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Determination of the belt force before the gross slip

Vlado A. Lubarda *

Department of NanoEngineering, University of California, San Diego, La Jolla, CA 92093-0448, USA
Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA
Montenegrin Academy of Sciences and Arts, Rista Stijovica 5, 81000 Podgorica, Montenegro

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ABSTRACT

The mechanics of belt friction before the state of gross slip is considered. The variation of the belt force within the contact region is evaluated based on the assumption of gradual growth of slip from the pull-end to the hold-end of the belt, as the pull force increases towards its Euler's value. The local pressure and friction forces exerted by the belt on the cylinder are also determined. Both flat and V-shaped belts are considered. The total pressure and friction forces are evaluated at an arbitrary stage of slip growth. They are neither proportional nor orthogonal to each other, unless the state of gross slip is reached throughout the contact range.

1. Introduction

Flexible belts, cables and ropes have wide applications in engineering, where they are used as belt drives for power transmission between rotating shafts, band breaks to reduce angular speed of rotating machine parts, hoist devices for lifting or lowering loads in construction or mining industry, devices for fastening marine vessels to the dock (bollards or capstans), conveyors, and magnetic tape drives, etc. In all cases, they operate by friction between the belt and the surface of a drum or a pulley. The frictional power transmission between pulleys through flexible elastic belts and the design of transmission systems which reduce the energy consumption are still problems of great interest for automobile and other industries [1–5]. Because of its wide importance, the mechanics of belt friction and the Euler's formula [6] relating the pull force to the hold force applied at two ends of the belt, at the instant of impending gross slip, are discussed in almost all textbooks of engineering mechanics and machine theory, e.g., [7–10]. The objective of this paper is to extend this analysis and determine the variation of the force in the belt before the state of impending slip is reached throughout the contact region between the belt and the cylinder. To the best of our knowledge, the analytical or experimental investigation of such force has not been previously reported in the literature. The analysis also delivers the local pressure and friction forces exerted by the belt on the cylinder, as well as their resultants.

2. Capstan equation

Fig. 1a shows a flat belt of negligible weight wrapped around a fixed circular disk or cylindrical drum with the contact (wrap) angle θ. The width of the rectangular cross section of the belt is a and its height h is assumed to be much smaller than the radius of the drum

* Department of NanoEngineering, University of California, San Diego, La Jolla, CA 92093-0448, USA.

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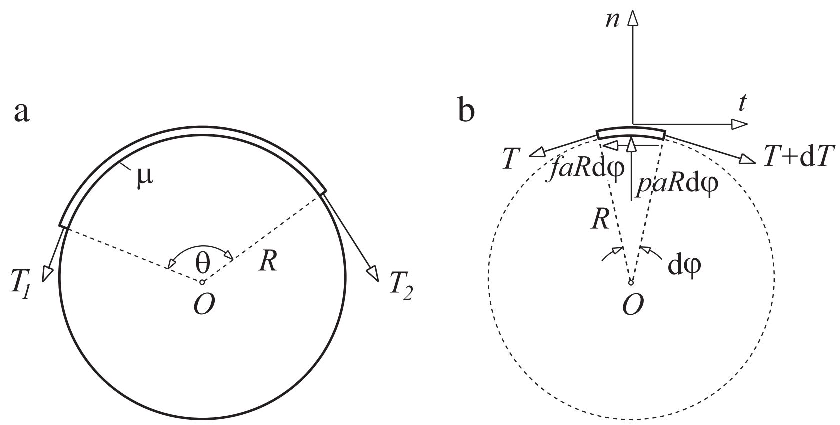

Fig. 1. (a) A flat belt wrapped around a fixed circular cylinder of radius R. The contact angle between the belt and the cylinder is θ. The hold force at the left end is T_1, and the pull force at the right end of the belt at the instant of its impending slip is T_2. The coefficient of static friction between the contacting surfaces is μ. (b) An infinitesimal belt segment subtended by the angle dφ. The tension in the belt at the left end of the segment is T and at the right end $T + \mathrm{d}T$. The total force from the pressure p acting along the contact area $aR\,\mathrm{d}\varphi$ is $p(\varphi)aR\mathrm{d}\varphi$ (neglecting the small quantities of higher order). The width of the belt is a. Likewise, the total force from the local friction f is $f(\varphi)aR\mathrm{d}\varphi$. The tangential and normal directions at the considered point of the belt are denoted by t and n.

($h \ll R$).[1] The hold force at the left-end (low-tension side) is T_1, and the pull force at the right-end (high-tension side) of the belt is T_2. The free body diagram of the belt segment of length $R\mathrm{d}\varphi$ is shown in Fig. 1b. The local pressure is denoted by $p = p(\varphi)$ and the local friction force by $f = f(\varphi)$ (both per unit contact area). For equilibrium, the sum of forces in the normal (n) and tangential (t) direction must vanish, which gives $T = paR$ and $\mathrm{d}T = faR\mathrm{d}\varphi$. These two equilibrium conditions involve three unknown quantities, the force in the belt $T = T(\varphi)$, the pressure $p = p(\varphi)$, and the friction force $f = f(\varphi)$. The problem is statically indeterminate, unless the pull force is increased to produce the state of impending gross slip of the belt over the cylinder. In the latter case, adopting the Amontons–Coulomb law of dry friction [11,12], the local friction force is $f = \mu p$, where μ is the coefficient of static friction between the belt and the cylinder. This makes the system of equations statically determinate. The increment of the force in the belt is then $\mathrm{d}T = \mu T\,\mathrm{d}\varphi$, and the integration gives the Euler formula [6], or the belt friction (capstan) equation,

$$T_2 = T_1 \exp(\mu\theta). \tag{1}$$

If a weightless belt actually slips over the cylindrical surface, the coefficient of kinetic friction μ_k, rather than the coefficient of static friction μ, should be used in Eq. (1).[2]

One could also proceed with the analysis by deriving the differential equation for the pressure, rather than the force in the belt. In this case, from $T = paR$ the increment of the force can be expressed as $\mathrm{d}T = aR\mathrm{d}p$, and the substitution into the second equilibrium condition ($\mathrm{d}T = faR\,\mathrm{d}\varphi$) gives $\mathrm{d}p = f\mathrm{d}\varphi$. In the state of impending slip this becomes $\mathrm{d}p = \mu p\mathrm{d}\varphi$. The integration gives $p(\varphi) = p(0)\exp(\mu\varphi)$, where $p(0) = T_1/(aR)$ is the pressure at the contact point $\varphi = 0$.

3. Force in the belt before the gross slip

Determination of the force in the belt $T(\varphi)$ in the contact region $\varphi \in [0, \theta]$ before the state of impending (gross) slip throughout the contact region is reached, i.e., when the applied pull force T_2 is less than its Euler's value $T_2^{\mathrm{E}} = T_1 \exp(\mu\theta)$, is statically indeterminate because of the lack of the relationship between the local friction force $f(\varphi)$ and the local contact pressure $p(\varphi)$. The deformability of the belt and the cylinder must be included in the analysis and the solution of a difficult elastic contact mechanics problem [19,20] is needed to obtain the expression for the tangential force in the belt. However, an approximate analysis can proceed by considering the stretchability of the belt only. Suppose that the belt, wrapped around a cylinder over the contact angle θ, is initially under equal pull and hold forces ($T_2 = T_1$), with no friction force developed between the belt and the cylinder. If the pull force T_2 is then increased by an infinitesimal amount ΔT, only the portion of the belt of length $R\Delta\theta_s$, adjacent to the pull-end, will additionally stretch, as the friction force develops within that portion of the belt balancing the force increment ΔT. The actual stretching of the belt relative to the rigid cylinder implies that the friction force (per unit contact area) within the stretched portion $R\Delta\theta_s$ is μp, where $p = T_1/(aR)$ is the corresponding average pressure (to first-order accuracy). The friction force elsewhere in the contact region is equal to zero.[3] With further increase of the pull force, the extent of the stretched zone of the belt increases and at an arbitrary value of the pull

[1] The V-shaped belts will be considered in Section 3.1.

[2] The coefficient of friction is taken to be constant along the contact. There has been a large amount of research devoted to the analysis in which the coefficient of friction was assumed to depend on the ratio of the actual and apparent contact areas and thus contact pressure [13,14]. Furthermore, the generalizations of the Euler formula were proposed to include the longitudinal and shear deformability of the belt. The so-called creep theory of the belt-drive mechanics, proposed by Reynolds [15] and further developed by Swift [16], was considered in [17] and [18], among others.

[3] In fact, for any increment ΔT of the pull force, there is an adjacent part of the contact angle near the pull-end where the state of impending slip is reached, so that the Euler's formula applies in the form $T_1 + \Delta T = T_1 \exp(\mu\Delta\theta_s)$. This gives $\Delta\theta_s = (1/\mu)\ln[1 + (\Delta T)/T_1]$. If $\Delta T \ll T_1$ then $\Delta\theta_s = (\Delta T)/(\mu T_1)$, to first order.

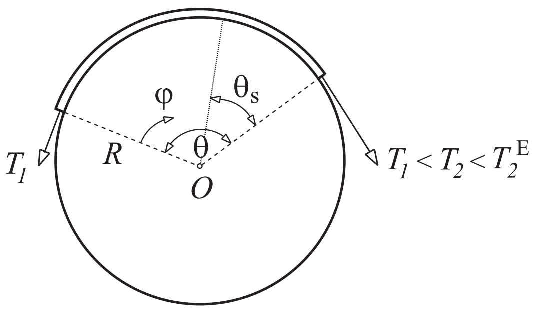

Fig. 2. The loaded belt before the state of gross slip. Under the pull force $T_1<T_2<T_2^{\mathrm{E}}$, the state of impending slip extends within the slip angle θ_s, and there the friction force is $f = \mu p$. Elsewhere within the contact angle $(0 \le \varphi \le \theta - \theta_s)$ the friction force vanishes and the belt force is constant.

force $T_1<T_2<T_2^{\mathrm{E}}$ it subtends the (slip) angle θ_s (Fig. 2). The friction force within this angle is $f(\varphi) = \mu p(\varphi)$, and the pressure is $p(\varphi) = T(\varphi)/(aR)$. Outside of this angle, i.e., within the inactive slip range $(0 \le \varphi < \theta - \theta_s)$, the friction force identically vanishes.[4] The force in the belt in the entire contact region is therefore specified by

$$T(\varphi) = \begin{cases} T_1, & \text{if } 0 \le \varphi \le \theta - \theta_s, \\ T_1 \exp\{\mu[\varphi - (\theta - \theta_s)]\}, & \text{if } \theta - \theta_s \le \varphi \le \theta. \end{cases} \tag{2}$$

In particular, $T_2 = T_1 \exp(\mu\theta_s)$, so that the angle of slip θ_s corresponding to the applied force $T_2 \in [T_1, T_2^{\mathrm{E}}]$ is

$$\theta_s = \frac{1}{\mu} \ln \frac{T_2}{T_1} = \theta - \frac{1}{\mu} \ln \frac{T_2^{\mathrm{E}}}{T_2}. \tag{3}$$

Consequently, Eq. (2) can be rewritten as

$$T(\varphi) = \begin{cases} T_1, & \text{if } 0 \le \varphi \le \theta - \theta_s, \\ T_2 \exp[\mu(\varphi - \theta)], & \text{if } \theta - \theta_s \le \varphi \le \theta. \end{cases} \tag{4}$$

Fig. 3 shows the variation of the belt force $T(\varphi)/T_1$ with φ/θ for three selected values of the ratio $T_2/T_2^{\mathrm{E}} = 1/m$, where $1 \le m \le \exp(\mu\theta)$ in order that $T_1 \le T_2 \le T_2^{\mathrm{E}}$. In this case, from Eq. (3), the active slip angle is $\theta_s = \theta - (\ln m)/\mu$, so that Eq. (4) becomes

$$T(\varphi) = \begin{cases} T_1, & \text{if } 0 \le \varphi \le \dfrac{1}{\mu} \ln m, \\ \dfrac{T_1}{m} \exp(\mu\varphi), & \text{if } \dfrac{1}{\mu} \ln m \le \varphi \le \theta. \end{cases} \tag{5}$$

The coefficient of friction used to construct the plots shown in Fig. 3 was taken to be $\mu = 1/\theta$. In this case, the maximum value of m is equal to $e = 2.718\ldots$ (base of the natural logarithm), while $T(\varphi) = T_1$ in the entire contact range $0 \le \varphi \le \theta$ (dotted horizontal line in Fig. 3).

The experimental verification of the proposed model leading to Eqs. (3) and (4) would involve the stretch measurements by means of strain gauges attached to the surface of the belt. Although the strain is greater for softer belts, the product of elastic stiffness of the belt and its local strain may depend only on the coefficient of friction, and applied pull and hold forces. For a given magnitude of the pull force $T_1<T_2<T_2^{\mathrm{E}}$, the inactive arc of contact where the slip does not occur $R(\theta - \theta_s)$ can be experimentally determined by detecting the portion of the belt for which the strain gauges record zero elongation, relative to the initial state with equal tension at both sides of the belt. A review of experimental investigations of the tension in a flat belt during power transmission has recently been presented in [5].

3.1. V-shaped belts

Many industrially important belt drives consist of belts with a V-shaped cross-section running on grooved disks or pulleys. Due to wedging effect in the groove, V-shaped belts can transmit more power without slip than flat belts [2,4]. The extension of the analysis to V-shaped belts with a trapezoidal cross-section, such as those shown in Fig. 4 is straightforward. If a denotes the contact length between the belt and the groove on each side of the groove, and if 2α is the angle subtended by two sides of the groove, the average

[4] In the mechanics of belt drive by rotating pulleys, the length $R\theta_s$ is called an active arc of contact, while $R(\theta - \theta_s)$ is referred to as an inactive (or idle) arc of contact [5,14].

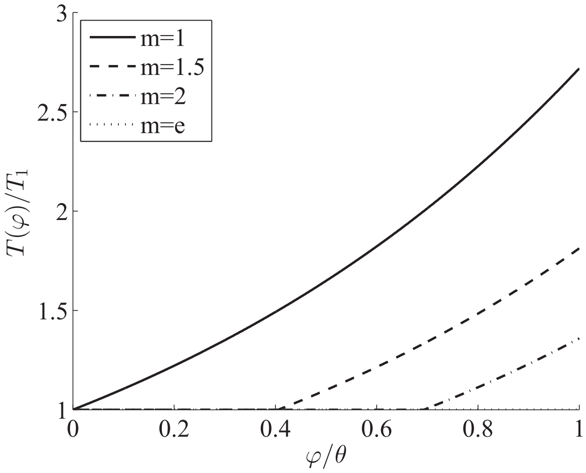

Fig. 3. The variation of the force in the belt $T(\varphi)$ (scaled by T_1) within the angle $0 \le \varphi \le \theta$. The pull force T_2 is taken to be $(1/m)$ times the pull force required to slip the belt, $T_2^{\mathrm{E}} = T_1 \exp(\mu\theta)$. The coefficient of friction is taken to be the reciprocal of the contact angle ($\mu\theta = 1$). The three shown curves are for $m = 1$ (impending slip), $m = 1.5$, and $m = 2$.

pressure over the contact side ($\overline{p}$) is related to the belt force by $T = 2\overline{p}a\rho\sin\alpha$, where ρ is the radius of curvature, measured from the centroid of the cross section to the center of the disk. This is

$$\rho = \rho_0 + \frac{b_1}{2\tan\alpha} + \frac{3b_1 + 4h\tan\alpha}{b_1 + h\tan\alpha}\frac{h}{6}, \tag{6}$$

expressed in terms of the prescribed depth h of the cross-section, its shorter base b_1, and the angle α. The radius of curvature ρ_0 is measured from the tip of the V groove to the center of the disk. The average friction force (per unit contact area) within the slip region defined by the angle θ_s is $\overline{f} = \mu\overline{p}$. It readily follows that

$$T(\varphi) = \begin{cases} T_1, & \text{if } 0 \le \varphi \le \theta - \theta_s, \\ T_2 \exp\left[\dfrac{\mu(\varphi-\theta)}{\sin\alpha}\right], & \text{if } \theta - \theta_s \le \varphi \le \theta \end{cases} \tag{7}$$

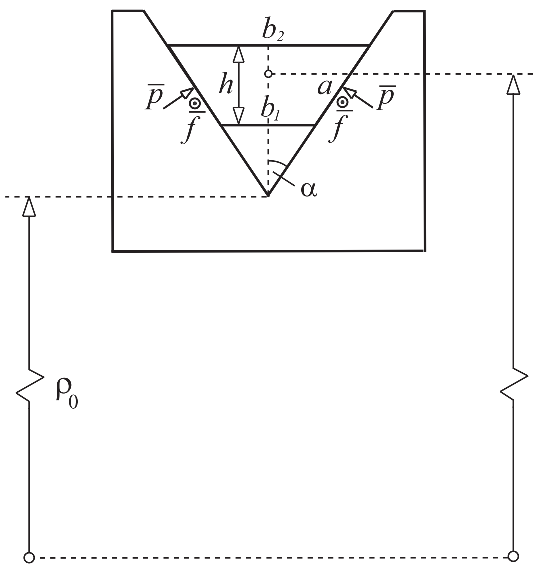

Fig. 4. The trapezoidal cross-section of the V-shaped belt. The depth of the cross-section is h, and its bases are b_1 and $b_2 = b_1 + 2h\tan\alpha$, where 2α is the groove angle. The radius of curvature, measured from the centroid of the trapezoidal cross-section to the center of the disk, is ρ. The average pressure over the contact surfaces is $\overline{p}$ and the average friction force (per unit contact area) is $\overline{f}$. The contact length on each side of the groove is $a = h/\cos\alpha$.

where

$$\theta_s = \frac{\sin\alpha}{\mu}\ln\frac{T_2}{T_1} = \theta - \frac{\sin\alpha}{\mu}\ln\frac{T_2^E}{T_2}. \tag{8}$$

These expressions are analogous to expressions (3) and (4), and can be deduced from them by replacing the actual coefficient of friction μ with the effective coefficient of friction $\mu/\sin\alpha$.

4. Total forces due to pressure and friction

The total pressure force is $\mathbf{P} = \mathbf{P}' + \mathbf{P}''$, where $\mathbf{P}'$ is the force from the pressure acting along the inactive slip range $(\theta - \theta_s)$, and $\mathbf{P}''$ is the force from the pressure acting along the slip range θ_s (Fig. 5). The former is

$$\mathbf{P}' = -P'\cos\left(\theta_0 + \frac{\theta-\theta_s}{2}\right)\mathbf{i} + P'\sin\left(\theta_0 + \frac{\theta-\theta_s}{2}\right)\mathbf{j}, \quad P' = 2T_1\sin\frac{\theta-\theta_s}{2}, \tag{9}$$

which acts along the direction through the middle of the angle $(\theta - \theta_s)$, because the pressure is constant and equal to $T_1/(aR)$ within the inactive slip range in the case of a flat belt, and $T_1/(2a\rho\sin\alpha)$ in the case of a V-shaped belt. The unit vectors in the horizontal and vertical direction are $\mathbf{i}$ and $\mathbf{j}$. The total pressure force in the slip range θ_s is

$$\mathbf{P}'' = \int_{\theta-\theta_s}^{\theta}\mathbf{n}(\varphi)p(\varphi)aRd\varphi = \int_{\theta-\theta_s}^{\theta}\mathbf{n}(\varphi)T(\varphi)d\varphi, \tag{10}$$

where $\mathbf{n}(\varphi) = -\cos(\theta_0+\varphi)\mathbf{i} + \sin(\theta_0+\varphi)\mathbf{j}$ is the unit vector orthogonal to the cylinder at an arbitrary contact point. By substituting expression (2) for $T(\varphi)$ and integrating, it follows that $\mathbf{P}'' = P''_x\mathbf{i} + P''_y\mathbf{j}$, where

$$\begin{aligned} P''_x &= -\frac{1}{1+\mu^2}\{T_2[\mu\cos(\theta_0+\theta) + \sin(\theta_0+\theta)] - T_1(\mu\cos\omega_0 + \sin\omega_0)\}, \\ P''_y &= \frac{1}{1+\mu^2}\{T_2[\mu\sin(\theta_0+\theta) - \cos(\theta_0+\theta)] - T_1(\mu\sin\omega_0 - \cos\omega_0)\}. \end{aligned} \tag{11}$$

The angle ω_0 is defined by $\omega_0 = \theta_0 + \theta - \theta_s$. The magnitude of the total pressure force, including the contributions from both $\mathbf{P}'$ and $\mathbf{P}''$, is

$$P = \frac{1}{\sqrt{1+\mu^2}}\left\{F_R^2 + 2\mu T_1^2[\mu - \sin(\theta-\theta_s) - \mu\cos(\theta-\theta_s)] + 2\mu T_1T_2(\sin\theta - \sin\theta_s)\right\}. \tag{12}$$

The magnitude of the resultant of the pull and hold forces acting at two ends of the belt is $F_R = (T_1^2 + T_2^2 - 2T_1T_2\cos\theta)^{1/2}$. In the case of a V-shaped belt, μ in the above expressions should be replaced with $\mu/\sin\alpha$.

Since there are no distributed friction forces within $(\theta - \theta_s)$, the total friction force is orthogonal to $\mathbf{P}''$ and equal to $\mathbf{F} = -\mu P''_y\mathbf{i} + \mu P''_x\mathbf{j}$. Its magnitude is

$$F = \mu P'', \quad P'' = \frac{1}{\sqrt{1+\mu^2}}\left(T_1^2 + T_2^2 - 2T_1T_2\cos\theta_s\right)^{1/2}. \tag{13}$$

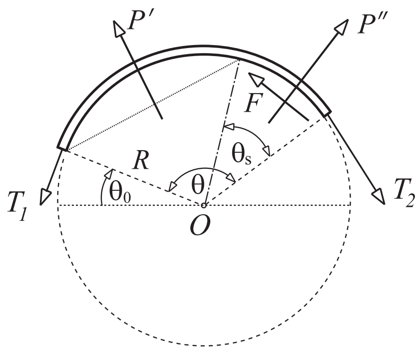

Fig. 5. The belt under the pull force T_2 for which the state of impending slip extends within the slip angle θ_s. The total friction force there is F, and the total pressure force P''. Elsewhere within the contact angle, the friction force vanishes, while the pressure force is P'.

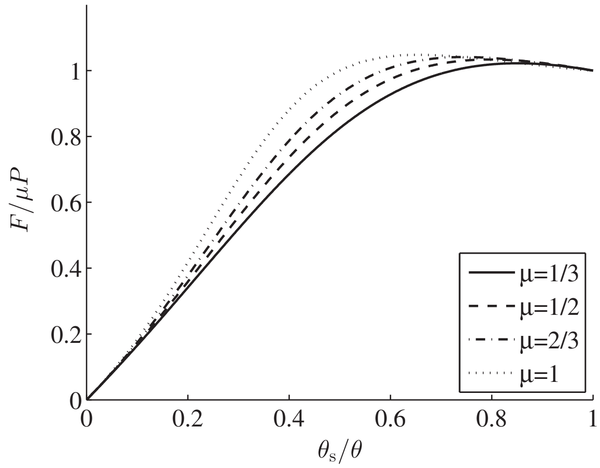

Fig. 6. The variation of the ratio $F/(\mu P)$ with θ_s/θ in the case $\theta = \pi$, for the four indicated values of the coefficient of friction μ.

The total friction and pressure forces are neither proportional nor orthogonal to each other, unless the state of gross slip throughout the entire contact range is reached. Fig. 6 shows the variation of the ratio $F/(\mu P)$ with the angle of slip (θ_s/θ) in the case $\theta = \pi$ and four selected values of the coefficient of friction. In all cases, $F/(\mu P)$ vanishes for $\theta_s = 0$ and is equal to 1 for $\theta_s = \theta$.

If the state of impending gross slip is reached throughout the contact range, then $\theta_s = \theta$, $\omega_0 = \theta_0$, and

$$F = \mu P, \quad P = \frac{1}{\sqrt{1+\mu^2}} F_R. \tag{14}$$

In this case, P and F are orthogonal to each other, because f is proportional and orthogonal to p throughout the contact range. Fig. 7 shows the corresponding variation of P and F with μ, illustrating what portion of the total reactive force F_R is due to pressure and what portion due to friction. The two contributions are equal to each other when $\mu = 1$.

5. Conclusions

A simplified analysis is presented to determine the variation of the belt force within the contact region before the state of impending gross slip. The analysis is based on the assumption of gradual growth of the slip range from the pull-end to the hold-end of the belt, as the pull force increases towards its Euler's value required for the gross slip. If the pull force is $T_2 < T_2^E = T_1$ exp

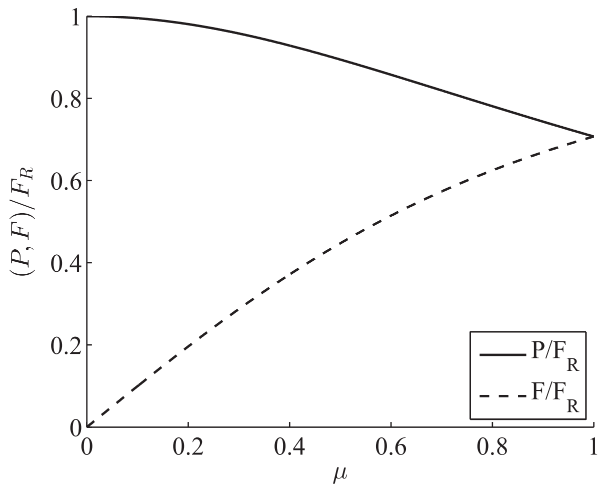

Fig. 7. The variations of the total pressure force P and total friction force F (scaled by the total reactive force F_R) with the coefficient of friction μ, in the case of impending gross slip. The pressure and friction contribution to F_R are equal to each other when $\mu = 1$.

$(\mu\theta)$ and the hold force is $T_1 < T_2$, the force in the belt is $T(\varphi) = T_1$ within the inactive slip range, and $T(\varphi) = T_2 \exp[\mu(\varphi - \theta)]$ within the active slip range. The extent of the active slip range within the contact angle θ is specified by $\theta_s = (1/\mu)\ln(T_2/T_1)$, where μ is the coefficient of static friction. As a consequence of the introduced model, the local belt force depends on the pull and hold forces, and the coefficient of friction, but not on the elastic stiffness of the belt. The local pressure and friction forces exerted by the belt on the cylinder are determined. Both flat and V-shaped belts are considered. The total pressure and friction forces are evaluated at an arbitrary stage of slip growth. They are neither proportional nor orthogonal to each other, unless the state of impending gross slip is reached throughout the contact range. In this case, the total pressure and friction force are $P = F_R/(1 + \mu^2)^{1/2}$ and $F = \mu P$, where F_R is the magnitude of the resultant of the pull and hold forces.

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