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Short communication

Stability of the B-spline basis via knot insertion

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Abstract

We derive the stability inequality $\|C\| \leqslant \gamma \| \sum_i c_i b_i \|$ for the B-splines b_i from the formula for knot insertion. The key observation is that knot removal increases the norm of the B-spline coefficients $C = \{c_i\}_{i \in \mathbb{Z}}$ at most by a constant factor, which is independent of the knot sequence. As a consequence, stability for splines follows from the stability of the Bernstein basis. © 2000 Elsevier Science B.V. All rights reserved.

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Let U be an increasing biinfinite knot sequence with $u_i < u_{i+n+1}$, and denote by $b_{i,U}$ the corresponding B-splines of degree n, normalized so that they form a partition of unity, i.e.,

$$\sum_{i=-\infty}^{\infty} b_{i,U}(x) = 1, \quad x \in \mathbb{R}. \tag{1}$$

A remarkable property of the B-spline basis is its uniform stability. There exists a constant γ, which depends only on the degree n, so that

$$\|p\| \leqslant \|C\| \leqslant \gamma \|p\|, \quad p = \sum_i c_i b_{i,U}, \tag{2}$$

where $\|\ \|$ denotes the maximum norm of functions defined on $\mathbb{R}$ or biinfinite sequences. Of course, the left inequality is an immediate consequence of (1) because of the positivity of the B-splines. The right inequality is more subtle, since it is by no means obvious that the estimate does not depend on the knot sequence U. The classical proof of de Boor (1968, 1976) is based on the construction of dual functionals. It involves divided differences and

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some elementary approximation theory. We give in this paper a different derivation, which uses only the formula for knot insertion.

We first dispose of a simple special case. If each knot of U has multiplicity $n+1$, i.e., if

$$\cdots = u_{-1} < u_0 = \cdots = u_n < u_{n+1} = \cdots,$$

the B-splines coincide with the Bernstein polynomials. For example, with $r = u_0$ and $s = u_{n+1}$,

$$b_i(x) = \binom{n}{i} \left(\frac{s-x}{s-r}\right)^{n-i} \left(\frac{x-r}{s-r}\right)^i, \quad 0 \leqslant i \leqslant n,$$

for $x \in [r, s)$, and $b_i(x) = 0$ outside this interval. Hence, the B-spline basis is decoupled, and (2) holds with γ equal to the stability constant γ_b of the Bernstein basis. By transforming each knot interval to $[0, 1)$ with a linear change of variables, we see that γ_b does not depend on the length of the intervals.

To prove (2) in general, we transform the spline p to Bernstein form by a finite number of knot sequence refinements. We will show that for each refinement

$$U \to V, \qquad p = \sum_i c_i b_{i,U} \to p = \sum_i d_i b_{i,V}$$

the norm of the coefficients decreases at most by a factor $1/\gamma_r$, i.e.,

$$\|C\| \leqslant \gamma_r \|D\|. \tag{3}$$

Hence, if k refinements are used, the norm of the coefficients of the spline p is less than γ_r^k times the norm of the coefficients of its Bernstein form. By the above remark, the Bernstein coefficients are bounded by $\gamma_b \|p\|$, so that (2) holds with

$$\gamma = \gamma_r^k \gamma_b.$$

We will now describe the argument in detail.

First, we recall the famous formula for knot insertion discovered by Böhm (1980).

Theorem 1. *Let V be a refinement of U, obtained by adding the knot $t = v_{\ell+1} \in [u_\ell, u_{\ell+1})$, and denote by c_i, d_i the coefficients of a spline corresponding to U and V respectively. Then, the coefficients with index $\leqslant \ell - n$ or $\geqslant \ell$ do not change,*

$$\ldots,\ d_{\ell-n-1} = c_{\ell-n-1}, \quad d_{\ell-n} = c_{\ell-n}, \quad d_{\ell+1} = c_\ell, \quad d_{\ell+2} = c_{\ell+1}, \ldots,$$

and the coefficients

$$d_i = \frac{t-u_i}{u_{i+n}-u_i} c_i + \frac{u_{i+n}-t}{u_{i+n}-u_i} c_{i-1}, \quad \ell - n < i \leqslant \ell, \tag{4}$$

replace $c_{\ell-n+1}, \ldots, c_{\ell-1}$.

We stated this theorem in a slightly simplified form. If $t = u_\ell$, fewer coefficients change. More precisely, if

$$u_{\ell-m} < u_{\ell-m+1} = \cdots = u_\ell = t < u_{\ell+1},$$

then the first fraction in (4) vanishes for $i = \ell - m + 1, \ldots, \ell$, so that $d_i = c_{i-1}$ for those indices. Although it is this case which we will use, we do not pay attention to multiplicities for the sake of simplicity. The formulas are valid in any case. By ignoring multiplicities, we consider more coefficients than necessary, which does not lead to optimal constants in the estimates.

The coefficients d_i, corresponding to the refined knot sequence, are convex combinations of the coefficients c_j. Therefore, $\|D\| \leqslant \|C\|$. For our proof, we need an inequality in the opposite direction. Estimates of this type have been analyzed in detail by Lyche and Mørken (1993). We include here a discussion of the simplest case for convenience of the reader.

Lemma 1. *The coefficients in Theorem* 1 *satisfy*

$$\max_{\ell-n<i<\ell} |c_i| \leqslant \gamma_r \max_{\ell-n\leqslant i\leqslant\ell} |d_i|,$$

where the constant γ_r depends only on n.

To prove this lemma, we invert the formula (4). This can be done in two ways, yielding the recursions

$$c_i = \frac{u_{i+n} - u_i}{t - u_i} d_i - \frac{u_{i+n} - t}{t - u_i} c_{i-1} \tag{5}$$

and

$$c_{i-1} = \frac{u_{i+n} - u_i}{u_{i+n} - t} d_i - \frac{t - u_i}{u_{i+n} - t} c_i. \tag{6}$$

Using (5), we can compute the coefficients c_i starting from $c_{\ell-n} = d_{\ell-n}$, and, using (6), starting from $c_\ell = d_{\ell+1}$. Which possibility we choose depends on the size of the fractions in the recursions. Let j be the largest index i, for which

$$t - u_i > u_{i+n} - t,$$

and, as a consequence,

$$|u_{i+n} - u_i| \leqslant |u_{i+n} - t| + |t - u_i| \leqslant 2|t - u_i|.$$

Then, for $\ell - n < i \leqslant j$, the absolute value of the fractions multiplying d_i and c_{i-1} in recursion (5) is bounded by 2 and 1 respectively. Hence, with $\delta = \max_{\ell-n\leqslant i\leqslant\ell} |d_i|$, it follows that

$$\begin{aligned}
|c_{\ell-n+1}| &\leqslant 2|d_{\ell-n+1}| + |d_{\ell-n}| \leqslant 3\delta, \\
|c_{\ell-n+2}| &\leqslant 2\delta + 3\delta = 5\delta, \\
&\ \ldots \\
|c_j| &\leqslant (2(j - \ell + n) + 1)\delta.
\end{aligned}$$

Similarly, for $j < i \leqslant \ell$, the recursion (6) yields a bound for $c_j, \ldots, c_{\ell-1}$. Both bounds are independent of the knot sequence, which proves the lemma. In the worst case, only one of the recursions is used, so that $\gamma_r \leqslant 2n - 1$.

Inserting a knot $t \in [u_\ell, u_{\ell+1})$ only involves the coefficients $c_{\ell-n}, \dots, c_\ell$ and the knots $u_{\ell-n+1}, \dots, u_{\ell+n}$. Hence, if we insert simultaneously knots in any of the intervals

$$[\ell + i(n+1), \ell + i(n+1) + 1), \quad i \in \mathbb{Z},$$

the computations are completely independent. Therefore, the lemma implies (3) for such knot sequence refinements.

To convert to Bernstein form, we need to increase the multiplicity of each knot to $n+1$. This requires at most n refinements of the above type for each knot subsequence

$$u_{\ell+i(n+1)}, \quad i \in \mathbb{Z}.$$

Altogether we need at most $k = n(n+1)$ refinements, which completes the proof of the stability inequality.

The constant γ obtained by our approach grows faster than $(2n)^{n(n+1)}$, which is a huge overestimate—the price paid for a short argument. However, with this small note we just want to establish a simple connection between knot insertion and stability. The precise asymptotic behavior of the condition number was recently determined by Scherer and Shadrin (1999), who showed that

$$\gamma \leqslant \text{const}\, n 2^n.$$

Except for the linear factor n, this remarkable estimate is optimal, since the condition number of the Bernstein basis satisfies $\gamma_b \sim 2^n$ (Lyche, 1978; Lyche and Scherer, t.a.).

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