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Existence Result to Boundary Value Problem for Fractional Differential Equations with Integral Boundary Conditions

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Abstract: By the means of the Green's function, the boundary value problem of fractional differential equation can be reduced to the equivalent integral equation. Recently, this method is used successfully to discuss the existence of the solution to boundary value problem of nonlinear fractional differential equation. By applying $Carath\acute{e}odory$ conditions and continuous condition on the nonlinear terms f, we obtain an existence results for solution. Our analysis relies on the concept of measures of noncompactness, the $M\ddot{o}nch's$ fixed point theorem and the Schaefer's fixed point theorem. Example is provided to illustrate the theory.

Keywords: Boundary value problem; fractional differential equations; Caputo fractional derivative; measure of noncompactness; $Carath\acute{e}odory$ condition

1 Introduction

This paper is mainly concerned with the existence of solutions of boundary value problems (BVP for short) for a nonlinear fractional differential equation

$$ {}^cD^q_{0+}u(t) = f(t,u(t)), \ \ t \in J := [0,T]. \tag{1}$$

$$ u(0) + \beta \int_0^T u(s)ds = u(T), \tag{2}$$

where $0 < q \leq 1$ is a real number, ${}^cD^q_{0+}$ is the Caputo's fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, and $\mathbb{R}$ is a Banach space with norm $\|\cdot\|$, $\beta > 0$ is real number.

In the last few years, fractional differential equations (in short FDEs) have been studied extensively. The motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the readers to the monographs by Kilbas *et al* [1], Miller and Ross [2], Oldham and Spanier [3], Podlubny [4] and Samko *et al* [5].

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [6–8], Babakhani and Daftardar-Gejji [9–11] and Bai [12], and so on. Also, there are some papers which deal with the existence and multiplicity of solutions(or positive solution) for nonlinear FDE of BVPs by using techniques of nonlinear analysis(fixed-point theorems, Leray-Shauder theory, topological degree theory, etc.). see [13], [14], [15]–[19], [21], [22], [24]–[26] and the references therein.

In [15], based on the $Krasnoselskii's$ fixed point theorem and *Leggett-Williams* fixed point theorem, Bai and $L\ddot{u}$ obtained positive solutions of the two-point BVP of FDE

$$D^q_{0+}u(t) + f(t,u(t)) = 0, \ u(0) = u(1) = 0, \ 0 < t < 1, \ 1 < q \leq 2.$$

D^q_{0+} is the standard *Riemann-Liouville* fractional derivative.

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In [19], Daftardar-Gejji extended the results in [24] to show the existence of at least one positive solution of the system of fractional differential equations

$$D^{q_i}u_i = f_i(t, u_1, u_2, \cdots, u_n),\ u_i(0) = 0,\ 0 < q_i < 1,\ 1 \le i \le n.$$

In [25, 26], Zhang discussed the existence of solutions of the nonlinear FDE

$$^cD^q_{0+}u(t) = f(t, u(t)),\ 0 < t < 1,\ 1 < q \le 2. \tag{3}$$

with the boundary conditions

$$u(0) = v \neq 0,\ u(1) = \rho \neq 0, \tag{4}$$

$$u(0) + u'(0) = 0,\ u(1) + u'(1) = 0, \tag{5}$$

respectively. Since conditions (4) and (5) are nonzero boundary values, the *Riemann-Liouville* fractional derivative D^q_{0+} is not suitable. Therefore, the author investigated the BVPs (3)-(4) and (3)-(5) by involving in the Caputo fractional derivative $^cD^q_{0+}$.

From above works, we can see a fact, although the BVPs of nonlinear FDE have been studied by some authors, to the best of our knowledge, under the noncompactness measure condition of nonlinearity $f(t, u)$, it is seldom considered for FDE. Motivated by the above mentioned work, the main aim of this paper is to study the existence of the problem (1)-(2) under the new conditions via applying the specified Kuratowski measure of noncompactness and $M\ddot{o}nch's$ fixed point theorem.

The remainder of this article is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts and various lemmas which will be used throughout this paper. In Section 3, we give main result of the problem (1)-(2). The last section is devoted to an example illustrating the applicability of the imposed conditions. The result can be considered as a contribution to this emerging field.

2 Preliminaries and lemmas

Let $J := [0, T]$. By $C(J, \mathbb{R})$ denote the Banach space of all continuous mapping $u : J \to \mathbb{R}$ with norm $\|u\|_\infty := \sup\{\|u(t)\| : t \in J\}$. $L^1(J, \mathbb{R})$ denote the Banach space of measurable functions $u : J \mapsto \mathbb{R}$ which are Bochner integrable, equipped with the norm $\|u\|_{L^1} := \int_J \|u(t)\|dt$. $L^\infty(J, \mathbb{R})$ be the Banach space of measurable functions $y : J \mapsto \mathbb{R}$ which are Bounded, equipped with the norm $\|y\|_{L^\infty} := \inf\{c > 0 : \|y(t)\| \le c, \quad a.e.\ t \in J\}$. $AC^1(J, \mathbb{R})$ be the space of functions $y : J \mapsto \mathbb{R}$, whose first derivative is absolutely continuous. Moreover, for a given set V of functions $v : J \mapsto \mathbb{R}$, let us denote by $V(t) = \{v(t), v \in V\}, t \in J$, and $V(J) = \{v(t) : v \in V, t \in J\}$.

Definition 1 *A map $f : J \times \mathbb{R} \mapsto \mathbb{R}$ is said to be $Carath\acute{e}odory$ if f satisfying the following conditions:*

(i) for each $u \in \mathbb{R}$, the mapping $f(\cdot, u) : t \in J \mapsto f(t, u)$ is measurable;

(ii) for almost all $t \in J$, the mapping $f(t, \cdot) : u \in \mathbb{R} \mapsto f(t, u)$ is continuous.

Definition 2 *[1, 4] The fractional order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha_a h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 3 *[1, 4] For a function h given on the interval $[a, b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by*

$$(D^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 4 *[1, 4] For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of order α of h, is defined by*

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha]+1$.

Definition 5 *A function $u \in AC^1(J,\mathbb{R})$ whose q-derivative exists on J is said to be a solution of the problem* (1)–(2) *if u satisfies the equation ${}^cD^q_{0+}u(t) = f(t,u(t))$ on J, and satisfy the condition* (2).

Definition 6 *[20] Assume that S is a bounded set in $\mathbb{R}$. Let*

$\alpha(S)$=inf $\Big\{\delta > 0$: *S can be expressed as the union $S = \bigcup_{i=1}^m S_i$ of a finite number of sets S_i with diameter* $diam(S_i) \le \delta\Big\}$.

$\alpha(S)$ is said to be the Kuratowski *measure of noncompactness and is called the noncompactness measure for short. For details and properties of the noncompactness measure see [20].*

The following lemmas are of great importance in the proof of our main results.

Lemma 1 *[23] Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let* A *be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}\mathrm{A}(V) \;\; or \;\; V = \mathrm{A}(V)\cup\{0\} \Rightarrow \alpha(V) = 0.$$

holds for every subset V of D, then A *has a fixed point.*

Lemma 2 *[23] Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J\times J$, and a function $f: J\times E \to E$ which satisfies the $Carath\acute{e}odory$ conditions, and there exists $p \in L^1(J,\mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset E$ one has*

$$\lim_{k\to 0^+}\alpha\Big(f(J_{t,k}\times B)\Big) \le p(t)\alpha(B); \;\; where \;\; J_{t,k} = [t-k,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\Big(\{\int_J G(s,t)f(s,y(s))ds : y \in V\}\Big) \le \int_J \|G(t,s)\|p(s)\alpha(V(s))ds.$$

Lemma 3 *[26] Let $\alpha > 0$, then differential equation*

$${}^cD^{\alpha}_{0+}u(t) = 0,$$

has solutions

$$u(t) = C_0 + C_1t + C_2t^2 + \cdots + C_{n-1}t^{n-1}\;, \;\; C_i \in \mathbb{R}, i = 0,1,2,\cdots,n.\; n = [\alpha]+1.$$

Lemma 4 *[26] Assume that $h \in C[0,1]\cap L(0,1)$ with a derivative of order q that belongs to $C[0,1]\cap L(0,1)$. Then*

$$I^q_{0+}\,{}^cD^q_{0+}h(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, where n is the smallest integer greater than or equal to q.

Lemma 5 *Let $h(t) \in C(J,\mathbb{R})$ and $0 < q \le 1$, then the unique solution of*

$${}^cD^q_{0+}u(t) = h(t), \;\; 0 < t < T, \tag{6}$$

$$u(0) + \beta\int_0^T u(s)ds = u(T), \;\; 0 < \beta \in \mathbb{R}. \tag{7}$$

is given by

$$u(t) = \int_0^T G(t,s)h(s)ds, \tag{8}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{qT(t-s)^{q-1}-(T-s)^q}{T\Gamma(q+1)} + \frac{(T-s)^{q-1}}{\beta T\Gamma(q)}, & if \;\; 0 \le s \le t \le T, \\ \frac{-(T-s)^q}{T\Gamma(q+1)} + \frac{(T-s)^{q-1}}{\beta T\Gamma(q)}, & if \;\; 0 \le t \le s \le T, \end{cases} \tag{9}$$

Proof. By the Lemma 4, we can reduce the equation of problem (6) to an equivalent integral equation

$$u(t) = I_{0+}^q h(t) - c_0 = -c_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds. \tag{10}$$

for some constants $c_0 \in \mathbb{R}$.

Applying the boundary conditions (7) , we have

$$c_0 = \frac{1}{T} \int_0^T \left[\frac{(T-s)^q}{\Gamma(q+1)} - \frac{(T-s)^{q-1}}{\beta\Gamma(q)} \right] h(s) ds.$$

Therefore, the unique solution of problem (6)-(7) is

$$\begin{aligned} u(t) &= -c_0 + \tfrac{1}{\Gamma(q)} \textstyle\int_0^t (t-s)^{q-1} h(s) ds \\ &= -\tfrac{1}{T} \textstyle\int_0^T \left[\tfrac{(T-s)^q}{\Gamma(q+1)} - \tfrac{(T-s)^{q-1}}{\beta\Gamma(q)} \right] h(s) ds + \tfrac{1}{\Gamma(q)} \textstyle\int_0^t (t-s)^{q-1} h(s) ds \\ &= \textstyle\int_0^t \left[\tfrac{qT(t-s)^{q-1} - (T-s)^q}{T\Gamma(q+1)} + \tfrac{(T-s)^{q-1}}{\beta T\Gamma(q)} \right] h(s) ds + \textstyle\int_t^T \left[\tfrac{-(T-s)^q}{T\Gamma(q+1)} + \tfrac{(T-s)^{q-1}}{\beta T\Gamma(q)} \right] h(s) ds \\ &= \textstyle\int_0^T G(t,s) h(s) ds. \end{aligned}$$

which completes the proof. ■

Remark 1 *From the expression of $G(t,s)$, it is obvious that $G(t,s)$ is continuous on $J \times J$. Denote by* $G^* = \sup \Big\{ \|G(t,s)\| : (t,s) \in J \times J \Big\}$.

3 Main results

In this section, we present and prove our main results. The first result is based on concept of measures of noncompactness and $M\ddot{o}nch's$ fixed point theorem.

Theorem 1 *Assume that:*

(H1) The function $f : J \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the $Carath\acute{e}odory$ conditions.

(H2) There exists $p_f \in L^\infty(J, \mathbb{R}_+)$, such that
$\|f(t,u)\| \le p_f(t) \cdot \|u\|$, for a.e. $t \in J$ and each $u \in \mathbb{R}$.

(H3) For almost each $t \in J$ and each bounded set $B \subset \mathbb{R}$. one has

$$\lim_{k \to 0^+} \alpha\Big(f(J_{t,k} \times B) \Big) \le p_f(t) \cdot \alpha(B); \;\; where \;\; J_{t,k} = [t-k, t] \cap J.$$

If

$$G^* \int_0^T p_f(s) ds < 1. \tag{11}$$

then the problem (1)-(2) *has at least one solution on J.*

Proof. Let the operator $\mathcal{A} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by the formula

$$(\mathcal{A}u)(t) := \int_0^T G(t,s) f(s, u(s)) ds. \tag{12}$$

Where $G(t,s)$ is given by (9). It is well known the fixed points of the operator $\mathcal{A}$ are solutions of the problem (1)-(2). Let $R > 0$ and consider the set $D_R = \Big\{ y \in C(J, \mathbb{R}) : \|y\|_\infty \le R \Big\}$. It was patently obvious that D_R is closed, bounded and convex. We will show that $\mathcal{A}$ satisfies the assumptions of Lemma 1. The proof will be given in three steps.

Step 1: We will show that the operator $\mathcal{A} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous.

For any $u_n, u \in C(J, \mathbb{R}), n = 1, 2, 3, \cdots$ with $\lim_{n \to \infty} \|u_n - u\| = 0$, we get

$$\lim_{n \to \infty} u_n(t) = u(t), \; t \in J.$$

Thus, by the condition (H1), we have

$$\lim_{n\to\infty} f(t,u_n(t)) = f(t,u(t)),\ a.e. \quad t\in J.$$

So, for $a.e.\ t \in J$, we can conclude that

$$\|f(t,u_n(t)) - f(t,u(t))\| \to 0, \quad \text{as } n\to\infty. \tag{13}$$

On the other hand,

$$\begin{aligned}\|(\mathcal{A}u_n)(t) - (\mathcal{A}u)(t)\| &= \|\textstyle\int_0^T G(t,s)\cdot[f(s,u_n(s)) - f(s,u(s))]ds\| \\ &\le \textstyle\int_0^T \|G(t,s)\|\cdot\|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\le G^*\cdot \textstyle\int_0^T \|f(s,u_n(s)) - f(s,u(s))\|ds.\end{aligned}$$

Hence, by (13) and the Lebesque dominated convergence theorem, we have

$$\|\mathcal{A}(u_n) - \mathcal{A}(u)\|_\infty \to 0 \quad \text{as } n\to\infty.$$

This means that the operator $\mathcal{A}$ is continuous.

Step 2: We will show that the operator $\mathcal{A}$ maps bounded sets D_R into itself.

For each $u\in D_R$, by the condition (H2) and (11), for each $t\in J$, we have

$$\begin{aligned}\|(\mathcal{A}u)(t)\| &\le \textstyle\int_0^T \|G(t,s)\|\cdot\|f(s,u(s))\|ds \\ &\le \textstyle\int_0^T \|G(t,s)\|\cdot p_f(s)\cdot\|u\|ds \\ &\le RG^* \textstyle\int_0^T p_f(s)ds \\ &\le R.\end{aligned}$$

This means that $\|\mathcal{A}(u)(t)\|\le R$, that is, the operator $\mathcal{A}$ maps D_R into itself.

Step 3: We will show that $\mathcal{A}(D_R)$ is bounded and equicontinuous.

By Step 2 we have $\mathcal{A}(D_R) = \{\mathcal{A}(u) : u\in D_R\}\subset D_R$. Thus, for each $u\in D_R$ we have $\|\mathcal{A}(u)\|_\infty \le R$ which means that $\mathcal{A}(D_R)$ is bounded.

Now, we shall show that the equicontinuity of $\mathcal{A}(D_R)$. Let $\tau_1,\tau_2\in J,\ \tau_1<\tau_2$ and $\forall u\in D_R$, then we have the estimate

$$\begin{aligned}\|\mathcal{A}(u)(\tau_2) - \mathcal{A}(u)(\tau_1)\| &= \|\textstyle\int_0^T [G(\tau_2,s) - G(\tau_1,s)]\cdot f(s,u(s))ds\| \\ &\le \textstyle\int_0^T \|G(\tau_2,s) - G(\tau_1,s)\|\cdot p_f(s)\cdot\|u\|ds \\ &\le R\|p_f\|_\infty \textstyle\int_0^T \|G(\tau_2,s) - G(\tau_1,s)\|ds.\end{aligned}$$

Since $G(t,s)$ is continuous in $J\times J$, as $\tau_1\to\tau_2$, we can conclude that the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V\subset \overline{conv}\Big(N(V)\cup\{0\}\Big)$. V is bounded and equicontinuous and therefore the function $v\to v(t) = \alpha(V(t))$ is continuous on J. By the condition (H3), Lemma 2 and the properties of the noncompactness measure, we have, for each $t\in J$,

$$\begin{aligned}v(t) &\le \alpha\Big((\mathcal{A}V)(t)\cup\{0\}\Big) \\ &\le \alpha\Big((\mathcal{A}V)(t)\Big) \\ &\le \textstyle\int_0^T \|G(t,s)\|\cdot p_f(s)\cdot\alpha(V(s))ds \\ &\le \|v\|_\infty\cdot G^* \textstyle\int_0^T p_f(s)ds.\end{aligned}$$

which gives

$$\|v\|_\infty \le \|v\|_\infty\cdot G^*\int_0^T p_f(s)ds.$$

This means that

$$\|v\|_\infty\cdot\left[1 - G^*\int_0^T p_f(s)ds\right]\le 0.$$

By (11) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in $\mathbb{R}$. By the $Ascoli - Arzel\grave{a}$ theorem, V is relatively compact in D_R. In view of lemma 1, we deduce that $\mathcal{A}$ has a fixed point which is obviously a solution of the problem (1)-(2). This completes the proof. ■

The second result is based on Schaefer's fixed point theorem.

Theorem 2 *Assume that:*

(H4) The function $f : J \times \mathbb{R} \longmapsto \mathbb{R}$ is continuous.

(H5) There exists a constant $M > 0$ such that

$$\|f(t, u)\| \le M \quad for\ each\ t \in J\ and\ all\ u \in \mathbb{R}.$$

then the problem (1)-(2) *has at least one solution on J.*

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathcal{A}$ defined by (12) has a fixed point. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is continuous.

Let $u_n, u \in C(J, \mathbb{R}), n = 1, 2, 3, \cdots$ with $\lim_{n\to\infty} \|u_n - u\| = 0$. Then for each $t \in J$ we get

$$\begin{aligned}\|(\mathcal{A}u_n)(t) - (\mathcal{A}u)(t)\| &= \| \textstyle\int_0^T G(t,s) \cdot [f(s, u_n(s)) - f(s, u(s))]ds\| \\ &\le \textstyle\int_0^T \|G(t,s)\| \cdot \|f(s, u_n(s)) - f(s, u(s))\|ds \\ &\le G^* \cdot \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\|_\infty.\end{aligned}$$

Since f is a continuous function, we have

$$\|\mathcal{A}(u_n) - \mathcal{A}(u)\|_\infty \le G^* \cdot \|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\|_\infty \to 0 \quad \text{as } n \to \infty.$$

This means that the operator $\mathcal{A}$ is continuous.

Step 2: We will show that the operator $\mathcal{A}$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant $\xi > 0$ such that for each $u \in B_\eta = \Big\{u \in C(J, \mathbb{R}) : \|u\|_\infty \le \eta\Big\}$, we have $\|\mathcal{A}(u)\|_\infty \le \xi$.

By the condition (H5), for each $t \in J$, we have

$$\begin{aligned}\|(\mathcal{A}u)(t)\| &\le \textstyle\int_0^T \|G(t,s)\| \cdot \|f(s, u(s))\|ds \\ &\le \textstyle\int_0^T \|G(t,s)\| \cdot M ds \\ &\le G^* \cdot TM.\end{aligned}$$

Thus

$$\|\mathcal{A}(u)\|_\infty \le G^* \cdot TM := \xi.$$

This means that $\|\mathcal{A}(u)\|_\infty \le \xi$, that is, the operator $\mathcal{A}$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Step 3: $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $\tau_1, \tau_2 \in J,\ \tau_1 < \tau_2$, B_η be a bounded set of $C(J, \mathbb{R})$ as in Step 2, and let $u \in B_\eta$. Then we have the estimate

$$\begin{aligned}\|\mathcal{A}(u)(\tau_2) - \mathcal{A}(u)(\tau_1)\| &\le \| \textstyle\int_0^T [G(\tau_2, s) - G(\tau_1, s)] \cdot f(s, u(s))ds\| \\ &\le M \textstyle\int_0^T \|G(\tau_2, s) - G(\tau_1, s)\|ds.\end{aligned}$$

Since $G(t, s)$ is continuous in $J \times J$, as $\tau_1 \to \tau_2$, we can conclude that the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the $Ascoli - Arzel\grave{a}$ theorem, we can conclude that $\mathcal{A} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and completely continuous.

Step 4: $\mathcal{A}$ priori bounds.

Now it remains to show that the set

$$\rho = \Big\{u \in C(J, \mathbb{R}) : u = \lambda\mathcal{A}(u)\ for\ some\ 0 < \lambda < 1\Big\}.$$

is bounded.

Let $u \in \rho$, then $u = \lambda\mathcal{A}(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda(\mathcal{A}u)(t) = \lambda \int_0^T G(t,s) f(s, u(s))ds.$$

This implies by (H3) that for each $t \in J$ we have

$$\begin{aligned} \|(\mathcal{A}u)(t)\| &\le \int_0^T \|G(t,s)\| \cdot \|f(s,u(s))\| ds \\ &\le \int_0^T \|G(t,s)\| \cdot M ds \\ &\le G^* \cdot TM. \end{aligned}$$

Thus for every $t \in J$, we have

$$\|\mathcal{A}(u)\|_\infty \le G^* \cdot TM := R.$$

This shows that the set ρ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that $\mathcal{A}$ has a fixed point which is obviously a solution of the problem (1)-(2). This completes the proof. ■

4 An example

In this section we give an example to illustrate the usefulness of theorem 1.

Example 1 *Let us consider the following fractional boundary value problem*

$$^cD^q u(t) = \frac{2}{19+e^t} u(t), \quad t \in J := [0,1],\ 0 < q \le 1, \tag{14}$$

$$u(0) + \int_0^1 u(s)ds = u(1). \tag{15}$$

Here $f(t,u(s)) = \frac{2}{19+e^t} u(s)$, $(t,u) \in J \times \mathbb{R}$, $\beta = 1$, $T = 1$. *Clearly condition* $(H1)$, $(H2)$ *hold with* $p_f(t) = \frac{2}{19+e^t}$. *we have*

$$G(t,s) = \begin{cases} \frac{q(t-s)^{q-1} - (1-s)^q}{\Gamma(q+1)} + \frac{(1-s)^{q-1}}{\Gamma(q)}, & \text{if } 0 \le s \le t \le 1, \\ \frac{-(1-s)^q}{\Gamma(q+1)} + \frac{(1-s)^{q-1}}{\Gamma(q)}, & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{16}$$

From (16)*, we have*

$$\begin{aligned} \int_0^1 G(t,s)ds &= \int_0^t \left[\frac{q(t-s)^{q-1} - (1-s)^q}{\Gamma(q+1)} + \frac{(1-s)^{q-1}}{\Gamma(q)}\right] ds + \int_t^1 \left[\frac{-(1-s)^q}{\Gamma(q+1)} + \frac{(1-s)^{q-1}}{\Gamma(q)}\right] ds \\ &= -\frac{t^q}{\Gamma(q+1)} - \frac{q[1-(1-t)^{q+1}]}{\Gamma(q+2)} + \frac{[1-(1-t)^q]}{\Gamma(q+1)} - \frac{q(1-t)^{q+1}}{\Gamma(q+2)} + \frac{(1-t)^q}{\Gamma(q+1)}. \end{aligned}$$

A simple computation gives

$$G^* < \frac{4}{\Gamma(q+1)} + \frac{3}{\Gamma(q+2)}. \tag{17}$$

We shall check that condition (11) *is satisfied. Indeed*

$$G^* \int_0^1 p_f(s)ds < 1 \Longleftrightarrow \frac{4}{\Gamma(q+1)} + \frac{3}{\Gamma(q+2)} < 10. \tag{18}$$

which is satisfied for some $q \in (0,1]$. *Then by Theorem 1 the problem* (14)-(15) *has at least one solution on* $[0,1]$ *for values of q satisfying (18).*

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