Research Article

An Improvement of Global Error Bound for the Generalized Nonlinear Complementarity Problem over a Polyhedral Cone

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We consider the global error bound for the generalized nonlinear complementarity problem over a polyhedral cone (GNCP). By a new technique, we establish an easier computed global error bound for the GNCP under weaker conditions, which improves the result obtained by Sun and Wang (2013) for GNCP.

1. Introduction

Let $\mathcal{K} = \{v \in R^m \mid Av \geq 0, Bv = 0\}$ be a polyhedral cone in R^m for matrices $A \in R^{s \times m}$, $B \in R^{t \times m}$, and let $\mathcal{K}^\circ$ be its dual cone; that is,

$$\mathcal{K}^\circ = \left\{ u \in R^m \mid u = A^\top \lambda_1 + B^\top \lambda_2, \lambda_1 \in R_+^s, \lambda_2 \in R^t \right\}. \quad (1)$$

For continuous mappings, $F, G : R^n \to R^m$, the generalized nonlinear complementarity problem, abbreviated as GNCP, is to find vector $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \qquad G(x^*) \in \mathcal{K}^\circ, \qquad F(x^*)^\top G(x^*) = 0. \quad (2)$$

Throughout this paper, the solution set of the GNCP, denoted by X^*, is assumed to be nonempty.

The GNCP is a direct generalization of the classical nonlinear complementarity problem and a special case of the general variational inequalities problem [1]. The GNCP was deeply discussed [2–7] after the work in [8]. It is well known that the global error bound is an important tool for theoretical analysis and numerical treatment for a mathematical problem [9, 10]. The global error bound estimation for GNCP with the mapping being γ-strongly G-monotone was discussed in [4], a global error bound estimation for GNCP with the mapping being γ-strongly monotone and Hölder continuous was established in [5], and a global error bound for the GNCP

for the linear and monotonic case was also established in [6, 7].

This paper is a follow-up to [4, 5, 11], as in these papers we will establish the global error bound estimation of the GNCP under weaker conditions than that needed in [4, 5, 11]. Based on a new technique, we establish a global error bound for the GNCP in terms of an easier computed residual function. The results obtained in this paper can be taken as an improvement of the existing results for GNCP and variational inequalities problem [4, 5, 11–13].

To end this section, we give some notations used in this paper. Vectors considered in this paper are taken in Euclidean space R^n equipped with the usual inner product, and the Euclidean 2-norm and 1-norm of vector in R^n are, respectively, denoted by $\|\cdot\|$ and $\|\cdot\|_1$. We use R_+^n to denote the nonnegative orthant in R^n and use x_+ and x_- to denote the vectors composed by elements $(x_+)_i := \max\{x_i, 0\}$, $(x_-)_i := \max\{-x_i, 0\}$ and $1 \leq i \leq n$, respectively. For simplicity, we use $(x; y)$ to denote vector $(x^\top, y^\top)^\top$, use I to denote the identity matrix with appropriate dimension, use $x \geq 0$ to denote a nonnegative vector $x \in R^n$, and use $\mathrm{dist}(x, X^*)$ to denote the distance from point x to the solution set X^*.

2. Global Error Bound for the GNCP

First, we give the following definition used in the subsequent.

Definition 1. The mapping $F : R^n \to R^m$ is said to be γ-uniform p-function with respect to $G : R^n \to R^m$ if there are constants $c_1 > 0$ and $\gamma > 0$ such that

$$\max_{1 \le i \le m} \left\{ [F(x) - F(y)]_i [G(x) - G(y)]_i \right\}$$

$$\ge c_1 \|G(x) - G(y)\|^{1+\gamma}, \tag{3}$$

$$\forall x, y \in R^n.$$

Remark 2. Based on this definition, γ-uniform p-function with respect to G is weaker than γ-strongly G-monotonicity by Definition 1 in [4], and if

$$F(x) = Mx + p, \qquad G(x) = Nx + q \tag{4}$$

with $M, N \in R^{m \times n}$, $p, q \in R^m$, then the above definition is equivalent in which the matrix $M^\top N$ is a p-matrix [14]. For example, let

$$M = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad p = q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

By Theorem 2.1.15 in [14], it is easy to verify that $M^\top N$ is a p-matrix. However, letting $x = (1; 2)$, we note that $x^\top M^\top N x = -1 < 0$ which shows that $M^\top N$ is not positive definite; that is, F is not strongly monotonicity with respect to mapping G.

Now, we give some assumptions for our analysis based on Definition 1.

Assumption 3. For mappings F, G and matrix A involved in the GNCP, we assume that

   (A1) mapping F is γ-uniform p-function with respect to mapping G;

   (A2) matrix $A^\top$ has full-column rank.

In the following, we give the conclusion established in [2].

Theorem 4. *A point $x^* \in R^n$ is a solution of the GNCP if and only if there exist $\lambda_1^* \in R^s$ and $\lambda_2^* \in R^t$, such that*

$$AF(x^*) \ge 0,$$

$$BF(x^*) = 0,$$

$$\lambda_1^* \ge 0, \tag{6}$$

$$(F(x^*))^\top G(x^*) = 0,$$

$$G(x^*) = A^\top \lambda_1^* + B^\top \lambda_2^*.$$

From Theorem 4, under Assumption 3 (A2), similar to discussion in [4], we can transform the system (6) into the following system in which neither λ_1 nor λ_2 is involved:

$$AF(x) \ge 0,$$

$$BF(x) = 0,$$

$$(F(x))^\top G(x) = 0, \tag{7}$$

$$UG(x) \ge 0,$$

$$VG(x) = 0,$$

where

$$U = \left\{ -A_L^{-1} B^\top \left[\left(A^\top A_L^{-1} - I \right) B^\top \right]^+ \left[A^\top A_L^{-1} - I \right] + A_L^{-1} \right\},$$

$$V = \left\{ A^\top \left\{ -A_L^{-1} B^\top \left[\left(A^\top A_L^{-1} - I \right) B^\top \right]^+ \left[A^\top A_L^{-1} - I \right] + A_L^{-1} \right\} \right.$$

$$\left. + B^\top \left[\left(A^\top A_L^{-1} - I \right) B^\top \right]^+ \left[A^\top A_L^{-1} - I \right] - I \right\}. \tag{8}$$

For the ease of description, let $\mu = F(x)$ and $\nu = G(x)$. Then, system (7) can be written as

$$A\mu \ge 0,$$

$$B\mu = 0,$$

$$\mu^\top \nu = 0, \tag{9}$$

$$U\nu \ge 0,$$

$$V\nu = 0,$$

where the solution set of (9) is denoted by Ω^*.

In the following, we give the error bound for a single quadratic function to reach our aims.

Lemma 5. *Let $S_1 := \{ \omega \in R^{2m} \mid f(\omega) = 0 \}$. Then, one has*

$$\operatorname{dist}(\omega, S_1) \le \tau \|f(\omega)\|^{1/2}, \tag{10}$$

where $\tau > 0$ is a constant, $\omega = (\mu; \nu)$, and $f(\omega) = \mu^\top \nu$.

Proof. For any $\omega \in R^{2m}$, let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)^\top$, $\nu = (\nu_1, \nu_2, \ldots, \nu_m)^\top$, and

$$\omega_i = \begin{cases} \mu_i & 1 \le i \le m, \\ \nu_i & m+1 \le i \le 2m. \end{cases} \tag{11}$$

Set $\mu_i = \xi_i + \xi_{m+i}$, $\nu_i = \xi_i - \xi_{m+i}$, $i = 1, 2, \ldots, m$, and $\xi = (\xi_1, \xi_2, \ldots, \xi_{2m})^\top$. Obviously, this linear transformation is an invertible; that is, there exists an invertible matrix $P \in R^{2m \times 2m}$ such that $\omega = P\xi$, and one has

$$f(\omega) = \mu^\top \nu = \sum_{i=1}^m \mu_i \nu_i = \sum_{i=1}^m \xi_i^2 - \sum_{i=m+1}^{2m} \xi_i^2 =: g(\xi). \tag{12}$$

Without loss of generality, we assume $f(\omega) > 0$. Define

$$\theta = \left(\frac{\sum_{i=m+1}^{2m} \xi_i^2}{g(\xi) + \sum_{i=m+1}^{2m} \xi_i^2} \right)^{1/2} = \left(\frac{\sum_{i=m+1}^{2m} \xi_i^2}{\sum_{i=1}^m \xi_i^2} \right)^{1/2},$$

$$\bar{\xi}_i = \begin{cases} \theta \xi_i & 1 \le i \le m, \\ \xi_i & m+1 \le i \le 2m. \end{cases} \tag{13}$$

It is easy to verify that $0 \leq \theta \leq 1$ and

$$
\begin{aligned}
g\left(\overline{\xi}\right) &= \theta^2 \sum_{i=1}^{m} \xi_i^2 - \sum_{i=m+1}^{2m} \xi_i^2 \\
&= \left(\frac{\sum_{i=m+1}^{2m} \xi_i^2}{\sum_{i=1}^{m} \xi_i^2} \right) \sum_{i=1}^{m} \xi_i^2 - \sum_{i=m+1}^{2m} \xi_i^2 \\
&= \sum_{i=m+1}^{2m} \xi_i^2 - \sum_{i=m+1}^{2m} \xi_i^2 \\
&= 0.
\end{aligned}
\tag{14}
$$

Let $\overline{\omega} = P\overline{\xi}$, and one has $f(\overline{\omega}) = g(\overline{\xi}) = 0$. Therefore, $\overline{\omega} \in S_1$. Moreover, one has

$$
\begin{aligned}
&\|\omega - \overline{\omega}\| \\
&= \left\| P\xi - P\overline{\xi} \right\| \\
&\leq \|P\| \left\| \xi - \overline{\xi} \right\| \\
&= \|P\| \left(\sum_{i=1}^{2m} \left(\xi_i - \overline{\xi}_i \right)^2 \right)^{1/2} \\
&= \|P\| \left(\sum_{i=1}^{m} \left(\xi_i - \theta \xi_i \right)^2 \right)^{1/2} \\
&= \|P\| \left(1 - \theta \right) \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} \\
&= \|P\| \frac{\left(1 - \theta^2 \right)}{1 + \theta} \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} \\
&= \frac{\|P\|}{1 + \theta} \left(1 - \frac{\sum_{i=m+1}^{2m} \xi_i^2}{g(\xi) + \sum_{i=m+1}^{2m} \xi_i^2} \right) \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} \\
&= \frac{\|P\| \, g(\xi)}{(1 + \theta) \left(\sum_{i=1}^{m} \xi_i^2 \right)} \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} \\
&= \frac{\|P\| \, g(\xi)}{(1 + \theta) \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2}} \\
&= \frac{\|P\| \, g(\xi)}{\left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} + \theta \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2}} \\
&= \frac{\|P\| \, g(\xi)}{\left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} + \left(\left(\sum_{i=m+1}^{2m} \xi_i^2 \right) / \left(\sum_{i=1}^{m} \xi_i^2 \right) \right)^{1/2} \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2}} \\
&= \frac{\|P\| \, g(\xi)}{\left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2} + \left(\sum_{i=m+1}^{2m} \xi_i^2 \right)^{1/2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\|P\| \, g(\xi)}{\left(\sum_{i=1}^{m} \xi_i^2 + \sum_{i=m+1}^{2m} \xi_i^2 \right)^{1/2}} \\
&\leq \frac{\|P\| \, g(\xi)}{g(\xi)^{1/2}} = \|P\| \, g(\xi)^{1/2} = \|P\| \, f(\omega)^{1/2},
\end{aligned}
\tag{15}
$$

where the third equality follows from the definition of $\overline{\xi}$, the sixth and tenth equations are due to the definition of θ, respectively, the second inequality follows from the fact that

$$
a^{1/2} + b^{1/2} \geq (a + b)^{1/2}, \quad \forall a, b \in R_+,
\tag{16}
$$

and the third inequality follows from the fact that

$$
g(\xi) \leq \sum_{i=1}^{m} \xi_i^2 + \sum_{i=m+1}^{2m} \xi_i^2.
\tag{17}
$$

And, letting $\tau = \|P\|$, then the desired result follows. $\qquad \square$

To establish a global error bound for GNCP, we also give the following result from [15] on the error bound for a polyhedral cone.

Lemma 6. *For polyhedral cone $P = \{x \in R^n \mid D_1 x = d_1, B_1 x \leq b_1\}$ with $D_1 \in R^{l \times n}$, $B_1 \in R^{m \times n}$, $d_1 \in R^l$, and $b_1 \in R^m$, there exists a constant $c_2 > 0$ such that*

$$
\mathrm{dist}\,(x, P) \leq c_2 \left[\left\| D_1 x - d_1 \right\| + \left\| \left(B_1 x - b_1 \right)_+ \right\| \right], \quad \forall x \in R^n.
\tag{18}
$$

Before proceeding, we present the following definition introduced in [16] with constant $\delta = 1$.

Definition 7. The mapping $G : R^n \to R^m$ is said to be δ-strongly nonexpanding with a constant $\alpha > 0$, if $\|G(x) - G(y)\| \geq \alpha \|x - y\|^\delta$, where $\delta > 0$.

Now, we are at the position to state our main results in this paper.

Theorem 8. *Suppose that F is γ-uniform p-function with respect to mapping G with positive constants c_1 and γ, respectively, and G is δ-strongly nonexpanding with positive constants α and δ, respectively. Then there exists constant $\rho_1 > 0$ such that*

$$
\begin{aligned}
\mathrm{dist}\,(x, X^*) \leq \rho_1 \Big\{ &\left\| [AF(x)]_- \right\| + \|BF(x)\| \\
&+ \left\| [UG(x)]_- \right\| + \|VG(x)\| \\
&+ \left\| \left[F(x)^\top G(x) \right] \right\|^{1/2} \Big\}^{2/(1+\gamma)\delta}, \quad \forall x \in R^n.
\end{aligned}
\tag{19}
$$

Proof. Using Lemma 5, for any $\omega \in R^{2m}$, there exists $\overline{\omega} \in S_1$ such that

$$
\|\omega - \overline{\omega}\| \leq \tau \left\| [f(\omega)] \right\|^{1/2},
\tag{20}
$$

where S_1 is defined in Lemma 5. Let

$$\Omega = \left\{ \omega \in R^{2m} \mid A(I,0)\,\omega \geq 0, B(I,0)\,\omega = 0, \right.$$
$$\left. U(0,I)\,\omega \geq 0, V(0,I)\,\omega = 0 \right\}. \tag{21}$$

From (9), we have $\Omega^* = \Omega \bigcap S_1$. For convenience, we also let

$$\Psi(\omega) = (-A(I,0)\,\omega, -B(I,0)\,\omega, -U(0,I)\,\omega,$$
$$-V(0,I)\,\omega, B(I,0)\,\omega, V(0,I)\,\omega)_+. \tag{22}$$

Using Lemma 6, for any $\omega \in S_1$, there exists $\omega^* \in \Omega^*$ such that

$$\|\omega - \omega^*\| \leq c_3 \left[\|(-A(I,0)\,\omega)_+\| + \|(-U(0,I)\,\omega)_+\| \right.$$
$$\left. + \|B(I,0)\,\omega\| + \|V(0,I)\,\omega\| \right]$$
$$\leq c_3 \left[\|(-A(I,0)\,\omega)_+\| + \|(-U(0,I)\,\omega)_+\| \right.$$
$$+ \|(B(I,0)\,\omega)_+\|$$
$$+ \|(-B(I,0)\,\omega)_+\| + \|(V(0,I)\,\omega)_+\|$$
$$\left. + \|(-V(0,I)\,\omega)_+\| \right]$$
$$\leq c_3 \left\{ \|(-A(I,0)\,\omega)_+\|_1 + \|(-U(0,I)\,\omega)_+\|_1 \right.$$
$$+ \|(B(I,0)\,\omega)_+\|_1$$
$$+ \|(-B(I,0)\,\omega)_+\|_1 + \|(V(0,I)\,\omega)_+\|_1$$
$$\left. + \|(-V(0,I)\,\omega)_+\|_1 \right\}$$
$$= c_3 \|\Psi(\omega)\|_1$$
$$\leq c_3 \sqrt{2s + 2t + 2m}\, \|\Psi(\omega)\|, \tag{23}$$

where c_3 is a positive constant and the third and fourth inequalities follow from the fact that $\|x\| \leq \|x\|_1 \leq \sqrt{n}\|x\|$, for all $x \in R^n$.

Furthermore,

$$\|\Psi(\omega) - \Psi(\overline{\omega})\|$$
$$= \left\| (-A(I,0)\,\omega, -B(I,0)\,\omega, -U(0,I)\,\omega, \right.$$
$$-V(0,I)\omega, B(I,0)\omega, V(0,I)\omega)_+$$
$$- (-A(I,0)\,\overline{\omega}, -B(I,0)\,\overline{\omega}, -U(0,I)\,\overline{\omega},$$
$$\left. -V(0,I)\overline{\omega}, B(I,0)\overline{\omega}, V(0,I)\overline{\omega})_+ \right\|$$
$$= \left\| P_{R_+^{2s+2t+2m}} \{(-A(I,0)\,\omega, -B(I,0)\,\omega, -U(0,I)\,\omega, \right.$$
$$- V(0,I)\,\omega, B(I,0)\,\omega, V(0,I)\,\omega)\}$$
$$- P_{R_+^{2s+2t+2m}} \{(-A(I,0)\,\overline{\omega}, -B(I,0)\,\overline{\omega}, -U(0,I)\,\overline{\omega},$$
$$\left. - V(0,I)\,\overline{\omega}, B(I,0)\,\overline{\omega}, V(0,I)\,\overline{\omega})\} \right\|$$

$$\leq \left\| \{(-A(I,0)\,\omega, -B(I,0)\,\omega, -U(0,I)\,\omega, \right.$$
$$- V(0,I)\,\omega, B(I,0)\,\omega, V(0,I)\,\omega)\}$$
$$- \{(-A(I,0)\,\overline{\omega}, -B(I,0)\,\overline{\omega}, -U(0,I)\,\overline{\omega},$$
$$\left. - V(0,I)\,\overline{\omega}, B(I,0)\,\overline{\omega}, V(0,I)\,\overline{\omega})\} \right\|$$
$$\leq \|A(I,0)\,\omega - A(I,0)\,\overline{\omega}\| + 2\|B(I,0)\,\omega - B(I,0)\,\overline{\omega}\|$$
$$+ \|U(0,I)\,\omega - U(0,I)\,\overline{\omega}\| + 2\|V(0,I)\,\omega - V(0,I)\,\overline{\omega}\|$$
$$\leq (\|A(I,0)\| + 2\|B(I,0)\| + \|U(0,I)\|$$
$$+ 2\|V(0,I)\|)\, \|\omega - \overline{\omega}\|, \tag{24}$$

where the second equality follows from the fact that

$$\min\{a,b\} = a - P_{R_+}(a - b), \quad \forall a, b \in R, \tag{25}$$

and the first inequality is by nonexpanding property of projection operator. Combining (24), one has

$$\|\Psi(\overline{\omega})\| \leq \|\Psi(\omega)\|$$
$$+ (\|A(I,0)\| + 2\|B(I,0)\| \tag{26}$$
$$+ \|U(0,I)\| + 2\|V(0,I)\|)\, \|\omega - \overline{\omega}\|.$$

Combining (23) with (26), for any $\omega \in R^{2m}$, we have

$$\|\omega - \omega^*\| \leq \|\omega - \overline{\omega}\| + \|\overline{\omega} - \omega^*\|$$
$$\leq \|\omega - \overline{\omega}\| + \sigma \|\Psi(\overline{\omega})\|$$
$$\leq \|\omega - \overline{\omega}\|$$
$$+ \sigma (\|\Psi(\omega)\| + (\|A(I,0)\| + 2\|B(I,0)\|$$
$$+ \|U(0,I)\|$$
$$+ 2\|V(0,I)\|)\, \|\omega - \overline{\omega}\|)$$
$$\leq \sigma \|\Psi(\omega)\|$$
$$+ [\sigma (\|A(I,0)\| + 2\|B(I,0)\|$$
$$+ \|U(0,I)\| + 2\|V(0,I)\|) + 1]\, \|\omega - \overline{\omega}\|$$
$$\leq \sigma \|\Psi(\omega)\|$$
$$+ [\sigma (\|A(I,0)\| + 2\|B(I,0)\| + \|U(0,I)\|$$
$$+ 2\|V(0,I)\|) + 1]\, \tau \left\| [f(\omega)] \right\|^{1/2}$$
$$\leq \eta \left(\|\Psi(\omega)\| + \left\| [f(\omega)] \right\|^{1/2} \right)$$
$$\leq \eta \left(\|\Psi(\omega)\|_1 + \left\| [f(\omega)] \right\|^{1/2} \right)$$
$$\leq \eta \left(\|(-A(I,0)\,\omega)_+\|_1 + \|(-U(0,I)\,\omega)_+\|_1 \right.$$
$$+ \|B(I,0)\,\omega\|_1$$
$$\left. + \|V(0,I)\omega\|_1 + \left\| [f(\omega)] \right\|^{1/2} \right)$$

$$\leq \eta \left(\sqrt{s} \left\| (-A(I,0)\,\omega)_+ \right\| + \sqrt{s} \left\| (-U(0,I)\,\omega)_+ \right\| \right.$$

$$+ \sqrt{t} \left\| B(I,0)\,\omega \right\|$$

$$\left. + \sqrt{m} \left\| V(0,I)\,\omega \right\| + \left\| [f(\omega)] \right\|^{1/2} \right)$$

$$\leq c_4 \left(\left\| (-A(I,0)\,\omega)_+ \right\| + \left\| (-U(0,I)\,\omega)_+ \right\| \right.$$

$$+ \left\| B(I,0)\,\omega \right\|$$

$$\left. + \left\| V(0,I)\,\omega \right\| + \left\| [f(\omega)] \right\|^{1/2} \right), \tag{27}$$

where the second inequality follows from (23) with constants $\sigma = c_3 \sqrt{2s + 2t + 2m}$ and $\omega = \overline{\omega}$, the third inequality uses (26), the fifth inequality follows from (20), the sixth inequality follows from the fact that

$$\eta = \max \{ \sigma, [\sigma (\|A(I,0)\| + 2\|B(I,0)\|$$

$$+ \|U(0,I)\| + 2\|V(0,I)\|) + 1]\,\tau \}, \tag{28}$$

the seventh and ninth inequalities follow from the fact that

$$\|x\| \leq \|x\|_1 \leq \sqrt{n}\,\|x\|, \quad \forall x \in R^n, \tag{29}$$

and the last inequality follows by letting $c_4 = \eta \max\{\sqrt{s}, \sqrt{t}, \sqrt{m}, 1\}$.

For any $x \in R^n$, let $\omega = (\mu, \nu) = (F(x), G(x)) \in R^{2m}$. Then there exists $\omega^* = (\mu^*, \nu^*) = (F(x^*), G(x^*)) \in \Omega^*$ such that

$$\text{dist}^{(1+\gamma)\delta}(x, X^*)$$

$$\leq \|x - x^*\|^{(1+\gamma)\delta}$$

$$\leq \frac{1}{\alpha^{1+\gamma}} \|G(x) - G(x^*)\|^{1+\gamma}$$

$$\leq \frac{1}{c_1 \alpha^{1+\gamma}} \max_{1 \leq i \leq n} \{ [F(x) - F(y)]_i [G(x) - G(y)]_i \}$$

$$\leq \frac{1}{c_1 \alpha^{1+\gamma}} \|F(x) - F(x^*)\| \|G(x) - G(x^*)\|$$

$$\leq \frac{1}{2c_1 \alpha^{1+\gamma}} \left\{ \|F(x) - F(x^*)\|^2 + \|G(x) - G(x^*)\|^2 \right\}$$

$$= \frac{1}{2c_1 \alpha^{1+\gamma}} \|\omega - \omega^*\|^2$$

$$\leq \frac{1}{2c_1 \alpha^{1+\gamma}} c_4^2 \left(\left\| (-A(I,0)\,\omega)_+ \right\| + \left\| (-U(0,I)\,\omega)_+ \right\| \right.$$

$$+ \left\| B(I,0)\,\omega \right\|$$

$$\left. + \|V(0,I)\omega\| + \|[f(\omega)]\|^{1/2} \right)^2$$

$$\leq \frac{1}{2c_1 \alpha^{1+\gamma}} c_4^2 \left(\left\| (A(I,0)\,\omega)_- \right\| + \left\| (U(0,I)\,\omega)_- \right\| \right.$$

$$+ \left\| B(I,0)\,\omega \right\|$$

$$\left. + \|V(0,I)\omega\| + \|[f(\omega)]\|^{1/2} \right)^2$$

$$\leq \frac{1}{2c_1 \alpha^{1+\gamma}} c_4^2 \left\{ \left\| [A\mu]_- \right\| + \|B\mu\| + \left\| [U\nu]_- \right\| + \|V\nu\| \right.$$

$$\left. + \left\| \mu^\top \nu \right\|^{1/2} \right\}^2, \tag{30}$$

where the second inequality follows from Definition 7 with constant $\alpha > 0$, the third inequality follows from Definition 1 with constants $c_1 > 0$ and $\gamma > 0$, the fifth inequality follows from the fact that $a^2 + b^2 \geq 2ab$, for all $a, b \in R$, and the sixth inequality is by (27). By (30) and letting $\rho_1 = \{(1/(2c_1 \alpha^{1+\gamma}))c_4^2\}^{1/(1+\gamma)\delta}$, then the desired result follows. □

Remark 9. Firstly, from remark of Definition 1, the conditions that F is γ-uniform p-function with respect to mapping G and G is δ-strongly nonexpanding in Theorem 8 are weaker than the conditions that F is γ-strongly G-monotone and G is strongly nonexpanding (i.e., $\delta = 1$) in Theorem 13 in [4]. In addition, the result in Theorem 8 is stronger than that in Theorem 13 in [4]. Thus, Theorem 8 is stronger than Theorem 13 in [4].

In the following, we also present an example to compare the condition in Theorem 8 in this paper and that in Theorem 13 in [4].

Example 10. When $\mathscr{K} = R^2_+$, $F(x) = Mx$, and $G(x) = x$, the (2) reduces to the linear complementarity problem (LCP) of finding vector $x^* \in R^n$ such that

$$F(x^*) \geq 0, \qquad G(x^*) \geq 0, \qquad F(x^*)^\top G(x^*) = 0, \tag{31}$$

where $M = \left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix} \right)$.

It is easy to see that the solution set of the LCP $X^* = \{0\}$. In fact,

$$X^* = \left\{ x \in R^2 \mid x \geq 0, Mx \geq 0, x^\top Mx = 0 \right\}$$

$$= \left\{ (x_1, x_2) \in R^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 \geq 3x_2, \right.$$

$$\left. x_1 = \frac{3 - \sqrt{5}}{2} x_2 \text{ or } x_1 = \frac{3 + \sqrt{5}}{2} x_2 \right\} \tag{32}$$

$$= \{0\}.$$

Clearly, M is a P matrix [17]. Thus, there exists constant $\tau > 0$ such that

$$\max_{1 \leq i \leq 2} \{(x - y)_i (Mx - My)_i\} \geq \tau \|x - y\|^2. \tag{33}$$

For any $x(\epsilon) := (-\epsilon; \epsilon)$, $\epsilon \geq 0$. By Theorem 8, with $\gamma = 1$, $\delta = 1$, $A = I$, $U = I$, $B = 0$, and $V = 0$ and letting $\varphi_1(x) := \|[F(x)]_-\| + \|[G(x)]_-\| + \|[F(x)^\top G(x)]\|^{1/2}$, we can obtain

$$\frac{\|x(\epsilon) - 0\|}{\varphi_1(x(\epsilon))} = \frac{\sqrt{2}\epsilon}{4\epsilon + \epsilon + \sqrt{5}\epsilon} \longrightarrow \frac{\sqrt{2}}{5 + \sqrt{5}} \tag{34}$$

as $\epsilon \to 0$. Thus, Theorem 8 provides a global error bound for this LCP.

However, letting $x = (1; 1)$, we note that $x^\top Mx = -1 < 0$ which shows that M is not positive definite, so the condition that F is strongly monotone in Theorem 13 in [4] does not hold. Thus, the result of Theorem 13 in [4] fails in providing an error bound for this LCP.

Secondly, if F is γ-strongly G-monotone and G is strongly δ-nonexpanding, then it is easy to verify that

$$\langle F(x) - F(y), G(x) - G(y) \rangle$$
$$\geq c_5 \|G(x) - G(y)\|^{1+\gamma} \tag{35}$$
$$\geq c_5 \alpha^{1+\gamma} \|x - y\|^{(1+\gamma)\delta}, \quad \forall x, y \in R^n,$$

where $c_5 > 0$ is constant. Moreover, the conditions that both F and G are Hölder continuous (or both F and G are Lipschitz continuous) in Theorem 8 in this paper are removed. Thus, Theorem 8 is stronger than Theorem 2.5 in [5]. Furthermore, by Theorem 2.1 in [5], the GNCP can be reformulated as general variational inequalities problem, and the conditions in Theorem 8 are also weaker than those in Theorem 3.1 in [11], Theorem 2 in [12], Theorem 3.1 in [13], and Theorem 3.1 in [16], respectively.

In the end of this paper, we will consider a special case of GNCP which was discussed in [11].

When $\mathcal{K} = R_+^m$, the (2) reduces to the generalization of the classical nonlinear complementarity problem of finding vector $x^* \in R^n$ such that

$$F(x^*) \geq 0, \qquad G(x^*) \geq 0, \qquad F(x^*)^\top G(x^*) = 0. \tag{36}$$

Combining this with Theorem 8, we can also immediately obtain the following conclusion.

Corollary 11. *Suppose that the hypotheses of Theorem 8 hold. Then there exists constant $\rho_2 > 0$ such that*

$$\mathrm{dist}\,(x, X^*)$$
$$\leq \rho_2 \left\{ \|[F(x)]_-\| + \|[G(x)]_-\| + \|[F(x)^\top G(x)]\|^{1/2} \right\}^{2/(1+\gamma)\delta},$$
$$\forall x \in R^n. \tag{37}$$

Remark 12. It is clear that if F is γ-uniform p-function and G is strongly δ-nonexpanding, for any $x, y \in R^n$, then

$$\max_{1 \leq i \leq m} \left\{ [F(x) - F(y)]_i [G(x) - G(y)]_i \right\}$$
$$\geq c_1 \|G(x) - G(y)\|^{1+\gamma} \geq c_1 \alpha^{1+\gamma} \|x - y\|^{(1+\gamma)\delta}, \tag{38}$$

so the condition in Corollary 11 is largely extended than the condition that F is a uniform p-function with respect to G (i.e., $\gamma = 1, \delta = 1$) in Theorem 3.3 in [11]. Moreover, the conditions that both F and G are Lipschitz continuous in Theorem 3.3 in [11] are removed. Thus, Corollary 11 is stronger than Theorem 3.3 in [11].

3. Conclusion

In this paper, we established a global error bound on the generalized nonlinear complementarity problems over a polyhedral cone, which improves the result obtained for variational inequalities and the GNCP in [4, 5, 11–13] by weakening the assumptions. Surely, we may use the error bound estimation to establish quick convergence rate of the methods for the GNCP under milder conditions. This is a topic for future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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