Applied Mathematical Sciences, Vol. 7, 2013, no. 92, 4595 - 4604
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.36348

Lie Symmetries Analysis for SIR Model of Epidemiology

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Abstract

In this paper a system of nonlinear ordinary differential equations arising from SIR model of epidemiology is transformed into a system of one equation of second order and one of first order. We use the property of the Lie generators algebras for any two dimensional Lie algebra to solve the first equation of the system. Then, the Lie point symmetry method is applied and differential invariants are used to obtain some exact solutions of the model.

Mathematics Subject Classification: 35Bxx, 35Dxx, 92Bxx

Keywords: Lie Symmetry method, SIR epidemiological model, Exact solutions.

1 Introduction

The epidemiological models consist of system of nonlinear differentials equations which describe the dynamics in each class. The simplest model was proposed in 1927 by Kermack and Mckendick [1]. This model is called the SIR model (S refers to all susceptible individuals), I (refers to infectious class) and finally R (refers to recovered class) [see [2] for more details].This model is called sometime a compartmental models because each letter is referring to a compartment in which an individual can reside. In fact, each individual can reside in exactly one compartment and can move from one compartment to another. This is a good and simple model for many infectious. Here we use the Lie symmetries methods [3, 5, 6, 7] for some various SIR type model systems. In fact, a system of nonlinear ordinary differential equations arising from this model is transformed into a system of one equation of second order and one of first order[4]. We analysis a family of two-dimensional subalgebra with the property $[v, w] = kv$ for any two dimensional Lie algebra generated by v and w. Then, the first equation is solved. Furthermore, the Lie point symmetry method is applied and the differential invariants are used to solve the system. The paper is arranged as follows: In section 2,the SIR type model systems proposed in this work is introduced. The Lie symmetry group method is applied for this model in section 3 and each one-parameter symmetry group is presented. In section 4, by using the reduction technics some exact solutions of the system are given. Finally, the conclusion contains remarks about this work.

2 Lie point symmetries

2.1 The proposed model

Among the various SIR type model systems[8, 9], we will be interested in the model

$$\begin{cases} \frac{dS}{dt} &= \frac{-\beta SI}{S+I+R}, \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I+R} - \nu I, \\ \frac{dR}{dt} &= \nu I - \alpha R, \end{cases}$$

where the involved parameters α, β and ν are related to the dynamics and nature of the model and $S = S(t), I = I(t), R = R(t)$ represent respectively[8] the time variation of the susceptible individuals, infectious class and recovered class.

To apply Lie symmetry method, first we derive R from the first equation in the above system, i.e:

$$R = -\frac{\beta SI + (S + I)}{\dot{S}},$$

with $\dot{S}$ denoted $\frac{dS}{dt}$. Hence, we get

$$\frac{d^2S}{dt^2} = \left[\alpha\beta SI(\frac{dS}{dt}) + \alpha S(\frac{dS}{dt})^2 + \alpha I(\frac{dS}{dt})^2 - \beta\nu SI(\frac{dS}{dt}) - \beta S(\frac{dS}{dt})^2 + \beta I(\frac{dS}{dt})^2\right] / \beta SI,$$

and

$$\frac{dv}{dt} = -\nu v + \nu y,$$

with $v(t) = S(t) + I(t)$. If we put $y(t) = S(t)$, and $\alpha = \beta$ the system M is transformed into a system of one equation of second order in y, and one of first order in v:

$$\frac{d^2y}{dt^2} = \left[(\alpha - \nu)y\frac{dy}{dt} + 2(\frac{dy}{dt})^2\right]/y, \tag{1}$$

$$\frac{dv}{dt} = -\nu v + \nu y. \tag{2}$$

3 Symmetry generators

In order to calculate the Lie point symmetries of equation (1), we consider the generator of the group of point transformations

$$X = \xi(t, y)\frac{\partial}{\partial t} + \varphi(t, y)\frac{\partial}{\partial y}.$$

We wish to determine all possible coefficient function ξ and φ so that the corresponding one-parameter group $\exp(\varepsilon X)$ is a symmetry group of equation (1). Then, we need to know the second prolongation of X which is given by [3, 5]

$$X^{(2)} = X + \varphi^t\frac{\partial}{\partial y_t} + \varphi^{tt}\frac{\partial}{\partial y_{tt}}.$$

The coefficients φ^t and φ^{tt} are calculated explicitly by the prolongation formula [3, 5]

$$\begin{aligned} \varphi^t &= D_t(\varphi - \xi y_t) + \xi y_{tt}, \\ \varphi^{tt} &= \varphi_{tt} + (2\varphi_{ty} - \xi_{tt})y_t + (\varphi_{yy} \\ &- 2\xi_{ty})y_t^2 - \xi_{yy}y_t^2 + (\varphi_y - 2\xi_t)y_{tt} - 3\xi_y y_t y_{tt}. \end{aligned} \tag{3}$$

According to the invariance criterion, X generates symmetries of the equation (1), if and only if

$$X^{(2)}[yy_{tt} - (\alpha - \nu))yy_t - 2(y_t)^2] = 0,$$

whenever $yy_{tt} - (\alpha - \nu))yy_t - 2(y_t)^2 = 0$. Hence, the invariance condition implies that,

$$\varphi(y_{tt-(\alpha-\nu)y_t}) + \varphi^t(-(\alpha - \nu)y - 4y_t) + \varphi^{tt}y = 0,$$

which must be satisfied whenever $yy_{tt} = (\alpha - \nu))yy_t + 2(y_t)^2$.

This haves a set of defining equations for ξ and φ, which may readily be solved [2]. We conclude that the most general infinitesimal symmetry of equation(1) has coefficient functions of the form

$$\begin{aligned} \xi(t,y) &= \frac{c_1}{y}e^{-(\alpha-\nu)t} + \frac{c_2}{y} + c_3 - c_4 e^{(\alpha-\nu)t} + c_5 e^{-(\alpha-\nu)t}, \\ \varphi(t,y) &= -c_2(\alpha-\nu) + c_6 e^{(\alpha-\nu)t}y^2 + c_7 y^2 + c_8 + c_4(\alpha-\nu)e^{(\alpha-\nu)t}y, \end{aligned} \tag{4}$$

where $c_1, \ldots, c_8$ are arbitrary constants. Thus the Lie algebra of infinitesimal symmetries of equation (1) is spanned by eight vector fields

$$\begin{aligned} X_1 &= \frac{1}{y}e^{-(\alpha-\nu)t}\frac{\partial}{\partial t} \\ X_2 &= \frac{1}{y}\frac{\partial}{\partial t} - (\alpha-\nu)\frac{\partial}{\partial y} \\ X_3 &= e^{(\alpha-\nu)t}y^2\frac{\partial}{\partial y} \\ X_4 &= y^2\frac{\partial}{\partial y} \\ X_5 &= y\frac{\partial}{\partial y} \\ X_6 &= \frac{\partial}{\partial t} \\ X_7 &= e^{(\alpha-\nu)t}(-\frac{\partial}{\partial t} + (\alpha-\nu)y\frac{\partial}{\partial y}) \\ X_8 &= e^{-(\alpha-\nu)t}\frac{\partial}{\partial t}. \end{aligned} \tag{5}$$

To obtain the group transformation generated by each infinitesimal symmetry X_k, we solve the system of first order ordinary differential equations,

$$\begin{aligned} \frac{d\widetilde{t}}{d\varepsilon} &= \xi(\widetilde{t},\widetilde{y}), \\ \frac{d\widetilde{y}}{d\varepsilon} &= \varphi(\widetilde{t},\widetilde{y}), \end{aligned}$$

subject to the initial conditions

$$\widetilde{t}(0) = t,\widetilde{}$$

The one-parameter groups G_i generated by X_i are given as follows. The entries gives the transformed point $\exp(\varepsilon X_i)(t, y) = (\widetilde{t}, \widetilde{y})$:

$$\begin{aligned}
G_1 &: (\frac{(\alpha-\nu)\varepsilon}{y} + t, y),\\
G_2 &: (\frac{1}{(\alpha-\nu)} \ln(\frac{y}{-(\alpha-\nu)\varepsilon + y}) + t, -(\alpha-\nu)\varepsilon + y),\\
G_3 &: (t, \frac{-y}{e^{(\alpha-\nu)t} y\varepsilon - 1}),\\
G_4 &: (t, \frac{-y}{\varepsilon y - 1}),\\
G_5 &: (t, ye^{\varepsilon}),\\
G_6 &: (\varepsilon + t, y),\\
G_7 &: (\frac{-1}{(\alpha-\nu)} \ln[(\alpha-\nu)\varepsilon + e^{-(\alpha-\nu)t}], (\alpha-\nu)e^{-(\alpha-\nu)t} y\varepsilon + y),\\
G_8 &: (\frac{1}{(\alpha-\nu)} \ln[(\alpha-\nu)\varepsilon + e^{(\alpha-\nu)t}], y).
\end{aligned}$$

As G_i is a symmetry group, so if $y(t)$ is a solution of equation (1), then are the transformed functions $g.y(t)$ which are given in the case of projectable symmetries $G_i, i = 3, ..., 8$:

$$\begin{aligned}
y_3(t) &= \frac{-y(t)}{e^{(\alpha-\nu)t} y(t)\varepsilon - 1},\\
y_4(t) &= \frac{-y(t)}{\varepsilon y(t) - 1},\\
y_5(t) &= y(t)e^{\varepsilon},\\
y_6(t) &= y(t-\varepsilon),\\
y_7(t) &= [(\alpha-\nu)e^{-(\alpha-\nu)t}\varepsilon - (\alpha-\nu)^2\varepsilon^2 + 1] y \left(\frac{-1}{(\alpha-\nu)} \ln[e^{-(\alpha-\nu)t} - (\alpha-\nu)\varepsilon] \right),\\
y_8(t) &= y(\frac{1}{(\alpha-\nu)} \ln[e^{(\alpha-\nu)t} - (\alpha-\nu)\varepsilon]),
\end{aligned}$$

where ε is any real number. Now we consider the applications of the above transformations. If we start from G_3. Take $y(t) = c$, where c is an arbitrary constant and also a trivial solution. We get a new nontrivial exact solution as:

$$y_3(t) = \frac{-c}{e^{(\alpha-\nu)t} c\varepsilon - 1}.$$

Further more, continue this iteration process, we can derive from G_7 with $c = \varepsilon = 1$ a new exact solution of equation (1)

$$y_7(t) = -\frac{1 + (\alpha-\nu)[e^{-(\alpha-\nu)t} - (\alpha-\nu)]}{1 - [e^{-(\alpha-\nu)t} - (\alpha-\nu)]}.$$

Now, we come back to the commutation relations between vector fields $X_k, k = 1, ..., 8$ which is given by the following table, the entry in row i and column j representing the Lie bracket $[X_i, X_j] := X_iX_j - X_jX_i$:

	X_1	X_2	X_3	X_4
X_1	0	0	$\theta X_5 + X_6$	X_8
X_2	0	0	$-X_7$	$-2\theta X_5$
X_3	$-\theta X_5 - X_6$	X_7	0	0
X_4	$-X_8$	$2\theta X_5$	0	0
X_5	$-X_1$	$-X_2$	X_3	X_4
X_6	$-\theta X_1$	0	θX_3	0
X_7	θX_2	0	0	θX_3
X_8	0	θX_1	θX_4	0

	X_5	X_6	X_7	X_8
X_1	X_1	θX_1	$-\theta X_2$	0
X_2	X_2	0	0	$-\theta X_1$
X_3	$-X_3$	$-\theta X_3$	0	$-\theta X_4$
X_4	$-X_4$	0	$-\theta X_3$	0
X_5	0	0	0	0
X_6	0	0	θX_7	$-\theta X_8$
X_7	0	$-\theta X_7$	0	$2\theta X_6 - \theta^2 X_5$
X_8	0	θX_8	$-2\theta X_6 + \theta^2 X_5$	0

where $\theta = \alpha - \nu$. It is known, that we can find a basis $\{v, w\}$ for any two dimensional Lie algebra with the property

$$[v, w] = kv.$$

Here, from the commutator table, it appears that, the Lie symmetry algebra admitted by equation (1), contains a very important family of two-dimensional Lie subalgebra $\{v, w\}$ with the property $[v, w] = kv$ without changing of the basis. Those subalgebras will be employed in the next section to reduce equation (1) and so to solve it by a pair of quadratures from the general solution of reduced equation.

4 Exact solutions of equation (1)

If we reduce the order of an ordinary differential equation using only a subgroups of the full symmetry group, we may very well lose any additional symmetry properties present in the full group [5, 6, 7]. Special subgroups, namely normal subgroups will enable the reduced equation to inherit the symmetry property from the full symmetry group.

In this section we will be interested to a family of two-dimensional subalgebra with the property $[v, w] = kv$.

4.1 Case 1: $[X_1, X_5] = X_1$

To reduce equation (1) using the corresponding two-parameters symmetry group, it is important to do this reduction in the right order. We need to first determine differential invariants of X_1, which are

$$I_1 = y, \qquad \text{and} \qquad I_2 = e^{(\alpha-\nu)t}[1 + (\alpha - \nu)\frac{y}{y_t}].$$

In terms of which equation (1) reduces to the ordinary differential equation

$$\frac{dI_2}{dI_1} = 0. \tag{6}$$

Integration of this equation yields,

$$y(t) = \frac{a}{b - e^{(\alpha-\nu)t}},$$

where a and b are arbitrary constants.

4.2 Case 2: $[X_2, X_5] = X_2$

The differential invariants of X_2 in these case are

$$I_1 = ye^{(\alpha-\nu)t}, \qquad \text{and} \qquad I_2 = \frac{1}{(\alpha - \nu)y} + \frac{1}{y_t}.$$

In terms of I_1 and I_2 equation (1) reduces to the equation

$$(\alpha - \nu)\frac{dI_2}{dI_1}I_1 + 1 = 0. \tag{7}$$

The solution of this equation is,

$$I_2 = -\frac{1}{(\alpha - \nu)}\ln I_1 + a_1,$$

where a_1 is an arbitrary constant. In this case, the general solution of equation (1) is defined implicitly by the first order differential equation

$$\frac{1}{y'} = -\frac{1}{(\alpha - \nu)y} - \frac{1}{(\alpha - \nu)}\ln y - t + a_1,$$

where primes denotes the differentiation with respect t.

4.3 Case 3: $[X_3, X_5] = -X_3$

The invariants of X_3 are:

$$I_1 = t, \qquad \text{and} \qquad I_2 = -\frac{y_t}{y^2} - \frac{(\alpha-\nu)}{y}.$$

The reduced equation obtained here is:

$$\frac{dI_2}{dI_1} = 0. \tag{8}$$

This equation gives $y(t)$ as solution of the Ricatti equation

$$y' = a_2 y^2 - (\alpha-\nu)y,$$

where a_2 an arbitrary constant. The general solution of this equation is

$$y(t) = \frac{1}{ke^{(\alpha-\nu)t} + a_2},$$

with k and a_2 are arbitrary constants.

4.4 Case 4: $[X_4, X_5] = -X_4$

The invariants of X_4 are:

$$I_1 = t, \qquad \text{and} \qquad I_2 = \frac{y^2}{y_t}.$$

In this case the equation (1) is taken into the differential equation

$$\frac{dI_2}{dI_1} + (\alpha-\nu)I_2 = 0. \tag{9}$$

Thus, $y(t)$ is solution of the first order differential equation

$$y' = a_3 e^{(\alpha-\nu)t} y^2,$$

where a_3 is an arbitrary constant. This case gives the same solution as in the previous case.

4.5 Case 5: $[X_8, X_6] = (\alpha-\nu)X_8$

The differential invariants are:

$$I_1 = y, \qquad \text{and} \qquad I_2 = y_t e^{-(\alpha-\nu)t}.$$

We obtain as in the previous case.

4.6 Case 6: $[X_7, X_6] = -(\alpha - \nu)X_7$

The invariants of X_7 are:

$$I_1 = ye^{(\alpha-\nu)t}, \qquad \text{and} \qquad I_2 = \frac{y_t}{y^2} + \frac{(\alpha - \nu)}{y}.$$

One finds that the reduced ordinary differential equation

$$\frac{dI_2}{dI_1} = 0,$$

and we obtain the same solution as in the previous case.

5 Conclusion

In this paper, a system of nonlinear ordinary differential equations arising from SIR model is transformed into a system of one equation of second order and one of first order. By using the Lie symmetries method the one-parameter and the corresponding generators are obtained. By using the reduction technics and the differential invariants some exact solutions of the system are obtained.

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Received: June 28, 2013