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Coupled Rocking and Translating Vibrations of a Buried Foundation[1]

The plane strain model of a rigid building foundation embedded below the surface of an elastic half space is treated. The foundation is assumed to be vibrating freely with arbitrary transient time-dependences U(t) and V(t) in the horizontal and vertical directions, respectively, and to be rocking about its mass center with magnitude Ω(t). The total restraining reactions exerted upon the foundation by the surrounding medium are determined exactly during the initial time period for a P-wave to traverse the foundation base width. Thereafter the results become approximate. In this manner the coupled equations of motion for free vibrations of the foundation are obtained. A numerical example presents the impulse response matrix for U, V, and Ω. Peak responses are found to occur during the early time period where the results are exact.

Introduction

In recent years there has been a mounting interest in improving methods for predicting earthquake responses of buildings by taking into account the seismic wave-foundation interaction. By evaluating the restraining reactions on a freely vibrating building foundation one can obtain the so-called "compliances" of the surrounding medium and can thus construct the equations of motion for the building which include the *soil-structure interaction* effects.

For surface-mounted footings on an elastic half space there have been many studies as listed in the recent paper by Jennings and Bielak [1].[2]

However, for the practical case of an embedded foundation there are relatively few analytical solutions. Luco [2] and Trifunac [3] have determined the steady-state response of a buried semicylinder to harmonic SH-waves; while the present authors [4] recently obtained the exact, early-time transient response of a buried rectangular foundation to an incident SH-wave.

The SH-wave or "antiplane strain" model, while being a useful initial, approximation, is rather limited. It permits only an antiplane translation response of the building to occur and thus excludes the coupled rocking and translation motions which are actually experienced by seismically loaded structures.

A plane-strain half-space model with an embedded foundation would thus become the next step for investigating stress-wave-foundation interaction. Thau [5] has obtained exact, early-time results for the coupled rocking and translation of a buried rigid strip. More recently, Umek [6] has derived explicitly the transient responses for vertical and horizontal translation and rocking rotation of an embedded rigid foundation subjected to incident elastic waves.

The present paper reports the work from Umek's thesis [6] on the free vibrations of the embedded foundation. The early-time transient solution for the soil-structure interaction forces, exerted on the foundation during its free vibrations in each of its three plane modes of rigid-body motion is derived. The three plane modes are the vertical translation and the coupled horizontal translation and rocking rotation. Equating the resistive reactions caused by the waves radiating from the vibrating foundation to the appropriate foundation inertia terms, yields the homogeneous equations of rigid-body motion for the buried foundation. The equations derived here are exact from the instant the foundation would start to move ($t = 0$) to the time required for a P-wave to traverse the base width of the foundation.

The analytical techniques involve application of Laplace and Kontorovich-Lebedev transforms to the governing equations for extracting effects at the lower corners of the foundation, and utilize previously derived quarter-space solutions from [5] for upper corner contributions. These procedures are thus similar to, but represent extensions of the short-time solution techniques employed in our previous embedded structure response studies [4, 5].

Description of Problem

The problem model in Fig. 1*(a)* consists of an elastic half space in which a rigid square foundation, 1 × 1 in dimensionless units, is embedded. A state of plane strain is assumed so that the foundation can experience horizontal and vertical translations and a rocking rotation about its mass center. The magnitudes of these motions are designated by the arbitrary causal functions $U(t)$, $V(t)$, and $\Omega(t)$, respectively.

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[2] Numbers in brackets designate References at end of paper.

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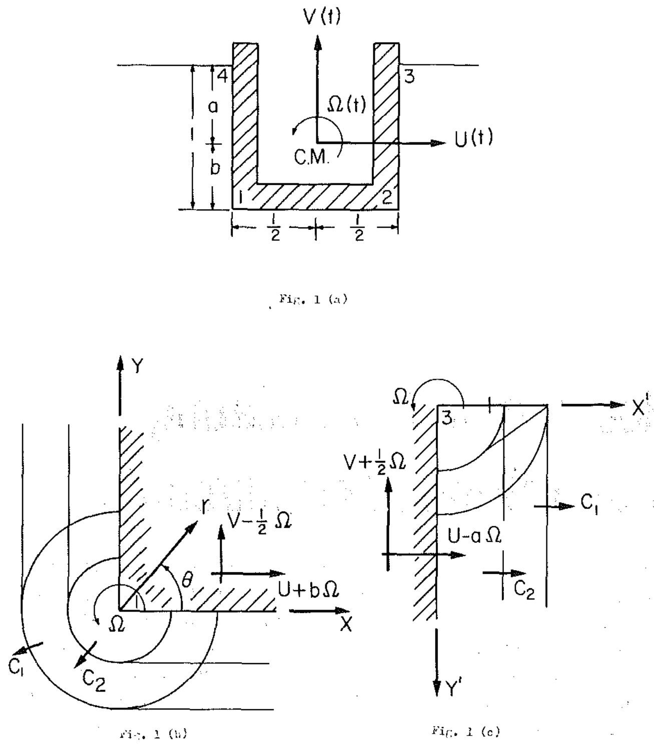

Fig. 1 Main problem and subproblem models—*(a)* Radiation from lower left corner; *(b)* Radiation from upper right corner

For convenience in later calculations, the origin of coordinates is placed at the lower corner 1 of the foundation. Its mass center is taken to lie midway between the side walls, but, as shown, is located at an arbitrary distance b above the base slab.

The P-waves $\phi(x, y, t)$ and S-waves $\psi(x, y, t)$ radiated into the half space from the vibrating foundation are governed by the equations of motion

$$(\nabla^2 - s^2)\bar{\phi}(x,y) = 0; \qquad (\nabla^2 - \kappa^2 s^2)\bar{\psi}(x,y) = 0 \tag{1}$$

in which the Laplace transform in time with parameter s, and denoted with an overbar, has been introduced. The P-wave speed c_1 is taken as unity so that the S-wave velocity c_2 becomes $1/\kappa$ where, in terms of Poisson's ratio ν,

$$\kappa = c_1/c_2 = [(2 - 2\nu)/(1 - 2\nu)]^{1/2} \tag{2}$$

Based on our units of distance and velocity, it follows that one unit of time is required for a P-wave to traverse the base width or embeddment depth of the foundation.

The transformed displacements $\bar{\mathbf{u}} = (\bar{u}, \bar{v}, 0)$ are given by

$$\bar{\mathbf{u}} = \vec{\nabla}\bar{\phi} + \vec{\nabla} \times \bar{\psi}\vec{\epsilon}_z \tag{3}$$

where ϵ_z is a unit, z-directed vector; while the transformed stresses $\bar{\sigma}_{ij}$, normalized by the shear modulus of the medium, are given in Cartesian tensor notation by

$$\bar{\sigma}_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i} + (\kappa^2 - 2)\delta_{ij}s^2\bar{\phi}. \tag{4}$$

The procedure followed in this analysis is first to solve equations (1) in the half-space medium in Fig. 1*(a)*, subject to the boundary conditions specified in the next section. Then, from equations (4), the stresses acting along the foundation sides are calculated, from which the net reactions acting at and about the mass center are derived. Equating these resistive reactions to the appropriate inertia terms for the foundation yields the homogeneous equations for its rigid-body motion.

Short-Time Analysis and Boundary Conditions

Even before the specification of boundary conditions, it is probably apparent that an analytical solution for the radiation problem in Fig. 1*(a)* is exceedingly difficult to obtain. However, by employing the short-time analysis method, as in our previous embedded foundation [4] and rigid-strip [5] studies, we are successful in obtaining exact solutions for a finite period of time. Theoretically, the analysis can be carried out for as long a time interval as desired, but the practical problem of performing requisite calculations limits the feasibility of the method to a short-time period. Here the short-time analysis will be performed in $0 \le t \le 1$ where we recall that $t = 1$ is the time for a P-wave to travel between any two adjacent corners of the foundation.

Briefly, and with respect to the foundation geometry at hand, the short-time analysis consists of obtaining the exact solution for the waves radiated locally from each foundation corner, before such waves interact with neighboring corners. In this manner, the problems for corners 1 and 2 are each equivalent to that for the radiation of waves into an infinite elastic medium from the vertex of a right angle wedge, Fig. 1*(b)*. Similarly, effects at corners 3 and 4 are determined from solutions of quarter-space problems with rigid vertical boundaries, Fig. 1*(c)*. Since a cylindrical P-wave emanating from a given corner and traveling with unit speed will not reach an adjacent corner at a unit distance away for one unit of time, the short-time analysis will provide exact results in $0 \le t \le 1$.

Boundary Conditions. Even with the "simpler" subproblems that arise in the short-time analysis method, neither exact nor manageable solutions can be derived under the boundary conditions of perfect bonding between the half space and foundation. This is true both for a quarter-space with a perfectly bonded, rigid, vertical boundary [7] and for a rigid wedge with displacement boundary conditions [8].

Therefore we relax the condition of perfect bonding and adopt the so-called "rigid-smooth" or "rigid-pressureless" conditions at the foundation boundary. By rigid-smooth we mean a boundary at which the normal component of motion of the elastic medium is equal to that of the foundation, but the shear stress is zero. At a rigid-pressureless boundary, the tangential displacement of the elastic medium follows that of the foundation under zero normal stress. In order to maximize the restraining effects of the half-space medium, we shall adopt rigid-smooth conditions at a given side when that side moves normal to itself and take rigid-pressureless conditions when the side moves tangentially. By this arrangement, normal or shearing stresses will be generated to resist normal or sliding motion, respectively, at each side. Furthermore, for both rigid-smooth and rigid-pressureless conditions, the wave potentials become uncoupled in the boundary conditions. That will render the short-time subproblems mathematically tractable.

In order to express the boundary conditions in a given corner subproblem, we must first refer the foundation rocking to a z-axis through that corner. Then the actual kinematics of each side can be determined. For example, at corner 1, for infinitesimal rocking amplitude the wedge motion consists of horizontal translation $\bar{U} + b\bar{\Omega}$; vertical translation $\bar{V} - (1/2)\bar{\Omega}$; and rocking rotation $\bar{\Omega}$ about corner 1 as indicated in Fig. 1*(b)*. Similarly, when viewed to rotate about upper corner 3, the right-side wall of the foundation has the rigid-body motions shown in Fig. 1*(c)*.

At each corner, each mode of motion is considered separately, because different boundary conditions apply for different motions. To repeat our convention: the displacement along each side must follow the given rigid-body foundation motion while the stress in the perpendicular direction to the motion is taken to vanish. With reference to corner 1, Fig. 1*(b)* and in terms of the polar coordinates (r, θ), the following boundary conditions are imposed:

Horizontal Translation

$$\bar{u}_\theta(r,\pi/2) = -(\bar{U} + b\bar{\Omega}); \quad \bar{\sigma}_{r\theta}(r,\pi/2) = 0$$
$$\bar{u}_r(r,2\pi) = \bar{U} + b\bar{\Omega}; \qquad \bar{\sigma}_{\theta\theta}(r,2\pi) = 0 \tag{5a}$$

Vertical Translation

$$\bar{u}_r(r,\pi/2) = \bar{V} - \frac{1}{2}\bar{\Omega}; \quad \bar{\sigma}_{\theta\theta}(r,\pi/2) = 0$$
$$\bar{u}(r,2\pi) = \bar{V} - \frac{1}{2}\bar{\Omega}; \quad \bar{\sigma}_{r\theta}(r,2\pi) = 0 \tag{5b}$$

Rocking Rotation

$$\bar{u}_\theta(r,\pi/2) = r\bar{\Omega}; \quad \bar{\sigma}_{r\theta}(r,\pi/2) = 0$$
$$\bar{u}_\theta(r,2\pi) = r\bar{\Omega}; \quad \bar{\sigma}_{r\theta}(r,2\pi) = 0 \tag{5c}$$

Note that rigid-smooth and rigid-pressureless conditions are mixed in the translation problems, whereas only rigid-smooth conditions occur for the rocking problem.

When conditions (5) are expressed in terms of the wave potentials they reduce to the simple forms:

Horizontal Translation

$$\bar{\phi}_\theta(r,\pi/2) = -r(\bar{U} + b\bar{\Omega}); \quad \bar{\phi}(r,2\pi) = 0$$
$$\bar{\psi}(r,\pi/2) = 0; \quad \bar{\psi}_\theta(r,2\pi) = r(\bar{U} + b\bar{\Omega}) \tag{6a}$$

Vertical Translation

$$\bar{\phi}(r,\pi/2) = 0; \quad \bar{\phi}_\theta(r,2\pi) = r\left(\bar{V} - \frac{1}{2}\bar{\Omega}\right)$$
$$\bar{\psi}_\theta(r,\pi/2) = r\left(\bar{V} - \frac{1}{2}\bar{\Omega}\right); \quad \bar{\psi}(r,2\pi) = 0 \tag{6b}$$

Rocking Rotation

$$\bar{\phi}_\theta(r,\pi/2) = \bar{\phi}_\theta(r,2\pi) = r^2\bar{\Omega}$$
$$\bar{\psi}(r,\pi/2) = \bar{\psi}(r,2\pi) = -2\bar{\Omega}/\kappa^2 s^2 \tag{6c}$$

in which subscripts on the potentials indicate partial differentiation.

The potentials are thus uncoupled in the boundary conditions as they are in the equations of motion (1). However, at the vertex of the wedge the "edge condition" which requires that the displacements (not potentials) be finite will, through equation (3), couple the potentials. This is an important point because it means that the finite, scalar, wedge radiation solutions are not the physically correct solutions to our problems. Instead, additional singular solutions of (1) will have to be superimposed to render the displacements finite at the vertex. This phenomenon in elastodynamic wedge and strip problems with relaxed boundary conditions has been discussed previously by Kostrov [9] and Thau [5].

The boundary conditions for the appropriate wedge problems at corner 2 are deduced in analogous fashion. It can be seen from the symmetry however, that the solutions for corner 2 can be directly obtained from those for corner 1.

To illustrate boundary conditions at an upper corner, we refer to corner 3 in Fig. 1*(c)*. The surface of the half space ($y' = 0$) is free of traction, i.e.,

$$\bar{\sigma}_{y'y'}(x',0) = \bar{\sigma}_{x'y'}(x',0) = 0. \tag{7a}$$

Along the vertical wall ($x' = 0$) conditions again depend on the type of motion.

Horizontal Translation and Rocking (Rigid-Smooth)

$$\bar{\phi}_x(0,y') = \bar{U} + (y' - a)\bar{\Omega}$$
$$\bar{\psi}(0,y') = 2\bar{\Omega}/\kappa^2 s^2 \tag{7b}$$

that is

$$\bar{u} = \bar{U} + (y' - a)\bar{\Omega}; \quad \bar{\sigma}_{x'y'} = 0$$

where a is the distance of the center of mass below the surface of the half space, Fig. 1*(a)*.

Vertical Motion (Rigid-Pressureless)

$$\bar{\phi}(0,y') = 0; \quad \bar{\psi}_x(0,y') = -\left(\bar{V} + \frac{1}{2}\bar{\Omega}\right)$$

that is (7c)

$$\bar{v} = -\left(\bar{V} + \frac{1}{2}\bar{\Omega}\right); \quad \bar{\sigma}_{x'x'} = 0$$

Subproblem Solutions

At each of the four corners of the foundation there are three subproblems, one each for horizontal and vertical translation, and for rocking about the corner. However, because of symmetry and previously derived upper corner radiation solutions [5], only three new boundary-value problems need to be solved: horizontal translation and rocking at corner 1 and vertical translation at corner 3.

Corner 1—Horizontal Translation. The boundary conditions for the wedge in Fig. 1*(b)* translating in the x-direction, are given by equations (6*a*) although for convenience we shall replace $\bar{U} + b\bar{\Omega}$ by unity to obtain the response for a unit (transformed) amplitude motion.

As in our previous foundation analysis [4], we apply the Kontorovich-Lebedev transform (abbreviated by *K-L*) in r to the equations of motion (1) to obtain

$$\bar{\phi}_{\theta\theta}{}^* - \nu^2\bar{\phi}^* = 0; \quad \bar{\psi}_{\theta\theta}{}^* - \nu^2\bar{\psi}^* = 0 \tag{8}$$

where ν is the *K-L* transform parameter and the asterisk denotes this transform as defined in [4].

K-L transforms with different kernels must be applied separately for $\bar{\phi}$ and $\bar{\psi}$ because of their different wave speeds. This hinders us from mixing these functions in boundary conditions, which as equations (6*a*) indicate is not required. However, it does explain why the perfectly bonded wedge problem has so far been unsolved.

The solutions of (8) satisfying the transformed boundary conditions (6*a*) are found as

$$\bar{\phi}^* = \frac{\pi}{2\nu s} \operatorname{sech}\frac{\nu\pi}{2} \operatorname{sech}\frac{3\nu\pi}{2} \sinh \nu(2\pi - \theta) \tag{9a}$$

$$\bar{\psi}^* = \frac{\pi}{2\kappa\nu s} \operatorname{sech}\frac{\nu\pi}{2} \operatorname{sech}\frac{3\nu\pi}{2} \sinh \nu(\theta - \pi/2) \tag{9b}$$

where complete details of the derivation are given in the thesis [6].

To isolate the cylindrical waves radiating from the corner, we subtract from the foregoing solutions the *K-L* transforms of the planar radiated waves which emanate from the sides of the wedge. The latter can be identified as

$$\bar{\phi}^{(P)} = \frac{1}{s}e^{sr\cos\theta} \text{ in } \frac{\pi}{2} \le \theta < \pi$$
$$\bar{\psi}^{(P)} = \frac{1}{\kappa s}e^{\kappa sr\sin\theta} \text{ in } \frac{3\pi}{2} < \theta \le 2\pi \tag{10}$$

which satisfy the inhomogeneous portions of the boundary conditions (6*a*). The *K-L* transforms of the cylindrical waves thus become

$$\bar{\phi}^{(c)*} = -\frac{\pi}{\nu s} \coth \nu\pi \operatorname{sech}\frac{3\nu\pi}{2} \cosh \nu\left(\theta - \frac{\pi}{2}\right); \quad \frac{\pi}{2} \le \theta < \pi$$

$$\bar{\psi}^{(c)*} = -\frac{\pi}{\kappa\nu s} \coth \nu\pi \operatorname{sech}\frac{3\nu\pi}{2} \cosh \nu(2\pi - \theta); \quad \frac{3\pi}{2} < \theta \le 2\pi \tag{11}$$

In the remaining angular sectors not specified in (11) the cylindrical wave potentials are equal to the total wave fields (9) because no plane waves for ϕ or ψ occur there.

Inversion of the cylindrical wave potentials in the appropriate angular sectors shows that the edge condition is not satisfied.

Consider the cylindrical waves along the vertical side $\theta = \pi/2$. By performing the inverse *K-L* transforms with contour integration techniques [6], we obtain familiar integral representations for the modified Bessel functions $K_n(\rho)$. In particular, at $\theta = \pi/2$, we find

$$\bar{\phi}_r^{(c)} = \frac{2}{3\pi}[K_{\frac{2}{3}}(\rho) + K_0(\rho)] \qquad (\rho = sr) \qquad (12)$$

$$\frac{1}{r}\bar{\psi}_\theta^{(c)} = \frac{2}{3\pi}[K_{\frac{2}{3}}(\kappa\rho) - K_0(\kappa\rho)]$$

Summing equations (15) to produce the radial displacement $\bar{u}_r$ and expanding the modified Bessel function for small argument produces the singular behavior as $\rho \to 0$,

$$\bar{u}_r^{(c)} \to \frac{2}{3^{3/2}}\left(\frac{2}{\rho}\right)^{2/3}\frac{1 + \kappa^{-2/3}}{\Gamma(1/3)} + \frac{2}{3\pi}\ln\kappa + O(\rho^{2/3}) \qquad (13)$$

To rectify this situation we add the following cylindrical waves to the field:

$$\bar{\phi}^{(c)} = AK_{1/3}(\rho)\ \cos\frac{1}{3}\left(\theta - \frac{\pi}{2}\right)$$

$$\bar{\psi}^{(c)} = BK_{1/3}(\kappa\rho)\ \sin\frac{1}{3}\left(\theta - \frac{\pi}{2}\right) \qquad (14)$$

These *singular* potentials satisfy the wave equations (1), satisfy the homogeneous version of boundary conditions (6*a*), and represent outgoing waves. They produce singular displacements of order $\rho^{-4/3}$ and $\rho^{-2/3}$ as $\rho = sr \to 0$. Consequently, the coefficients A and B can be determined from the condition that both singular terms in the *total* displacement must vanish. This yields the values

$$B = \kappa^{1/3}A = 4(3\pi s\kappa^{1/3})^{-1} \qquad (15)$$

It can be shown rigorously [6] that the additional waves (14) with coefficients (15) renders both displacements finite at $r = 0$ for any choice of θ. Hence the solution is now completed.

The reactions on the wedge due to the cylindrical waves from corner 1 (denoted with subscript 1) are calculated from

$$\bar{F}_{1x}^{(c)} = \int_0^\infty -[\bar{\sigma}_{rr}^{(c)}(r,\pi/2) + \bar{\sigma}_{r\theta}^{(c)}(r,2\pi)]dr$$

$$\bar{M}_{10}^{(c)} = \int_0^\infty r\bar{\sigma}_{rr}^{(c)}(r,\pi/2)dr; \qquad \bar{F}_{1y}^{(c)} = 0 \qquad (16)$$

where the moment is taken about the vertex, being positive in the counterclockwise sense.

The foregoing reactions can be evaluated in closed form. Details are presented in reference [6] for deriving the results,

$$\bar{F}_{1xx}^{(c)} = \frac{4}{3^{3/2}}\left[(1 + \kappa^2)\left(\frac{1}{3} + \frac{3^{1/2}}{2\pi}\right) - \kappa^{2/3}(1 + \kappa^{2/3})\right]$$

$$\bar{M}_{10x}^{(c)} = (27s)^{-1}[54 + 18\kappa^{-1} + 12\kappa^{4/3} - 9.5\kappa^2 - 72\kappa^{-1/3}] \qquad (17)$$

where a third subscript is appended to indicate reactions caused by x-directed unit motion.

Since the cylindrical waves from corner 1 do not reach either adjacent corner of the foundation before $t = 1$, the foregoing reactions, when multiplied by the actual horizontal translation of corner 1, $\bar{U} + b\bar{\Omega}$, become the total reactions on the full foundation contributed from corner 1 in $0 \le t \le 1$.

It is apparent from symmetry that, if the wedge modeling corner 1 were assumed to move in the y-direction with unit magnitude, then the cylindrical waves would exert the reactions

$$\bar{F}_{1yy}^{(c)} = \bar{F}_{1xx}^{(c)}; \qquad \bar{M}_{10y} = -\bar{M}_{10x} \qquad (18)$$

Similarly, from symmetry, we can evaluate the reactions at corner 2 due to cylindrical radiated waves caused by unit magnitude *x* and *y* directed translations:

$$\bar{F}_{2yy}^{(c)} = \bar{F}_{1xx}^{(c)}; \qquad \bar{F}_{2yy}^{(c)} = \bar{F}_{2xx}^{(c)}$$

$$\bar{M}_{20x}^{(c)} = \bar{M}_{10x}^{(c)}; \qquad \bar{M}_{20y}^{(c)} = \bar{M}_{20x}^{(c)} \qquad (19)$$

where the moments for the corner 2 problems are taken about corner 2.

Corner 2—Rocking Rotation. To complete the lower corner radiation studies, we consider the wedge modeling corner 1, Fig. 1(*b*), to be rotating about its vertex with infinitesimal amplitude $\bar{\Omega}(s)$. The boundary conditions for the radiated waves are rigid-smooth along each face as given by equations (5*c*) or equivalently (6*c*).

The solution of this problem is constructed from the outgoing wave solution to the following wedge radiation problem:

$$(\nabla^2 - s^2)\bar{S}(r,\theta;s) = 0 \ \text{in}\ 0 < r < \infty;\ \pi/2 \le \theta \le 2\pi \qquad (19a)$$

with boundary conditions

$$\bar{S}(r,\theta;s) = 1 \ \text{at}\ \theta = \pi/2 \ \text{and}\ \theta = 2\pi \qquad (19b)$$

and $\bar{S}$ remains finite as $r \to 0$.

It is clear from equations (6*c*) that in terms of $\bar{S}$,

$$\bar{\psi} = -2(\kappa s)^{-2}\bar{\Omega}\bar{S}(r,\theta;\kappa s) \qquad (20)$$

and we shall show that

$$\bar{\phi} = s^{-2}\bar{\Omega}\bar{S}_\theta(r,\theta;s) \qquad (21)$$

To prove the latter result (21) we note first that the θ-derivative of a solution of the Helmholtz equation (19*a*) is itself a solution. Then from (19*a*) and (19*b*) we have along the wedge faces, $\theta = \pi/2$ and $\theta = 2\pi$, that

$$\nabla^2\bar{S} = r^{-2}\bar{S}_{\theta\theta} = s^2\bar{S} = s^2 \qquad (22)$$

Consequently

$$\bar{\phi}_\theta = s^{-2}\bar{\Omega}\bar{S}_{\theta\theta} = r^2\bar{\Omega} \ \text{at}\ \theta = \pi/2 \ \text{and}\ 2\pi \qquad (23)$$

and so the boundary conditions (6*c*) for $\bar{\phi}$ are indeed satisfied.

The solution $\bar{S}$ is identical to that for radiation of SH-waves from the wedge which was derived in our previous paper [4]. The plane waves, radiating from the wedge sides can be subtracted from $\bar{S}$ leaving the cylindrical waves radiating from the vertex. They are given by equations (34) in [4].

Again, however, it is found that additional singular potentials must be superimposed here to render the displacements finite at the vertex. These are found to be [6],

$$\bar{\phi}^{(c)} = DK_{2/3}(\rho)\ \cos\frac{2}{3}\left(\theta - \frac{\pi}{2}\right)$$

$$\bar{\psi}^{(c)} = EK_{2/3}(\kappa\rho)\ \sin\frac{2}{3}\left(\theta - \frac{\pi}{2}\right) \qquad (24)$$

with

$$E = \kappa^{2/3}D = \frac{16(3 - \kappa^{4/3})\bar{\Omega}}{9\pi s^2(\kappa^{2/3} + \kappa^2)} \qquad (25)$$

which completes the solution.

The reactions on the wedge due to the total cylindrical wave field can be determined explicitly as [6]

$$\bar{F}_{1x\Omega}^{(c)} = -\bar{F}_{1y\Omega}^{(c)} = (54s)^{-1}\bar{\Omega}[5\kappa^2 - 108(1 - 2\kappa^{-1}) + 96(3\kappa^{2/3} - \kappa^2)(1 + \kappa^{4/3})^{-1}] \qquad (26)$$

$$\bar{M}_{10\Omega}^{(c)} = 16(3^{11/2}s^2)^{-1}\bar{\Omega}[(3^{3/2}\pi)^{-1}(\kappa^2 - 3\kappa^{-2}) - 4(5\kappa^2 - 27\kappa^{-2}) - 36(3 - \kappa^{4/3})^2(\kappa^{2/3} + \kappa^2)^{-1}] \qquad (27)$$

where the first subscript on the reactions indicates the corner number; the second indicates the direction of the force or the point about which the moment is taken; while the third refers to the type of motion causing the reactions.

By symmetry the reactions at and about corner 2 caused by rocking about this corner would become

$$\overline{F}_{2x\Omega}{}^{(c)} = \overline{F}_{2y\Omega}{}^{(c)} = \overline{F}_{1x\Omega}{}^{(c)}$$
$$\overline{M}_{20\Omega}{}^{(c)} = \overline{M}_{10\Omega}{}^{(c)} \qquad (28)$$

Corner 3—Translation and Rocking. Fig. 1*(c)* shows the waves radiating into the quarter-space formed by the intersection of the right side wall of the foundation and the surface of the half space. With the rocking axis taken at corner 3, the vertical wall has the translations, $\overline{U} - a\overline{\Omega}$ and $-\overline{V} - (1/2)\overline{\Omega}$, in the x' and y'-directions, respectively. From corner 3, in addition to cylindrical P and S-waves, the planar von Schmidt or "head" wave plus a Rayleigh surface wave will radiate.

In a previous paper [5] on the motion of a finite rigid strip in a half space, the radiation problems for horizontal motion and rocking were solved and the reactions exerted on the wall by the waves radiated from the corner were obtained. For unit magnitude horizontal translation the reactions are

$$\overline{F}_{3x'x'}{}^{(c)} = \kappa^2(\kappa^2 - 2)^2 I_1$$
$$\overline{M}_{30'x'}{}^{(c)} = s^{-1}(\kappa^2 - 2)[\kappa^2(\kappa^2 - 1)I_2 - (\kappa^2 + \kappa)^{-1}] \qquad (29)$$

while for rocking about corner 3 the reactions are

$$\overline{F}_{3x'\Omega}{}^{(c)} = s^{-1}[\kappa^2(\kappa^2 - 1)(\kappa^2 - 2)I_2 - (\kappa - 1) \times (0.5\kappa^3 + \kappa^2 - 2\kappa - 2) \cdot (\kappa^2 + \kappa)^{-1}] \qquad (30)$$
$$\overline{M}_{30'\Omega}{}^{(c)} = 4\kappa^2(\kappa^2 - 1)^2 I_3/s^2$$

where, as usual, the moments are taken about the corner being treated. The integrals $I_{1,2,3}$, defined in [5] with details for their numerical evaluation, are constants depending only on κ.

To solve the problem for vertical translation of the vertical wall with rigid-pressureless boundary conditions (7*c*) we follow the same procedures used in solving the horizontal translation and rocking problems in [5]. Complete details are given in [6] where the resultant reactions are shown to be

$$\overline{F}_{3y'y'}{}^{(c)} = -\kappa^6 I_4$$

with

$$I_4 = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\beta^3\Delta}$$
$$\Delta = (\beta^2 + \lambda^2)^2 - 4\lambda^2\alpha\beta \qquad (31)$$
$$\alpha^2 = \lambda^2 + 1;\ \beta^2 = \lambda^2 + \kappa^2$$

I_4 is rendered a proper integral by the substitution $\lambda = \kappa \tan\theta$ and it can then be evaluated numerically.

By symmetry, the reactions contributed by the waves radiating from corner 4 due to unit magnitude motions of the left wall can be deduced as

$$\overline{F}_{4x'x'}{}^{(c)} = \overline{F}_{3x'x'}{}^{(c)};\quad \overline{F}_{4y'y'}{}^{(c)} = \overline{F}_{3y'y'}{}^{(c)}$$
$$\overline{F}_{4x'\Omega}{}^{(c)} = \overline{F}_{3x'\Omega}{}^{(c)};\quad \overline{M}_{40'x'}{}^{(c)} = \overline{M}_{30'x'}{}^{(c)} \qquad (32)$$
$$\overline{M}_{40'\Omega}{}^{(c)} = \overline{M}_{30'\Omega}{}^{(c)}$$

Plane Wave Reactions—Base and Side Walls. So far we have extracted the plane wave fields in all of the foregoing radiation subproblems and have omitted their contributions to the reactions. It is not only more convenient to calculate the plane-wave reactions along the three sides of the foundation separately, but also these reactions, unlike those due to the cylindrical waves, are exact for all time. That is because the plane waves which issue from the sides are deduced solely by the given motion of the side and are not affected by the corners. As they radiate from the foundation, the plane waves do not interact with any corners.

To calculate the plane-wave effects is straightforward. For a given side we place the axis of rocking at the point where the normal projection of the mass center intersects that side. Then, with this point as origin, the exact horizontal and vertical translations of the side can be deduced. Finally, subject to the appropriate boundary conditions, rigid-smooth or rigid-pressureless, we can construct the plane radiated waves from the plane, rigid boundary.

For example, at the right side wall the translation becomes

$$\overline{u} = \overline{U} - y\overline{\Omega};\ \overline{v} = \overline{V} + \tfrac{1}{2}\overline{\Omega} \qquad (33)$$

where we refer to Cartesian coordinates parallel to those in Fig. 1*(a)*, but with origin on the right wall at a distance b above the base slab. It is then straightforward to verify that the radiated plane waves caused by the motions (33) become

$$\overline{\phi}^{(p)} = -\frac{1}{s}(\overline{U} - y\overline{\Omega})e^{-sx}$$
$$\overline{\psi}^{(p)} = \frac{1}{\kappa^2 s^2}\left[\kappa s\overline{V} + \left(1 + \frac{1}{2}\kappa s\right)\overline{\Omega}\right]e^{-\kappa s x}. \qquad (34)$$

The reactions produced by these waves are calculated by integrating the appropriate stresses over the actual length of this side, i.e., from $-b < y < a$.

Below are listed the total plane wave reactions from the three sides exerted at and about the mass center:

$$\overline{F}_x{}^{(p)} = -(\kappa + 2\kappa^2)s\overline{U} + [\kappa^2(a - b) - \kappa b]s\overline{\Omega}$$
$$\overline{F}_y{}^{(p)} = -(\kappa^2 + 2\kappa)s\overline{V} \qquad (35)$$
$$\overline{M}_c{}^{(p)} = [\kappa^2(a - b) - \kappa b]s\overline{U} - \left[\frac{2}{3}\kappa^2\left(a^2 + b^2 - ab + \frac{1}{8}\right) + \kappa\left(b^2 + \frac{1}{2}\right)\right]s\overline{\Omega}$$

Equations of Motion

The equations of motion for the foundation are constructed by equating the total reactions at and about its mass center to the appropriate inertia terms. The contributions of the cylindrical waves, equations (17)–(19), (26)–(32) are weighted by the actual translation magnitudes occurring at each respective corner and are then combined with the plane wave reactions (35). Numerical values are obtained with $\kappa^2 = 3$ (i.e., $\nu = 1/4$) and for $b = 0.3880$. The latter value is not only physically realistic, but it also causes the plane wave coupling terms in (35) between rocking and sliding to vanish. Finally, we choose the foundation material density to be 1.5 times that of the soil, but the foundation is taken to occupy one sixth of a unit soil cube. With a unit shear modulus and unit P-wave speed, the mass of a unit soil cube becomes $\kappa^2 = 3$. Thus the dimensionless foundation mass and polar (rocking) moment of inertia are calculated as 0.75 and 0.21, respectively.

With coefficients abbreviated here to two decimal places, the foundation equations become

$$(0.75s^2 + 7.73s + 1.35)\overline{U}(s) + (0.84 - 0.20s^{-1})\overline{\Omega}(s) = 0 \qquad (36a)$$
$$(0.84 - 0.24s^{-1})\overline{U}(s) + (0.21s^2 + 1.95s + 0.35 + 0.71s^{-1} - 0.11s^{-2})\overline{\Omega}(s) = 0 \qquad (36b)$$
$$(0.75s^2 + 6.46s + 0.90)\overline{V}(s) = 0 \qquad (36c)$$

Note that the coupling compliances between $\overline{U}$ and $\overline{\Omega}$ are not

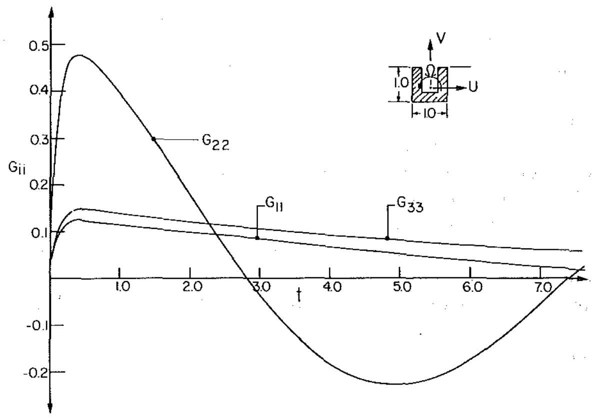

Fig. 2(a) Directly induced responses $G_{ii}(t)$

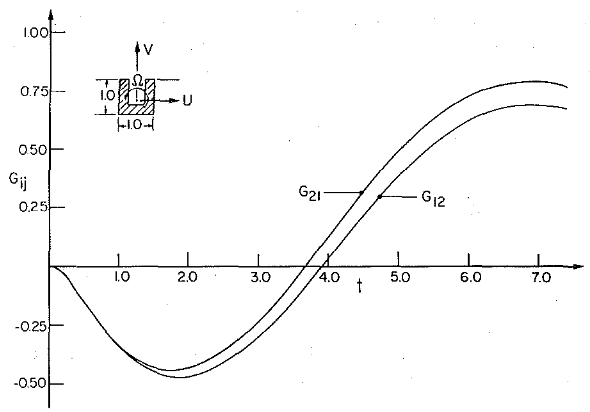

Fig. 2(b) Coupling responses $G_{ij}(t)$

Fig. 2 Impulse responses of foundation

symmetric because different relaxed boundary conditions at the lower corners are used for rocking than for horizontal translation. This phenomenon has also been reported for a surface mounted vibrating footing when different relaxed boundary conditions are used in the evaluation of the restraining reactions [10].

The uncoupled equation for $\bar{V}$ is identical to that for a damped, simple oscillator. The more complicated, coupled equations for $\bar{U}$ and $\bar{\Omega}$ also manifest radiation damping and soil-spring terms, but in addition, they contain memorylike $1/s$ and $1/s^2$ terms. Such terms have been interpreted physically in reference [5] in connection with their occurrence in the equations of motion for an embedded rigid strip. Their presence causes the characteristic determinant of equations (36*a*) and (36*b*) to have six roots. Unfortunately, just as in the strip problem [5], one of them is positive, indicating that the long-time limits of the solutions are unstable. While this feature has no significance at early times when the results are exact ($0 \leq t \leq 1$), it does prevent an accurate estimate of long-time responses.

To illustrate the characteristics of the foundation—half-space system, numerical results are presented for the impulse responses $G_{ij}(t)$, where $i = 1, 2, 3$ corresponds to U, Ω, and V, respectively, and $j = 1, 2, 3$ corresponds to a delta function applied on the right-hand side of equation (36*a*), (36*b*), and (36*c*), respectively. These results, shown in Figs. 2(*a, b*), constitute the temporal Green's function matrix for the system and illustrate free vibrations due to initial velocities.

Discussion of Results and Conclusions

It is seen from Fig. 2(*a*) that the diagonal responses G_{ii} peak in the initial time period where they are exact and then begin to decay as would be expected physically. Because their coefficients are so small, exponentially unstable terms in G_{11} and G_{22} are still insignificant up to $t = 7$. Hence, these results appear to be valid approximations well beyond the first unit of time. The $V = G_{33}$ response is overdamped. However, the coupled system for U and Ω has one pair of complex characteristic roots providing oscillations with a period of about 10. However, the coefficient of this damped oscillation in G_{22} is over 30 times greater than that in G_{11}. Consequently, the rocking oscillations in G_{22} are strikingly demonstrated, while the oscillations in G_{11} can barely be exhibited on the graphs. In fact the directly induced rocking response is considerably larger than that for translation—thus indicating the possible importance of rocking motion in building vibrations.

The coupling responses G_{12} and G_{21}, shown in Fig. 2(*b*), agree quite well at early times where their values are predicted most accurately. The average value of these responses would probably serve as a useful approximation for estimating the coupling compliance of the half space with an embedded foundation.

Since the Green's functions found here appear quite valid on physical grounds for a considerable length of time it follows that they can be used through the convolution integrals $G_{ij} * F_j$ (summation implied) to estimate the early to moderate time responses of the foundation to arbitrary loadings $F_j(t)$, including those produced by the scattering of incident seismic waves.

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