac{\partial\psi^-}{\partial\theta}=-\frac{A^-}{\cos(s\pi)}sr^{s-1}\cos\{s(\theta+\pi)\}. \quad (18)$$

Derivation of ψ^+ in the UHP. Motivated by Hui and Riedel (1981) and our anticipation that the stress and strain field will have same order of singularity in both UHP and LHP, we assume

$$\psi^+=\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}r^s\phi^+(\theta) \quad (19)$$

where $\phi^+(\theta)$ is independent of r. Using Eq. (4) we get

$$\tau_\theta^+(r,\theta)=\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}sr^{s-1}\phi^+(\theta) \quad (20)$$

$$\tau_r^+(r,\theta)=-\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}r^{s-1}\phi^{+\prime}(\theta) \quad (21)$$

and

$$\tau_e^+(r,\theta)=\sqrt{\tau_r^{+2}+\tau_\theta^{+2}}=\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}r^{s-1}\sqrt{\{\phi^{+\prime}\}^2+\{s\phi^+\}^2} \quad (22)$$

where prime (′) denotes derivative with respect to θ.

We now solve for $\phi^+(\theta)$ in the UHP. Substitution of ψ^+ from Eq. (19) in Eq. (8) gives

$$\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}\frac{\dot a}{G^+}r^{s-3}\{\sin(\theta)\phi^{+\prime\prime\prime}-(s-2)\cos(\theta)\phi^{+\prime\prime}+s^2\sin(\theta)\phi^{+\prime}-(s-2)s^2\cos(\theta)\phi^+\}$$
$$+\bar B\left(\frac{\dot a}{\bar B G^+}\right)^{n/(n-1)}r^{n(s-1)-1}[n(\phi^{+\prime})^2\phi^{+\prime\prime}+(\phi^{+\prime})^2\phi^+\{s^2(2n-1)-s(n-1)\}+s^2\phi^{+\prime\prime}\phi^{+2}+s^2\phi^{+3}\{ns^2-s(n-1)\}]=0 \quad (23)$$

which is valid for all r only when $s=(n-2)/(n-1)$ (Hui and Riedel, 1981). Equation (23) can be simplified to

$$\sin(\theta)\phi^{+\prime\prime\prime}+(2-s)\cos(\theta)\phi^{+\prime\prime}+s^2\sin(\theta)\phi^{+\prime}-(s-2)s^2\cos(\theta)\phi^++(\phi^{+\prime 2}+s^2\phi^{+2})^{(n-3)/2}\{\phi^{+\prime\prime}(s^2\phi^{+2}+n\phi^{+\prime 2})+s\phi^+\phi^{+\prime 2}(1+2ns-n-s)+s^3\phi^{+3}(1+ns-n)\}=0 \quad (24)$$

which is the governing differential equation for $\phi^+(\theta)$. Traction-free crack face and displacement and traction continuity across the interface provide the necessary boundary conditions for ϕ^+ as follows.

From the traction-free crack surface boundary condition (Eq. (10)), and using Eq. (20), we have

$$\phi^+(\pi)=0. \quad (25)$$

Using Eqs. (17) and (20) and continuity of traction across the interface, we obtain

$$A^-=\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}\frac{\phi_0}{\tan(s\pi)}, \quad (26)$$

where $\phi_0=\phi^+(\theta=0)$. Displacement continuity across the interface implies

$$\gamma_{r,r}^+=\gamma_{r,r}^-,\ \text{along}\ \theta=0. \quad (27)$$

$\gamma^-_{r,r}(r,\theta=0)$ can be obtained from Eq. (18),

$$\dot\gamma_{r,r}^-(r,\theta=0)=\frac{\partial}{\partial r}(\tau_r^-/G^-)=-A^-\frac{s(s-1)}{G^-}r^{s-2} \quad (28)$$

whereas $\gamma^+_{r,r}$ can be obtained using Eqs. (5), (7), (20), (21), and (22),

$$\gamma_{r,r}^+(r,\theta=0)=-\frac{1}{G^+}\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}(s-1)r^{s-2}\phi_0'+\frac{\bar B}{\dot a}\left(\frac{\dot a}{\bar B G^+}\right)^{n/(n-1)}r^{n(s-1)}\phi_0'\{(\phi_0')^2+(s\phi_0)^2\}(n-1)/2 \quad (29)$$

where $\phi_0'=\phi^{+\prime}(\theta=0)$. Substituting Eqs. (28) and (29) in Eq. (27) and denoting $\beta=G^+/G^-$ we have

$$(s-1)\phi_0'-\phi_0'\{(\phi_0')^2+(s\phi_0)^2\}^{(n-1)/2}=\beta s(s-1)\phi_0\frac{\cos(s\pi)}{\sin(s\pi)}, \quad (30)$$

from which ϕ_0' can be solved numerically for given values of ϕ_0, $\dot a$ and material properties. Note that the left side of Eq. (30) is a monotonically decreasing function of ϕ_0', so that for a given ϕ_0, if a solution for ϕ_0' exists, then it is unique. Also, Eq. (30) includes the boundary condition for the case of a rigid LHP because it gives $\phi_0'=0$ for $G^-\to\infty$ or $\beta=0$.

The boundary conditions (25) and (30) are insufficient for solving ϕ^+ because ϕ_0 is not known. However, from the boundedness of stresses at finite distance away from the crack tip, we can find another boundary condition, i.e.,

$$\phi^{+\prime\prime}(\theta=0)=-\frac{ns^2\phi_0+\{(\phi_0')^2+(s\phi_0)^2\}^{(n-3)/2}\{s\phi_0(\phi_0')^2(n^2-3n+1)-s^3\phi_0^3\}}{n+(n-1)\{s^2\phi_0^2+n(\phi_0')^2\}\{(\phi_0')^2+s^2\phi_0^2\}^{(n-3)/2}}. \quad (31)$$

A discussion of this condition is given by Hui and Riedel (1981) and Delph and Stengle (1989), but for completeness its derivation is given in the Appendix. The boundedness of $\phi^+(\theta=\pi)$ and $\phi^{+\prime}(\theta=\pi)$ has also been shown by Delph and Stengle (1989).

The numerical strategy for solving ϕ^+ can be outlined as follows. For a guessed value of ϕ_0, we solve for ϕ_0' and $\phi^{+\prime\prime}(\theta=0)$ using Eqs. (30) and (31), respectively. Using ϕ_0 ϕ_0' and $\phi^{+\prime\prime}(\theta=0)$ as initial conditions, ϕ^+ is solved numerically and $\phi_\pi=\phi^+(\theta=\pi)$ is obtained. ϕ_0 is then varied until the condition $\phi_\pi=0$ is satisfied.

Summary of Results.

Stresses and Strains in the UHP. With $s=(n-2)/(n-1)$ and ϕ^+ being known according to the scheme outlined above, we have

$$\psi^+=\left(\frac{\dot a}{\bar B G^+}\right)^{1/(n-1)}r^{(n-2)/(n-1)}\phi^+(\theta) \quad (32)$$

so that stresses are given by

$$\tau_j^+=\alpha_n\left(\frac{\dot a}{\bar B G^+ r}\right)^{1/(n-1)}\tilde\tau_e$$

and

$$\tau_e^+ = \alpha_n \left(\frac{\dot{a}}{\bar{B}G^+ r}\right)^{1/(n-1)} \tilde{\tau}_e$$

where $\alpha_n = s\phi_0$, $j = r$, θ and

$$\tilde{\tau}_r = \frac{\phi^{+\prime}(\theta)}{s\phi_0}$$

$$\tilde{\tau}_\theta = \frac{\phi^+(\theta)}{\phi_0}$$

$$\tilde{\tau}_e^2 = \tilde{\tau}_\theta^2 + \tilde{\tau}_r^2. \quad (33)$$

The strain-rate field is obtained by inserting the stress field into the material law (Eq. (5)). It has a singularity of the form $\dot{\gamma}_i \propto r^{-n/(n-1)}$. The asymptotic strain field in the UHP is obtained using the steady-state condition. The strain has the same type of singularity as the stress, viz.,

$$\gamma_m = \frac{\alpha_n}{G^+} \left(\frac{\dot{a}}{\bar{B}G^+ r}\right)^{1/(n-1)} \tilde{\gamma}_m(\theta), \quad (34)$$

$m = x, y$. The angular variation of $\gamma_m(r, \theta)$, $\tilde{\gamma}_m(\theta)$, can be written as $\tilde{\gamma}_m = \tilde{\gamma}_m^{el} + \tilde{\gamma}_m^{cr}$, where the elastic part is given by $\tilde{\gamma}_m^{el} = \tilde{\tau}_m$, and the creep part is given by

$$\tilde{\gamma}_m^{+cr} = \frac{\alpha_n^{(n-1)}}{(\sin\theta)^{1/(n-1)}} \int_0^\theta \frac{\tilde{\tau}_e^{n-1}\,\tilde{\tau}_m}{(\sin\theta')^{(n-2)/(n-1)}}\, d\theta' \quad (35)$$

where

$$\tilde{\tau}_x = \tilde{\tau}_r \cos\theta - \tilde{\tau}_\theta \sin\theta$$

$$\tilde{\tau}_y = \tilde{\tau}_r \sin\theta + \tilde{\tau}_\theta \cos\theta,$$

$\tilde{\tau}_r$ and $\tilde{\tau}_\theta$ are given by Eq. (33). The limiting values of $\tilde{\gamma}_x^{+cr}$ and $\tilde{\gamma}_y^{+cr}$ at $\theta = 0$ are $(n-1)\phi_0'(s^2\phi_0^2 + \phi_0'^2)^{(n-1)/2}/\alpha_n$ and $(n-1)(s^2\phi_0^2 + \phi_0'^2)^{(n-1)/2}$, respectively.

Stresses and Strains in the LHP. The stress function in the LHP is obtained by using Eqs. (14), (16), and (26), i.e.,

$$\psi^-(r, \theta) = \left(\frac{\dot{a}}{\bar{B}G^+}\right)^{1/(n-1)} \frac{\phi_0}{\sin(s\pi)}\, r^{(n-2)/(n-1)} \sin\{s(\theta+\pi)\}. \quad (36)$$

The associated stresses and strains are found to be

$$\tau_\theta^-(r, \theta) = \alpha_n \left(\frac{\dot{a}}{\bar{B}G^+ r}\right)^{1/(n-1)} \frac{\sin\{s(\theta+\pi)\}}{\phi_0 \sin(s\pi)}$$

$$\tau_r^-(r, \theta) = -\alpha_n \left(\frac{\dot{a}}{\bar{B}G^+ r}\right)^{1/(n-1)} \frac{\cos\{s(\theta+\pi)\}}{\phi_0 \sin(s\pi)}$$

$$\gamma_j^- = \tau_j^+/G^-, \quad j = r, \theta.$$

Figure 2(*a*–*d*) shows the angular functions $\tilde{\tau}_r(\theta)$, $\tilde{\tau}_\theta(\theta)$ and $\tilde{\tau}_e(\theta)$ for $n = 4$, 6 and $\beta = .5$, 1. As expected, for a given value of n, $\tilde{\tau}_r$ at $\theta = 0$ increases monotonically with β, with $\tilde{\tau}_r = 0$ at $\beta = 0$. It is interesting to note that $\tilde{\tau}_\theta(\theta)$ is quite insensitive to n and β. Figure 3 shows that α_n increases with both n and β, which implies that $\tau_\theta^+(r, \theta)$, $\tau_r^+(r, \theta)$ and $\tau_e^+(r, \theta)$ increase with n and β for a given crack speed.

The Case of $1 \le n < 3$

It has been shown by Hui and Riedel (1981) that for $n < 3$, the linear term in Eq. (8) dominates over the nonlinear term as $r \to 0$. The deletion of the nonlinear term from Eq. (8) leads to the elastic stress singularity $\sigma \propto r^{-1/2}$, and the asymptotic solution for both the lower and upper half-planes, $\psi^\pm$, is

$$\psi^\pm = Ar^{1/2} \cos\frac{\theta}{2}, \quad \text{as } r \to 0, \quad (37)$$

where A is an unspecified factor. The solution satisfies the governing differential equation for ψ^+ and ψ^- in the UHP and LHP, the traction-free crack surfaces and continuity of τ_θ across the interface. The factor A in Eq. (37) can be determined by specifying the complete problem including the remote

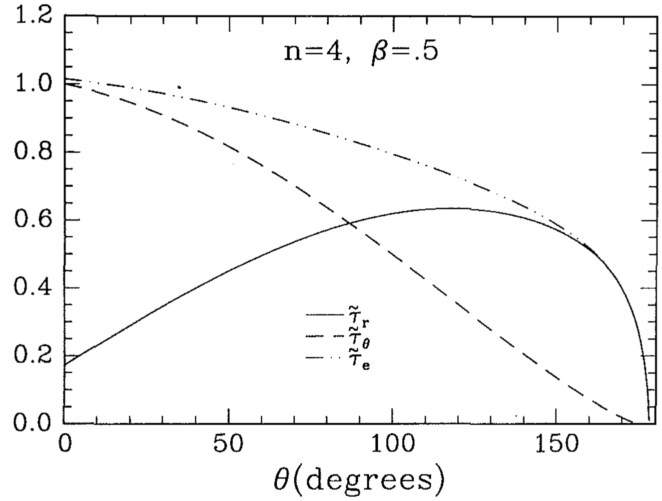

Fig. 2(*a*)

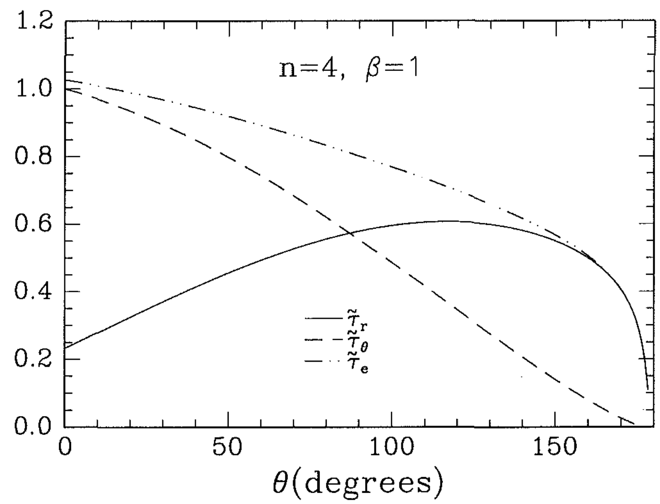

Fig. 2(*b*)

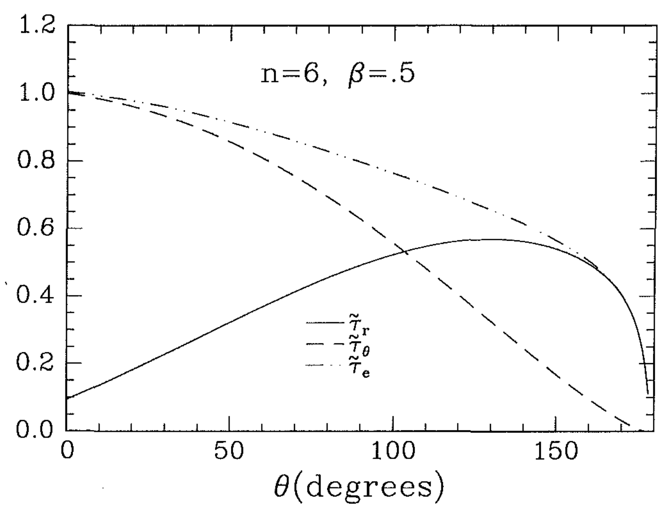

Fig. 2(*c*)

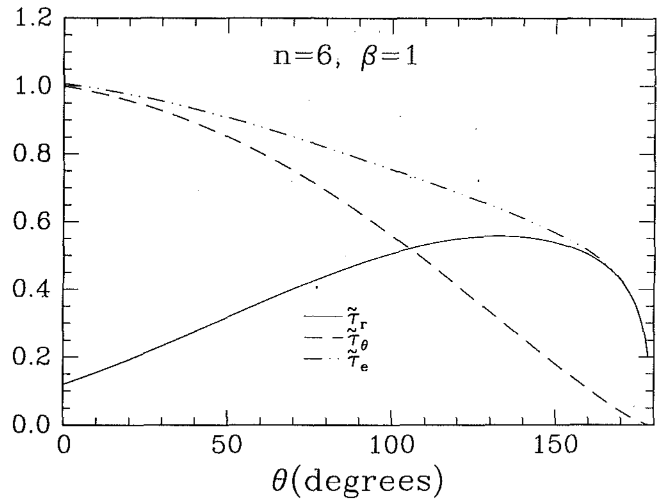

Fig. 2(*d*)

Fig. 2 Angular distribution of the stress components, $\tilde{\tau}_r$ and $\tilde{\tau}_\theta$, and of the equivalent shear stress, $\tilde{\tau}_e$, in the elastic power-law creeping material near the tip of a growing crack in mode III for $n = 4$, 6 and $\beta = G^+/G^- = .5$ and 1

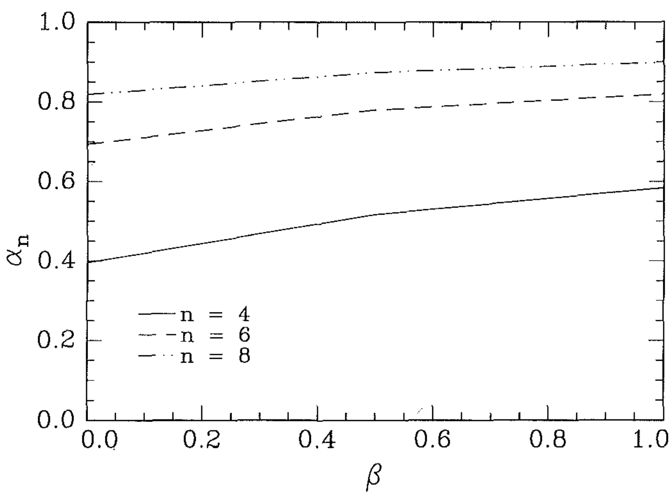

Fig. 3 Variation of $\alpha_n = s\phi_0$ with n and β

boundary conditions. In general, A will be a function of the applied load and the crack growth history.

Nonsteady Crack Growth

The solution for the stress function given in Eqs. (32) and (37) are also asymptotically valid for nonsteady crack growth with a time-dependent growth rate $\dot{a}(t)$. The proof is shown by Hui and Riedel (1981).

Conclusions

The asymptotic stress field near the tip of a quasi-statically growing crack along the interface between a linear elastic and an elastic power-law creeping material subjected to antiplane shear loading has been studied. For $n \geq 3$, the stress and strain field has the form $(\dot{a}/\overline{B}G^+r)^{1/(n-1)} f_i(\theta, n)$. The amplitude of this asymptotic field is thus independent of remote boundary conditions and crack growth history. For $n < 3$, the stress and strain fields are of the form $Ar^{-1/2}f_i(\theta)$, where the constant A is determined by remote boundary conditions and crack growth history.

Acknowledgments

C.-Y. Hui would like to thank Professor Shih of Brown University for his valuable suggestions. Thanks to the Materials Science Center, Cornell University, for providing computing facilities while conducting this work.

References

Bose, K., and Castañeda, P. Ponte, 1992, "Stable Crack Growth Under Mixed Mode Conditions," *J. Mech. Phys. Solids*, Vol. 40, No. 5, pp. 1053–1103.

Champion, C. R., and Atkinson, C., 1990, "A Mode III Crack at the Interface between Two Nonlinear Materials," *Proc. R. Soc. Lond.*, Vol. A429, pp. 247–257.

Champion, C. R., and Atkinson, C., 1991, "A Crack at the Interface between Two Power-Law Materials Under Plane Strain Loading," *Proc. R. Soc. Lond.*, Vol. A432, p. 547.

Chen, Xing-Yu, and Drugan, W. J., 1991, "Plane Strain Elastic-Ideally Plastic Crack Fields for Mode I Quasi-static Growth at Large Scale Yielding II. Global Analytic Solution for Finite Geometries," *J. of the Mechanics and Physics of Solids*, Vol. 39, No. 7, pp. 895–925.

Delph, T. J., and Stengle, G. A., 1989, "An Analysis of the Hui-Riedel Equation," *International Journal of Fracture*, Vol. 40, pp. 295–305.

Drugan, W. J., and Chen, Xing-Yu, 1989, "Plane Strain Elastic-Ideally Plastic Crack Fields for Mode I Quasi-static Growth at Large Scale Yielding I. A New Family of Asymptotic Solutions," *J. of the Mechanics and Physics of Solids*, Vol. 37, No. 1, pp. 1–26.

Drugan, W. J., 1991, "Near Tip Fields for Quasi-Static Crack Growth Along a Ductile-Brittle Interface," ASME Journal of Applied Mechanics, Vol. 58, pp. 111–119.

England, A. H., 1965, "A Crack Between Dissimilar Media," ASME Journal of Applied Mechanics, Vol. 32, pp. 400–402.

Hui, C. Y., and Riedel, H., "The Asymptotic Stress and Strain Field near the Tip of a Growing Crack Under Creep Conditions," *International Journal of Fracture*, Vol. 17, No. 4, Aug. 1981, pp. 409–425.

O'Dowd, N. P., Stout, M. G., and Shih, C. F., 1992, "Fracture Toughness of Alumina/Niobium Interfaces: Experiments and Analyses," *Philosophical Magazine*, Vol. 66, No. 6, pp. 1037–1064.

Rice, J. R., and Sih, G. C., "Plane Problems of Cracks in Dissimilar Media," ASME Journal of Applied Mechanics, Vol. 32, June 1965, pp. 418–423.

Rice, J. R., 1988, "Elastic Fracture Mechanics Concepts for Interfacial Cracks," ASME Journal of Applied Mechanics, Vol. 55, pp. 98–103.

Sharma, S. M., and Aravas, N., 1991, "Determination of Higher Order Terms in Asymptotic Elastoplastic Crack Tip Solutions," *J. Mech. Phys. Solids*, Vol. 39, No. 8, pp. 1043–1072.

Sharma, S. M., and Aravas, N., 1993, "On the Development of Variable-Separable Asymptotic Elastoplastic Solutions for Interfacial Cracks," *Int. J. Solids Structures*, Vol. 30, No. 5, pp. 695–732.

Shih, C. F., 1991, "Cracks on Bimaterial Interfaces: Elasticity and Plasticity Aspects," *Materials Science and Engineering*, Vol. A143, pp. 77–90.

Shih, C. F., and Asaro, R. J., 1988, "Elastic-Plastic Analysis of Cracks on Bimaterial Interfaces: Part I—Small Scale Yielding," ASME Journal of Applied Mechanics, Vol. 55, pp. 299–316.

Shih, C. F., and Asaro, R. J., and O'Dowd, N. P., 1989, "Elastic-Plastic Analysis of Cracks on Bimaterial Interfaces: Part II—Structure of Small Scale Yielding," ASME Journal of Applied Mechanics, Vol. 56, pp. 763–779.

Shih, C. F., and Asaro, R. J., 1991, "Elastic-Plastic Analysis of Cracks on Bimaterial Interfaces: Part III—Large Scale Yielding," ASME Journal of Applied Mechanics, June 1991, Vol. 58, pp. 450–463.

Tippur, H. V., and Rosakis, A. J., 1990, "Quasi-Static and Dynamic Crack Growth along Bimaterial Interfaces: A Note on Crack Tip Field Measurements Using Coherent Grading Sensing," SM Report 90-18, Aug., Graduate Aeronautical Laboratories, California Institute of Technology, Pasadena, CA.

Wang, T. C., 1990, "Elastic-Plastic Asymptotic Fields for Cracks on Bimaterial Interfaces," *Engineering Fracture Mechanics*, Vol. 37, No. 3, pp. 527–538.

Williams, M. L., 1959, "The Stresses Around a Fault or Crack in Dissimilar Media," *Bulletin of the Seismological Society of America*, Vol. 49, No. 2, Apr., pp. 199–204.

APPENDIX

Equation (24) suggests that $\phi^+(\theta)$ may have a singularity at $\theta = 0$ due to the vanishing coefficient, $\sin(\theta)$, of $\phi^{+'''}$ and $\phi^{+'}$. However, we do not expect the stresses τ_θ^+ and τ_r^+ to be singular away from the tip at finite values of r. Thus from Eqs. (20) and (21) we conclude that ϕ^+ and $\phi^{+'}$ are bounded at $\theta = 0$. Following Delph and Stengle (1989), we assume that

$$\phi^+ = \phi_0 + \phi_0'\theta + B_2\theta^{1+\alpha_1} + B_3\theta^2 + B_4\theta^{2+\alpha_2} + \cdots$$

as $\theta \to 0^+$, where B_i, $i = 2, 3, \ldots$ are constants and $0 < \alpha_1, \alpha_2, \ldots < 1$. Then

$$\phi^{+'} = \phi_0' + B_2(1+\alpha_1)\theta^{\alpha_1} + 2B_3\theta + B_4(2+\alpha_2)\theta^{1+\alpha_2} + \cdots$$

$$\phi^{''+} = B_2(1+\alpha_1)\alpha_1\theta^{\alpha_1-1} + 2B_3 + B_4(2+\alpha_2)(1+\alpha_2)\theta^{\alpha_2} + \cdots$$

$$\phi^{'''+} = B_2(\alpha_1^2-1)\alpha_1\theta^{\alpha_1-2} + B_4(2+\alpha_2)(1+\alpha_2)\alpha_2\theta^{\alpha_2-1} + \cdots$$

Thus as $\theta \to 0$, $\phi^{+''} \sim B_2(1+\alpha_1)\alpha_1\theta^{\alpha_1-1}$ and $(\sin(\theta)\phi^{+'''}) \sim B_2(\alpha_1^2-1)\alpha_1\theta^{\alpha_1-1}$. Using these asymptotic values in Eq. (24) as $\theta \to 0$ and retaining only the singular terms (order of θ^{α_1-1}), we get

$$\{\alpha_1 - s + 1 + \{(\phi_0^{+'})^2 + (s\phi_0)^2\}^{(n-3)/2}\{n(\phi_0')^2 + (s\phi_0)^2\}\}\theta^{\alpha_1-1} = 0$$

and

$$\alpha_1 = s - 1 - \{(\phi_0^{+'})^2 + (s\phi_0^+)^2\}^{(n-3)/2}\{n(\phi_0')^2 + (s\phi_0)^2\}$$

which is less than zero, because $s-1 = -1/(n-1) < 0$ $(n>1)$ and the quantities $(\cdot)^2$ are greater than zero. This contradicts our hypothesis of $\alpha_1 > 0$. Thus $\phi^{+''}$ must be bounded at $\theta = 0$, i.e., $\phi^+ = \phi_0 + \phi_0'\theta + B_3\theta^2 + B_4\theta^{2+\alpha_2} + \cdots$, and $(\sin(\theta)\phi^{+'''}) \sim B_4(2+\alpha_2)(1+\alpha_2)\alpha_2\theta^{\alpha_2}$ + higher order terms of θ, as $\theta \to 0$. Hence $\sin(\theta)\phi^{+'''} = 0$ at $\theta = 0$ and from Eq. (24),

$$\phi^{+''}(\theta=0) = -\frac{ns^2\phi_0 + \{(\phi_0')^2 + (s\phi_0)^2\}^{(n-3)/2}\{s\phi_0(\phi_0')^2(n^2-3n+1) - s^3\phi_0^3\}}{n + (n-1)\{s^2\phi_0^2 + n(\phi_0')^2\}\{(\phi_0')^2 + s^2\phi_0^2\}^{(n-3)/2}}$$

which gives an additional boundary condition for Eq. (24).