

# A New Method for Solving Integer Linear Programming Problems with Fuzzy Variables

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## Abstract

A new method namely, decomposition method for solving integer linear programming problems with fuzzy variables by using classical integer linear programming has been proposed. In the decomposition method, ranking functions are not used. The proposed method can serve managers by providing the best solution to a variety of integer linear programming problems with fuzzy variables in a simple and effective manner. With the help of numerical examples, the decomposition method is illustrated.

**Mathematics Subject Classifications:** 90C10, 90C70, 90C90, 65K05

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## 1 Introduction

Linear programming [2] has applications in many fields of operations research. It is concerned with the optimization of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions. In real world situation the available information in the system under consideration are not exact, therefore fuzzy linear programming (FLP) was introduced and studied by many researchers [10, 9, 4, 3, 6, 7, 8, 11]. Fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modeling and industrial applications. The concept of FLP on a general level was first proposed by Tanaka et al. [10]. FLP problems have an essential role in fuzzy modeling, which can formulate uncertainty in actual environment. Afterwards, many

authors have considered various types of the FLP problems and proposed several approaches for solving these problems. In particular, the most convenient methods are based on the concept of comparison of fuzzy numbers with help of ranking functions [3,7,8]. Usually in such methods authors define a crisp model which is equivalent to the FLP problem and then use optimal solution of the model as the optimal solution of the FLP problem. Mahdavi-Amiri and Nasseri [7] introduced a dual simplex algorithm for solving linear programming problem with fuzzy variables and its dual by using a general linear ranking function and linear programming directly.

Recently, Herrera and Verdegay [5] have proposed three methods for solving three models of fuzzy integer linear programming based on the representation theorem and on fuzzy number ranking method. Allahviranloo et al. [1] have proposed a new method based on fuzzy number ranking method for a FILP problem via crisp integer linear programming (ILP) problems. Nasseri [9] has proposed a new method for solving the FLP problems in which he has used the fuzzy ranking method for converting the fuzzy objective function into crisp objective function.

In this paper, we have proposed a new method namely, decomposition method for solving a ILP problem with fuzzy variables by using the classical ILP. The significance of this paper is providing a new method for solving ILP problems with fuzzy variables without using any ranking functions. This method can serve managers by providing the best solution to a variety of integer linear programming problems with fuzzy variables in a simple and effective manner. The decomposition method is illustrated with the help of numerical examples.

## 2 Preliminaries

We need the following definitions of the basic arithmetic operators on fuzzy triangular numbers based on the function principle which can be found in [9].

**Definition 1.** A fuzzy number  $\tilde{a}$  is a triangular fuzzy number denoted by  $(a_1, a_2, a_3)$  where  $a_1, a_2$  and  $a_3$  are real numbers and its membership function  $\mu_{\tilde{a}}(x)$  is given below.

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a_1)/(a_2 - a_1) & \text{for } a_1 \leq x \leq a_2 \\ (a_3 - x)/(a_3 - a_2) & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2.** Let  $(a_1, a_2, a_3)$  and  $(b_1, b_2, b_3)$  be two triangular fuzzy numbers. Then

$$(i) (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

- (ii)  $(a_1, a_2, a_3) \ominus (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$ .
- (iii)  $k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$ , for  $k \geq 0$ .
- (iv)  $k(a_1, a_2, a_3) = (ka_3, ka_2, ka_1)$ , for  $k < 0$ .

Let  $F(R)$  be the set of all real triangular fuzzy numbers.

**Definition 3.** Let  $\tilde{A} = (a_1, a_2, a_3)$  and  $\tilde{B} = (b_1, b_2, b_3)$  be in  $F(R)$ . Then ,

- (i)  $\tilde{A} = \tilde{B} \Leftrightarrow a_i = b_i$ , for all for  $i = 1$  to  $3$  and
- (ii)  $\tilde{A} \leq \tilde{B} \Leftrightarrow a_i \leq b_i$ , for all for  $i = 1$  to  $3$ .

**Definition 4.** Let  $\tilde{A} = (a_1, a_2, a_3)$  be in  $F(R)$ . Then,

- (i)  $\tilde{A}$  is said to be positive if  $a_i \geq 0$ , for all for  $i = 1$  to  $3$ ;
- (ii)  $\tilde{A}$  is said to be integer if  $a_i \geq 0, \forall i = 1$  to  $3$  are integers and
- (iii)  $\tilde{A}$  is said to be symmetric if  $a_2 - a_1 = a_3 - a_2$ .

**Definition 5.** A real fuzzy vector  $\tilde{b} = (\tilde{b}_i)_{m \times 1}$  is called nonnegative and denoted by  $\tilde{b} \geq 0$ , if each element of  $\tilde{b}$  is a nonnegative real fuzzy number, that is,  $\tilde{b}_i \geq 0, 1, 2, \dots, m$ .

Consider the following  $m \times n$  fuzzy linear system with nonnegative real triangular fuzzy numbers:

$$A\tilde{x} \leq \tilde{b}, \tag{1}$$

where  $A = (a_{ij})_{m \times n}$  is a nonnegative crisp matrix and  $\tilde{x} = (\tilde{x}_j), \tilde{b} = (\tilde{b}_i)$  are nonnegative fuzzy vectors and  $\tilde{x}_j, \tilde{b}_i \in F(R)$ , for all  $1 \leq j \leq n$  and  $1 \leq i \leq m$ .

**Definition 6.** A nonnegative fuzzy vector  $\tilde{x}$  is said to be a solution of the fuzzy linear system (1) if  $\tilde{x}$  satisfies equation (1).

Using the definitions 3 and 6 and the arithmetic operations on triangular fuzzy number, we obtain the following theorem.

**Theorem 1.** Let  $A\tilde{x} \leq \tilde{b}$  be an  $m \times n$  fuzzy linear system where  $A = (a_{ij})_{m \times n}$  is a nonnegative crisp matrix ,  $\tilde{x} = (\tilde{x}_j), \tilde{b} = (\tilde{b}_i)$  are nonnegative real triangular fuzzy vectors and  $\tilde{x}_j = (x_1^j, x_2^j, x_3^j)$  and  $\tilde{b}_i = (b_1^i, b_2^i, b_3^i) \in F(R)$ , for all  $1 \leq j \leq n$  and  $1 \leq i \leq m$ . If  $x_2^\circ = (x_2^{j^\circ})_{n \times 1}$  is a solution of the system  $Ax_2 \leq b_2, x_2 \geq 0$  where  $x_2 = (x_2^j)_{n \times 1}$  and  $b_2 = (b_2^i)_{m \times 1}$ ,  $x_1^\circ = (x_1^{j^\circ})_{n \times 1}$  is a solution of the system  $Ax_1 \leq b_1, x_1 \geq 0, x_1 - x_2^\circ \leq 0$  where  $x_1 = (x_1^j)_{n \times 1}$  and  $b_1 = (b_1^i)_{m \times 1}$  and  $x_3^\circ = (x_3^{j^\circ})_{n \times 1}$  is

a solution of the system  $Ax_3 \leq b_3$ ,  $x_3 - x_2^\circ \geq 0$  where  $x_3 = (x_3^j)_{mx1}$  and  $b_3 = (b_3^i)_{mx1}$ , then  $\tilde{x}^\circ = (\tilde{x}_j^\circ)$  is a solution of the system  $A\tilde{x} \leq \tilde{b}$  where  $\tilde{x}_j^\circ = (x_1^{j^\circ}, x_2^{j^\circ}, x_3^{j^\circ})$ .

**Note 1.** If  $\tilde{x} = (x_1, x_2, x_3)$  and  $\tilde{b} = (b_1, b_2, b_3)$  are symmetric, we can obtain  $x_3^\circ$  from the relation  $x_3^\circ = x_2^\circ + (x_2^\circ - x_1^\circ)$  without solving the last system  $Ax_3 = b_3$ ,  $x_3 - x_2^\circ \geq 0$ .

### 3 Fuzzy integer linear programming

Consider the following integer linear programming problem with fuzzy variables :

$$(P) \quad \text{Maximize} \quad \tilde{z} = c\tilde{x}$$

$$\text{subject to} \quad A\tilde{x} \leq \tilde{b}, \quad (2)$$

$$\tilde{x} \geq 0 \text{ and are integers,} \quad (3)$$

where the coefficient matrix  $A = (a_{ij})_{m \times n}$  is a nonnegative real crisp matrix, the cost vector  $c = (c_1, \dots, c_n)$  is nonnegative crisp vector and  $\tilde{x} = (\tilde{x}_j)_{nx1}$  and  $\tilde{b} = (\tilde{b}_i)_{mx1}$  are nonnegative real fuzzy vectors such that  $\tilde{x}_j, \tilde{b}_i \in F(R)$  for all  $1 \leq j \leq n$  and  $1 \leq i \leq m$ .

**Definition 7.** A fuzzy vector  $\tilde{x}$  is said to be a feasible solution of the problem (P) if  $\tilde{x}$  satisfies (2) and (3).

**Definition 8.** A feasible solution  $\tilde{x}$  of the problem (P) is said to be an optimal solution of the problem (P) if there exists no feasible  $\tilde{u} = (\tilde{u}_j)_{nx1}$  of (P) such that  $c\tilde{u} > c\tilde{x}$ .

Using the Theorem 1. and the arithmetic operations of fuzzy numbers, we can obtain the following result.

**Theorem 2.** A fuzzy vector  $\tilde{x}^\circ = (x_1^\circ, x_2^\circ, x_3^\circ)$  is an optimal solution of the problem (P) iff  $x_2^\circ$ ,  $x_1^\circ$  and  $x_3^\circ$  are optimal solutions of the following crisp integer linear programming problems (P<sub>2</sub>), (P<sub>1</sub>) and (P<sub>3</sub>) respectively where

$$(P_2) \quad \text{Maximize} \quad z_2 = cx_2$$

$$\text{subject to} \quad Ax_2 \leq b_2, x_2 \geq 0 \text{ are integers ;}$$

$$(P_1) \quad \text{Maximize} \quad z_1 = cx_1$$

$$\text{subject to} \quad Ax_1 \leq b_1, x_1 \geq 0, x_1 \leq x_2^\circ \text{ and are integers}$$

and

$$(P_3) \quad \text{Maximize} \quad z_3 = cx_3$$

$$\text{subject to} \quad Ax_3 \leq b_3, \quad x_3 \geq 0, \quad x_3 \geq x_2^\circ \text{ and are integers.}$$

**Proof:** Suppose that  $\tilde{x}^\circ = (x_1^\circ, x_2^\circ, x_3^\circ)$  is an optimal solution of the problem (P).

Let  $\tilde{x} = (x_1, x_2, x_3)$  be a feasible solution of the problem (P).

This implies that

$$c_1x_1 \leq c_1x_1^\circ; \quad c_2x_2 \leq c_2x_2^\circ; \quad c_3x_3 \leq c_3x_3^\circ;$$

$$Ax_1^\circ \leq b_1; \quad Ax_2^\circ \leq b_2; \quad Ax_3^\circ \leq b_3, \quad x_1^\circ, x_2^\circ, x_3^\circ \geq 0. \quad (4)$$

Let  $\tilde{z} = (z_1, z_2, z_3)$  be the objective function of the problem (P).

Now, from (4) we have,

$$\text{Max. } z_1 = c_1x_1^\circ; \quad \text{Max. } z_2 = c_2x_2^\circ; \quad \text{Max. } z_3 = c_2x_3^\circ \quad (5)$$

Now, from (4) and (5), we can conclude that  $x_2^\circ$ ,  $x_1^\circ$  and  $x_3^\circ$  are optimal solutions of the crisp integer linear programming problems (P<sub>2</sub>), (P<sub>1</sub>) and (P<sub>3</sub>).

Suppose that  $x_2^\circ$ ,  $x_1^\circ$  and  $x_3^\circ$  are optimal solutions of the crisp integer linear programming problems (P<sub>2</sub>), (P<sub>1</sub>) and (P<sub>3</sub>) with optimal values  $z_2^\circ$ ,  $z_1^\circ$  and  $z_3^\circ$  respectively. This implies that  $\tilde{x}^\circ = (x_1^\circ, x_2^\circ, x_3^\circ)$  is an optimal solution of the problem (P) with optimal value  $\tilde{z}^\circ = (z_1^\circ, z_2^\circ, z_3^\circ)$ .

Hence the theorem.

## 4 Numerical Examples

The proposed method is illustrated by the following examples.

**Example 1.** Consider the following integer linear programming problem with fuzzy variables:

$$(P) \quad \text{Maximize} \quad \tilde{z} = 10\tilde{x}_1 + 20\tilde{x}_2$$

$$\text{subject to}$$

$$6\tilde{x}_1 + 8\tilde{x}_2 \leq (46, 48, 60)$$

$$\tilde{x}_1 + 3\tilde{x}_2 \leq (7, 12, 20)$$

$$\tilde{x}_1, \tilde{x}_2 \geq 0 \text{ and are integers.}$$

Let  $\tilde{z} = (z_1, z_2, z_3)$ ,  $\tilde{x}_1 = (y_1, x_1, t_1)$  and  $\tilde{x}_2 = (y_2, x_2, t_2)$ .

Now, the problem (P<sub>2</sub>) is given below:

$$(P_2) \quad \text{Maximize} \quad z_2 = 10x_1 + 20x_2$$

subject to

$$6x_1 + 8x_2 \leq 48; \quad x_1 + 3x_2 \leq 12$$

$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Now, using an algorithm for ILP problem, the solution of the problem (P<sub>2</sub>) is  $x_1 = 5$ ,  $x_2 = 2$  and  $z_2 = 90$ .

Now, the problem (P<sub>1</sub>) is given below:

$$(P_1) \text{ Maximize } z_1 = 10y_1 + 20y_2$$

subject to

$$6y_1 + 8y_2 \leq 46; \quad y_1 + 3y_2 \leq 7; \quad y_1 \leq 5; \quad y_2 \leq 2$$

$$y_1, y_2 \geq 0 \text{ and are integers.}$$

Now, using an algorithm for ILP problem, the solution of the problem (P<sub>1</sub>) is

$$y_1 = 4, y_2 = 1 \text{ and } z_1 = 60.$$

Now, the problem (P<sub>3</sub>) is given below:

$$(P_3) \text{ Maximize } z_3 = 10t_1 + 20t_2$$

subject to

$$6t_1 + 8t_2 \leq 60; \quad t_1 + 3t_2 \leq 20; \quad t_1 \geq 5; \quad t_2 \geq 2$$

$$t_1, t_2 \geq 0 \text{ and are integers.}$$

Now, using an algorithm for ILP problem, the solution of the problem (P<sub>3</sub>) is

$$t_1 = 6, t_2 = 3 \text{ and } z_3 = 120.$$

Therefore, the solution for the given fuzzy integer linear programming problem is

$$\tilde{x}_1 = (y_1, x_1, t_1) = (4, 5, 6), \quad \tilde{x}_2 = (y_2, x_2, t_2) = (1, 2, 3) \text{ and } \tilde{z} = (60, 90, 120).$$

**Example 2.** Consider the following integer linear programming problem with fuzzy variables:

$$(P) \text{ Maximize } \tilde{z} = 4\tilde{x}_1 + 3\tilde{x}_2$$

subject to

$$\tilde{x}_1 + 2\tilde{x}_2 \leq (4, 8, 12)$$

$$2\tilde{x}_1 + \tilde{x}_2 \leq (6, 9, 12)$$

$$\tilde{x}_1, \tilde{x}_2 \geq 0 \text{ and are integers.}$$

Let  $\tilde{z} = (z_1, z_2, z_3)$ ,  $\tilde{x}_1 = (y_1, x_1, t_1)$  and  $\tilde{x}_2 = (y_2, x_2, t_2)$  and also,  $\tilde{x}_1$  and  $\tilde{x}_2$  be symmetric.

Now, the problem (P<sub>2</sub>) is given below:

$$(P_2) \text{ Maximize } z_2 = 4x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 8; \quad 2x_1 + x_2 \leq 9;$$

$$x_1, x_2 \geq 0 \text{ and are integers.}$$

Now, by using an algorithm for ILP problem, the solution of the problem (P<sub>2</sub>) is

$$z_2 = 19, x_1 = 4 \text{ and } x_2 = 1.$$

Now, the problem (P<sub>1</sub>) is given below:

$$(P_1) \text{ Maximize } z_1 = 4y_1 + 3y_2$$

subject to

$$y_1 + 2y_2 \leq 4; 2y_1 + y_2 \leq 6; y_1 \leq 4; y_2 \leq 1$$

$$y_1, y_2 \geq 0 \text{ and are integers.}$$

Now, using an algorithm for ILP problem, solution of the problem (P<sub>1</sub>) is

$$z_1 = 12, y_1 = 3 \text{ and } y_2 = 0.$$

Now, since  $\tilde{x}_1 = (y_1, x_1, t_1)$  and  $\tilde{x}_2 = (y_2, x_2, t_2)$  are symmetric, we have

$$t_1 = 5, t_2 = 2 \text{ and } z_3 = 26.$$

Therefore, the solution for the given fuzzy integer linear programming problem is

$$\tilde{x}_1 = (y_1, x_1, t_1) = (3, 4, 5); \tilde{x}_2 = (y_2, x_2, t_2) = (0, 1, 2) \text{ and } \tilde{z} = (12, 19, 26).$$

## 5 Conclusion

The decomposition method provides an optimal solution to FILP problems without using ranking functions and applying classical integer linear programming. This method can serve managers by providing the best solution to a variety of integer linear programming problems with fuzzy variables in a simple and effective manner.

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