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Implicit Curvilinear Interpolation

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Abstract

The shifting operator is applied to generate implicit interpolating equations for curves defined by three and four equidistant data. The equations trade the abscissa of the interpolated point for another curvilinear measurement. The new method yields expressions for estimating the extreme and mirror points on curves.

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1. Introduction

The interpolation of equidistant, curvilinear data is familiar in science and engineering. The data are represented by a preassigned expression such as a straight line or a parabola. The abscissas of intermediate points are substituted into the assigned expression to estimate responses between adjacent measurements. The explicit method uses two kinds of numbers: those for ordinates and those for abscissas. Another approach to curvilinear interpolation does not depend on a preassigned form of the interpolating instrument or on numerical values of abscissas. It depends on relationships among measurements. These relationships are established by means of the shifting operator, exp(hx)F(x)=F(x+h). The new method can be described as implicit curvilinear interpolation.

2. Three equidistant curvilinear points: first exponential representation

Let three equidistant, monotonic curvilinear data be denoted B, C, and D. The implicit method uses another curvilinear measurement, A, that lies on one side of the center of the curve. Suppose A lies to the left of C at a known fraction of the distance between B and C. Then the interpolated response E lies to the right of C between C and D at the same distance from C. If the abscissa of C is zero, and if A lies at a distance (–x) from C, then the interpolated value E lies on the curve at a distance (+x) from C.

For example, let B, C, and D be 3, 4, and 5, respectively. If the fourth measurement is A=3.3 then the implicitly interpolated value E is 4.7. The interpolating instrument is a particular relationship connecting the numerical values of A, B, C, and D to interpolated estimate E. One such relationship is Eq. (1) [1].

$$E^2(C^2 - DB) + AE(D^2 + B^2 - 2C^2) + A^2(C^2 - DB) - (C(D - B))^2 = 0 \qquad (1)$$

Eq. (1) is exact on linear numbers and on linear numbers as exponents in simple forms like 2^x. It is also exact on the circular and hyperbolic sines and cosines of linear arguments. Eq. (1) has the disadvantage that the interpolated value of E appears as the root of a quadratic equation. The proper choice for E must be decided in the context of the problem. Another disadvantage of Eq. (1) is its sensitivity to translation of the data. These disadvantages limit the application of Eq. (1) and make it inconvenient to apply. However, Eq. (1) illustrates the methodology of implicit curvilinear interpolation.

The derivative dE/dA can be obtained from Eq. (1) by implicit differentiation. The derivative can be use to obtain expressions for the extreme and mirror values on three-point curves when the curve has an interior extremum [1]. Typically, the extreme value lies on one side of curve midpoint C. The mirror value lies on the other side of midpoint C. The mirror point is the reflection of the extreme point through the midpoint of the curve. Extreme and mirror points are illustrated in Refs. [1,2].

3. Three equidistant curvilinear points: parabolic representation

Let a term T be added to every letter in Eq. (1). Now let T increase without limit. This process yields a new relationship among three curvilinear points B, C, D as illustrated by Eq. (2). That equation is used to obtain new formulas for estimating extreme and mirror points on curves [1,2].

$$(2C - B - D)(E - A)^2 + (E + A - 2C)(B - D)^2 = 0 \quad (2)$$

Eq. (2) is exact on linear numbers and their squares. It is applied to three equidistant curvilinear data B, C, D in the same manner as Eq. (1). Eq. (2) has the advantage that it is insensitive to translation of the data. It has the disadvantage that its estimate of E appears as the root of a quadratic expression. The proper choice of E must be selected in the context of the problem. Expressions for the extreme and mirror points on curve BCD, as rendered by Eq. (2), are illustrated in Ref. [2]. The distance between the extreme point and the midpoint of the curve is estimated by Eq. (26) in Ref. [1]. The cited Eq. (26) estimates the distance parameter that applies to both Eqs. (1) and (2).

4. Three equidistant curvilinear points: second exponential representation

Eq. (3) is an explicit expression for interpolated number E given three equidistant, monotonic, curvilinear data B, C, D and an arbitrary curvilinear point A.

$$E = [(D - A + B)C^2 - BD(2C - A)] / [A(B + D - 2C) - BD + C^2] \quad (3)$$

Eq. (3) is exact on linear numbers (denoted x) and on simple expressions like 2^x, 100/x, and tan(x) where ($-\Pi/2 < x < \Pi/2$). For example, let B=4, C=8, D=16. If A=2 then E=32. Eq. (3) is a simple, direct, easy-to-use, and versatile expression for the implicit interpolation of monotonic, equidistant, three-point curves. It does not yield expressions for the extreme and mirror points because the functions on which it is exact do not have extreme and mirror points. Eq. (3) is developed from Eq. (29) in Ref. [3].

5. Three equidistant curvilinear points: power method

Eq. (4) estimates curvilinear point E based on numbers for B, D, and A. It differs from previous methods in that it requires an estimate of the degree (N) of the curve passing through B, D, and A instead of a datum at center point C. The amount information required by Eq. (4) is the same as in the preceding methods but the applications of Eq. (4) are more restricted. It is a polynomial-type method.

$$E = (B^{(1/N)} + D^{(1/N)} - A^{(1/N)})^N \quad (4)$$

For example, let N=2, B=16, and D=36. Assign three successive values of A as 16, 25, and 36 in that order. Then E is 36, 25, and 16, respectively. Now assume the same numbers for B and D but let N=3. The three cited values for A yield 36, 24.33, and 16 for interpolated point E, respectively. If N=1/2, the same assignments for B, D, and A yield E as 36, 30.45, and 16, respectively. Given B and D, the assignment A=25 yields different values of E for different values of N. Sometimes N can be estimated by measuring E, using that value for A in Eq. (4), and subtracting E from the substituted equation. For example, let B=16, D=36, E=27. The degree of the interpolating curve is N~0.658.

Implicit interpolating equations are used primarily with positive, monotonic real numbers. They do not apply in all cases. When the equations are successful, one end-point datum of the curve reproduces the other end-point datum of the curve. Otherwise, they are not useful. When an interpolated number appears as the root of an equation, the same root must apply throughout the domain of the data: root-switching is not permitted. Eqs. (3) and (4) are not recommended for data exhibiting an interior extremum. In most cases, the easiest method to apply is Eq. (3). It has fewer restrictions and its unambiguous results are easier to interpret.

6. Four equidistant curvilinear points: exponential method

Let a curve be defined by four equidistant points B, C, D, E. If J is a curvilinear measurement to the left of the curve midpoint then K is the interpolated value an equal distance to the right of the midpoint. The relationship of the four curvilinear data is given by Eq. (5). The extreme ordinate on curve BCDE, if there is one, is denoted EXT. It is estimated by Eq. (6) and it lies on one side of the curve at a certain distance from the midpoint. The mirror point, denoted MIR, is estimated by Eq. (7). The mirror point lies on the other side of midpoint of curve BCDE. The extreme and mirror points are each at a distance (h) from the curve midpoint. The distance parameter is estimated by Eq. (11).

$$J^2(CD - BE) + JK(E^2 - C^2 - D^2 + B^2) + K^2(CD - BE) - CD(E^2 + B^2) + BE(C^2 + D^2) = 0 \quad (5)$$

$$EXT = [(+/-)2][SQRT(PR / (Q^2 - 4P^2))] \quad (6)$$

$$MIR = [(+/-)Q][SQRT(R / (P(Q^2 - 4P^2)))] \quad (7)$$

$$P = CD - BE \quad (8)$$

$$Q = E^2 + B^2 - C^2 - D^2 \quad (9)$$

$$R = BE(C^2 + D^2) - CD(E^2 + B^2) \quad (10)$$

$$h \sim 6(EXT - MIR) / (E + 9D - 9C - B) \quad (11)$$

Eq. (5) is exact on linear numbers and those numbers as exponents in simple expressions like 2^x. It is therefore considered an exponential-type relationship. It is also exact on simple functions like $(Y)\sin(x^o) + (Z)\cos(x^o)$ where Y and Z are real numbers.

When using the circular functions, the range of (x) is limited to avoid confusion caused by an extremum in the data. Eq. (5) is exact on (Y)sinh(x) + (Z)cosh(x) and it is sensitive to translation of the data. The values of EXT and MIR can be positive or negative as determined by the context of the problem. This ambiguity can be confusing when estimating (h), the distance parameter.

For example, let B, C, D, and E be the sines of 50^o, 60^o, 70^o, and 80^o, respectively. The extreme value on the curve is 1.000 corresponding to sin(90^o), while the mirror value is 0.6428 corresponding to sin(40^o). The distance parameter is estimated as 2.43 whereas the true value is 2.50. If B, C, D, and E are cosh(2/10), cosh(3/10), cosh(4/10), and cosh(5/10), respectively, the values of EXT and MIR are estimated as 1.000 and 1.255, respectively, and (h) is estimated as about 3.57 whereas the true value of (h) is 3.50.

7. Four equidistant curvilinear points: polynomial method

An implicit representation of curve BCDE is possible by means of the relationship in Eq. (12). It is exact on linear numbers and their squares and it is invariant under translation of the data. Eq. (12) is a polynomial-type estimator so it has limited accuracy on exponential-type functions like the hyperbolic sines and cosines. It yields expressions for the extreme and mirror points as in Eqs. (13) and (14), respectively.

$$(C + D - E - B)(P - Q)^2 - (P + Q)(B + C - D - E)(C + E - B - D) + (B + E)(C^2 + D^2) \\ - (E^2 + B^2)(D + C) + 2EB(D + C) - 2CD(B + E) = 0 \quad (12)$$

$$EXT = [(4)(K)(QC) - (LC)^2] / [8(LC)(QC)] \quad (13)$$

$$MIR = [3(LC)^2 + 4(K)(QC)] / [8(LC)(QC)] \quad (14)$$

$$QC = C + D - E - B \quad (15)$$

$$LC = (B + C - D - E)(C + E - B - D) \quad (16)$$

$$K = (B + E)(C^2 + D^2) - (E^2 + B^2)(D + C) + 2EB(D + C) - 2CD(B + E) \quad (17)$$

Eq. (12) is applied in the manner of Eq. (5). For example, let the data be 2, 3, 4, and 5 as B, C, D, and E, respectively. If P=1, then Q=6. If the data are the squares of the cited integers, and if P=1, then Q=36. Eqs. (13) and (14) do not apply to linear data because the straight line has no extreme or mirror points. If the foregoing, cited integers are squared, the value of EXT is zero and MIR is 49. The distance parameter (h) is 3.50 as estimated by Eq. (11). That is the correct value.

10. Discussion

As the number of equidistant curvilinear measurements increases, the applicability of the implicit interpolating equations decreases. Generally, four-point equations fail more often than three-point equations. The equations for equidistant, four-point curves are not unique [4]. Equations for more equidistant data can be developed.

Table 1 illustrates the application of the three-point implicit interpolating equations to equidistant data B, C, D denoted M(3), M(4), and M(5), respectively. If the operation applied to the data is squaring (M^2), the three initial data are 9, 16, and 25, respectively. The variable datum is M(3.5) so the interpolated number is M(4.5). The entries in Table 1 illustrate that Eqs. (1)-(3) usually render recognizable estimates of the true values.

The entries for Eq. (4) are obtained by assigning the exponent N as 2.0. This choice is arbitrary and it is not the best selection in every case. For example, exponential-type data like 2^M are not accurately approximated by a quadratic function. It is ordinarily difficult to immediately perceive the proper value of N for three curvilinear numbers. That is a disadvantage of Eq. (4). The examples illustrate the principle of implicit curvilinear interpolation. Equations for the implicit, curvilinear interpolation of experimental data seldom, if ever, appear in the literature of applied mathematics.

Table 1. Interpolation of three-point curves by four implicit equations. The equidistant data are B=M(3), C=M(4), and D=M(5). The value to be interpolated is halfway between the second and third points, M(4.5), so the fourth datum is halfway between first and second points, M(3.5). The exponent N in Eq. (4) is arbitrarily assigned as N=2.

Function, M	Eq. (1)	Eq. (2)	Eq .(3)	Eq. (4), N=2	True value
M	4.500	4.500	4.500	4.399	4.500
M^2	20.26	20.25	20.26	20.25	20.25
M^3	91.41	90.89	91.25	96.60	91.13
2^M	22.63	22.39	22.63	26.23	22.63
100/M	22.21	22.19	22.22	24.01	22.22
$\sin(10M^o)$	0.7071	0.7071	0.7071	0.6806	0.7071
$\cos(10M^o)$	0.7071	0.7071	0.7070	0.6844	0.7071
$\tan(10M^o)$	1.000	0.9996	1.000	1.030	1.000
ln(M+1)	1.705	1.705	1.705	1.663	1.705
cosh(M/5)	1.433	1.433	1.433	1.466	1.433

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