ON THE ISOPERIMETRIC PROBLEM WITH RESPECT TO A MIXED EUCLIDEAN-GAUSSIAN DENSITY

N. FUSCO, F. MAGGI, AND A. PRATELLI

ABSTRACT. The isoperimetric problem with respect to the product-type density $e^{-\frac{|x|^2}{2}}\,dx\,dy$ on the Euclidean space $\mathbb{R}^h \times \mathbb{R}^k$ is studied, with particular emphasis on the case $k = 1$. A conjecture about the minimality of large cylinders in the case $k > 1$ is also formulated.

1. Introduction

The isoperimetric problem in a manifold with density has received an increasing attention in recent times. In the case the ambient manifold is the Euclidean space $\mathbb{R}^n$, $n \geq 1$, this problem amounts to introduce notions of volume and perimeter weighted with respect to a positive density e^v, $v : \mathbb{R}^n \to \mathbb{R}$, and to formulate the variational problems

$$\inf\left\{\int_{\partial E} e^v\, d\mathcal{H}^{n-1} : \int_E e^v = m\right\}, \qquad m > 0\,. \tag{1.1}$$

The explicit characterization of *isoperimetric sets* (i.e., of minimizers in (1.1)) - at least in those situations where a definite structure of v makes plausible to achieve such a result - is still an open problem, with various results and conjectures scattered in the literature [14, 5, 12, 6]. We are concerned here with the basic model given by the cartesian product of two Euclidean spaces

$$\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k = \{z = (x, y) : x \in \mathbb{R}^h\,, y \in \mathbb{R}^k\}\,, \quad n = h + k \geq 1\,,$$

equipped with the product type density

$$\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{h/2}}\,, \quad (x, y) \in \mathbb{R}^n\,.$$

This leads to consider notions of volume and perimeter of "mixed" Euclidean-Gaussian type for subsets of $\mathbb{R}^n$. Precisely, if $E \subset \mathbb{R}^n$ has, say, C^1-boundary, then we are going to set

$$V_{mix}(E) = \frac{1}{(2\pi)^{h/2}} \int_E e^{-\frac{|x|^2}{2}} dz\,, \tag{1.2}$$

$$P_{mix}(E) = \frac{1}{(2\pi)^{h/2}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(z)\,, \tag{1.3}$$

and to cast the isoperimetric problems

$$\Lambda(m) = \inf\{P_{mix}(E) : V_{mix}(E) = m\}\,, \qquad m > 0\,. \tag{1.4}$$

The main goal of this paper is to give a description of the isoperimetric sets in (1.4).

To introduce our first result, Theorem 1.1 below, we start recalling the well-known situation in the "pure" Euclidan and Gaussian cases. Indeed, when $h = 0$, (1.4) reduces to the classical Euclidean isoperimetric problem for sets $E \subseteq \{0\} \times \mathbb{R}^k \approx \mathbb{R}^k$,

$$\inf\{\mathcal{H}^{k-1}(\partial E) : \mathcal{H}^k(E) = m\}\,,$$

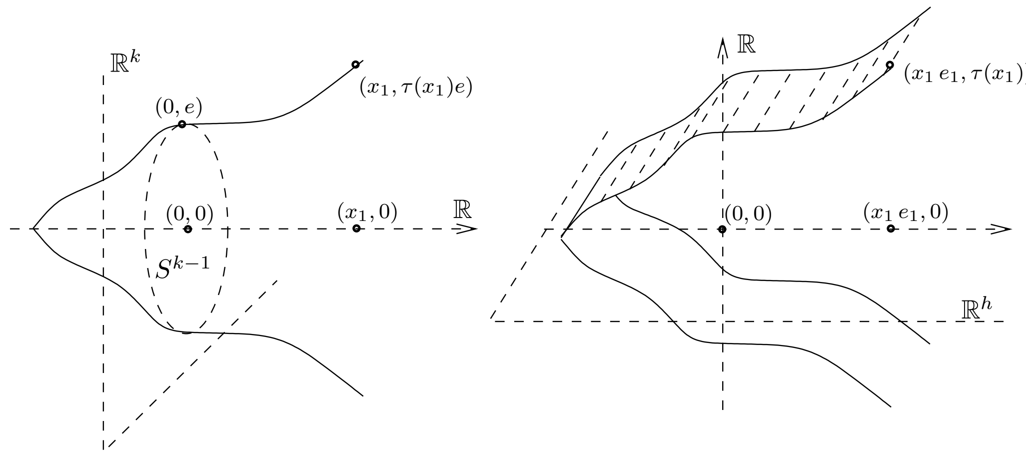

FIGURE 1. Examples of sets E associated to a non-negative and increasing function τ as in (1.5). On the left we consider the case $h=1$, $k>1$; on the right we have set $h>1$ and $k=1$.

and isoperimetric sets are Euclidean balls. When $k=0$, (1.4) becomes the Gaussian isoperimetric problem for sets $E \subseteq \mathbb{R}^h \times \{0\} \approx \mathbb{R}^h$,

$$\inf\left\{\frac{1}{(2\pi)^{h/2}}\int_{\partial E} e^{-\frac{|x|^2}{2}}d\mathcal{H}^{h-1}(x) : \frac{1}{(2\pi)^{h/2}}\int_E e^{-\frac{|x|^2}{2}}dx = m\right\},$$

and isoperimetric sets are known to be half-spaces (see for instance [15, 3, 4, 7]). Therefore, in the mixed cases where both $h \geq 1$ and $k \geq 1$ one could naively expect that, up to *vertical translations* of the form $z \mapsto z + (0, y_0)$, $y_0 \in \mathbb{R}^k$, and up to *horizontal rotations* of the form $z = (x, y) \mapsto (Qx, y)$, $Q \in \mathbf{O}(h)$, minimizers should be sets E of the form

$$E = \{(x,y) \in \mathbb{R}^n : |y| < \tau(x_1)\}, \tag{1.5}$$

for some non-negative *increasing* function $\tau : \mathbb{R} \to [0,\infty)$ (see Figure 1). We can visualize such a set E as a cylinder in the $(h-1)$-directions $x_2, \dots x_h$ over the axially symmetric set in $\mathbb{R}\times\mathbb{R}^k$ defined as

$$\{(s,y) \in \mathbb{R}\times\mathbb{R}^k : |y| < \tau(s)\}.$$

The following theorem, proved in sections 2 and 3, ensures in particular that isoperimetric sets have always this form.

Theorem 1.1 (Existence, symmetry and regularity). *Let $h \geq 1$, $k \geq 1$. For every $m > 0$, the variational problem* (1.4) *has at a least a solution in the class of sets of locally finite perimeter in $\mathbb{R}^n$. If E is such an isoperimetric set, then there exists an increasing function $\tau : \mathbb{R} \to [0,\infty)$ such that, up to a horizontal rotation and a vertical translation, we have*

$$E = \{(x,y) \in \mathbb{R}^n : |y| < \tau(x_1)\}. \tag{1.6}$$

Moreover, the function τ is locally absolutely continuous on $\mathbb{R}$ and

$$\partial E \setminus \{(x,y) \in \mathbb{R}^n : y = 0\}$$

is an analytic manifold. Finally, if $k < 7$, then ∂E is an analytic manifold.

Remark 1.2. In section 2.1 we are going to recall the notion of set of locally finite perimeter and to extend the definition of $P_{mix}(E)$ to Borel subsets $E \subset \mathbb{R}^n$. This shall be done in such a way that $P_{mix}(E) = P_{mix}(F)$ whenever the Borel sets E and F are equivalent with respect to the Lebesgue measure on $\mathbb{R}^n$.

Remark 1.3. Theorem 1.1 states the equivalence of the isoperimetric problem (1.4) with a one-dimensional variational problem that is *independent* of the "horizontal dimension" h. Indeed, if a set E satisfies (1.6) for some increasing function $\tau \in W^{1,1}_{loc}(\mathbb{R};[0,\infty))$, then the mixed-volume and the mixed-perimeter of E satisfy

$$V_{mix}(E) = \mathcal{V}(\tau)\,, \qquad P_{mix}(E) = \mathcal{P}(\tau)\,,$$

where the functionals $\mathcal{V}(\tau)$ and $\mathcal{P}(\tau)$ are defined as,

$$\mathcal{V}(\tau) = \frac{\omega_k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}}\, \tau(s)^k\, ds\,, \tag{1.7}$$

$$\mathcal{P}(\tau) = \frac{k\omega_k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}}\, \tau(s)^{k-1}\, \sqrt{1+\tau'(s)^2}\, ds\,. \tag{1.8}$$

Here ω_k denotes the Lebesgue measure of the unit ball in $\mathbb{R}^k$. Similar formulas hold if τ is just of locally bounded variation, see Lemma 2.10. In particular, the isoperimetric problem (1.4) is equivalent to a one-dimensional variational problem, i.e. we have

$$\Lambda(m) = \inf\left\{\mathcal{P}(\tau) : \tau \text{ is increasing, } \mathcal{V}(\tau) = m\right\}\,.$$

By (1.7) and (1.8) this last problem is independent of the value of h.

We next turn to the harder problem of a more explicit identification of isoperimetric sets. We present a rather complete picture of the situation in the case $k = 1$, together with some interesting remarks in the case $k > 1$. This is achieved through the analysis of the first and second order necessary conditions for minimality determined by the use volume-preserving variations. Whenever E is an open set with C^2-boundary, the first-order, stationarity condition (or Euler-Lagrange equation) for the isoperimetric problem (1.4) takes the form (see, e.g. [14, Proposition 3.2])

$$H_E(z) - (x,0)\cdot \nu_E(z) = \text{constant}\,, \qquad \forall z \in \partial E\,, \tag{1.9}$$

where H_E denotes the mean curvature of ∂E, and ν_E the outer unit normal to E. We now make two important remarks concerning the solutions to (1.9).

Remark 1.4 (Cylinders are always stationary)**.** It is easily seen that the "cylinders",

$$K_r = \{(x,y) \in \mathbb{R}^n : |y| < r\}\,, \qquad r > 0\,,$$

are always stationary for the isoperimetric problem (1.4) (note that K_r is obtained in (1.5) by setting $\tau(s) = r$ for every $s \in \mathbb{R}$). Thanks to the choice of the normalization constants in (1.2) and (1.3) we find that

$$V_{mix}(K_r) = \omega_k\, r^k\,, \qquad P_{mix}(K_r) = k\omega_k\, r^{k-1}\,, \qquad r > 0\,, k \geq 1\,.$$

In particular, if $k = 1$ then $P_{mix}(K_r) = 2$ for every $r > 0$, and the cylinders K_r with large r may enclose an arbitrarily large amount of mixed-volume by paying a constant amount of mixed-perimeter.

Remark 1.5 (A fundamental family of stationary sets, for $k = 1$)**.** A remarkable property of the Euler-Lagrange equation (1.9) in the case $k = 1$ is that it somehow possesses "very

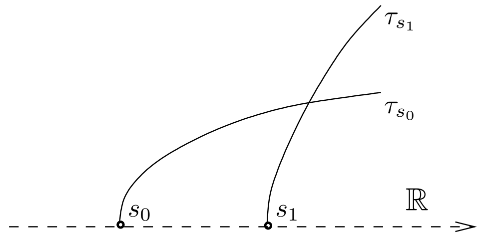

FIGURE 2. A qualitative picture of the functions τ_s corresponding to different values of s.

few" solutions. More precisely, let us introduce a one parameter family of functions $\{\tau_{s_0}\}_{s_0\in\mathbb{R}}$, $\tau_{s_0} : \mathbb{R} \to [0,\infty)$, by setting

$$\begin{aligned} \tau_{s_0}(s) &= 0 & s \le s_0\,, \\ \tau'_{s_0}(s) &= \frac{\zeta(s)}{\sqrt{\zeta(s_0)^2 - \zeta(s)^2}} & s > s_0\,, \end{aligned}$$

where $\zeta : \mathbb{R} \to (0,\infty)$ is defined as

$$\zeta(s) = e^{\frac{s^2}{2}} \int_s^\infty e^{-\frac{t^2}{2}}\,dt\,, \quad s \in \mathbb{R}\,,$$

(see step one in the proof of Lemma 4.5 for a description of ζ). Given $s_0 \in \mathbb{R}$, we now set

$$E(s_0) = \{(x,y) \in \mathbb{R}^h \times \mathbb{R} : |y| < \tau_{s_0}(x_1)\}\,,$$

so that $E(s_0)$ corresponds to the choice $\tau = \tau_{s_0}$ in (1.5). In Lemma 4.4 we are going to prove the following important property of the family of sets $\{E(s_0)\}_{s_0\in\mathbb{R}}$. If E is a stationary set that is associated to a non-negative, increasing function $\tau : \mathbb{R} \to [0,\infty)$ as in (1.5), and if $\{s \in \mathbb{R} : \tau(s) > 0\} = (s_0,\infty)$ then, up to a vertical translation and a horizontal rotation, we necessarily have $E = E(s_0)$ if $s_0 \in \mathbb{R}$, and $E = K_r$ if $s_0 = -\infty$. Various qualitative properties of τ_{s_0} are established in Lemma 4.5 (for example, τ_{s_0} is strictly increasing and strictly concave on $[s_0,\infty)$, see Figure 2).

We are now in the position to state our main result for the case $k = 1$. Let us recall that the isoperimetric function Λ defined in (1.4) is easily seen to be increasing and continuous, with $\Lambda(m) \to 0^+$ as $m \to 0^+$.

Theorem 1.6 (Isoperimetric function and isoperimetric sets for $k = 1$)**.** *Let $h \ge 1$, $k = 1$. There exists $m_0 > 0$ such that every isoperimetric set E with mass m, up to a vertical translation or a horizontal rotation, satisfies the following properties:*

(i) *if $m > m_0$, then $E = K_r$ for $r = m/\omega_1$;*
(ii) *if $m = m_0$, then either $E = K_r$ for $r = m_0/\omega_1$, or $E = E(s_0)$ for some $s_0 \in \mathbb{R}$ (and both possibilities occur);*
(iii) *if $m < m_0$, then $E = E(s_0)$ for some $s_0 \in \mathbb{R}$.*

Moreover, Λ is strictly increasing on $[0, m_0]$, is constantly equal to 2 on $[m_0,\infty)$, and is strictly concave on $(0, m_1)$, for some $m_1 \in (0, m_0]$.

Remark 1.7. It is clear from Remark 1.5 that a statement like Theorem 1.6 comes as a direct consequence of a careful study of the functions $v(s_0) = V_{mix}(E(s_0))$ and $p(s_0) = P_{mix}(E(s_0))$, $s_0 \in \mathbb{R}$. Theorem 1.6 essentially follows from the determination of the limits as $s_0 \to \pm\infty$ of $p(s_0)$ and $v(s_0)$ (see Lemma 4.6). A complete study of these functions seems to be a really subtle

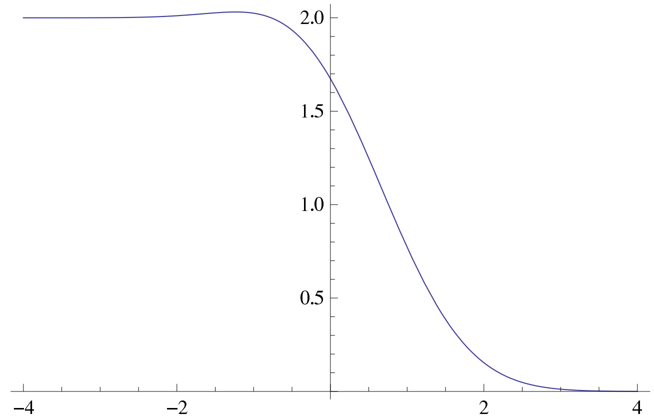

FIGURE 3. A qualitative picture of $p(s_0)$, drawn with Mathematica by Sergio Conti, suggests the validity of (1.10).

task, but would lead to strengthen the conclusions of Theorem 1.6. For example, it would suffice to prove the existence of $\overline{s} \in \mathbb{R}$ such that

$$\{s_0 \in \mathbb{R} : p(s_0) \geq 2\} = (-\infty, \overline{s}]\,, \qquad p'(s_0) < 0\,, \quad \forall s_0 > \overline{s}\,, \tag{1.10}$$

in order to infer (by a straightforward adaptation of the argument used in the proof of Theorem 1.6) that $m_1 = m_0$, and that for every $m \in (0, m_0]$ there exists only a single $s_0 = s_0(m)$ such that $E(s_0)$ is an isoperimetric set of mass m. In other words, we would achieve a uniqueness result for isoperimetric sets.

When $k > 1$ the Euler-Lagrange equation (1.9), even if restricted to sets E of the form (1.5), clearly admits a larger family of solutions, and we cannot expect to observe the relatively simple situation described in Remark 1.5. We can however learn something interesting concerning cylinders from the second order necessary condition for minimality. Let us recall that if E is an open set with C^2-boundary, the stability condition with respect to volume preserving variations leads as usual to a weighted Poincaré type inequality on the boundary of E (see section 4.3). If we assume that a cylinder K_r is stable, the resulting Poincaré inequality on ∂K_r is equivalent to the Poincaré inequality on $\mathbb{R}$ endowed with the Gaussian density, with a constant depending on the radius r and on the dimension k. By comparison with the sharp constant in this kind of inequality, one deduces the following result.

Theorem 1.8 (Stability of cylinders, $k > 1$)**.** *Let $k > 1$, $h \geq 1$. The cylinder K_r is stable if and only if $r \geq \sqrt{k-1}$. In particular, if $r < \sqrt{k-1}$, then K_r is not an isoperimetric set.*

Remark 1.9 (Are large cylinders isoperimetric regions?)**.** Starting from Theorem 1.8, and in analogy with the log-convexity conjecture [14, Conjecture 3.12], one may conjecture that if $k > 1$ and $r \geq \sqrt{k-1}$ then the cylinder K_r is an isoperimetric set. Having in mind the situation described in Theorem 1.6 for the case $k = 1$, it may as well be that the cylinders K_r are isoperimetric regions only for r larger than some critical radius $r_c > \sqrt{k-1}$. Indeed, in the case $k = 1$ it turns out that for *every* $r > 0$ the cylinder K_r is a local minimizer for the isoperimetric problem (1.4) (thus being "stable"), although we know from Theorem 1.6 that K_r is an isoperimetric set if and only if $r \geq \omega_1/m^+$. It may as well be unwise to trust too much in analogies, since the lack of connectedness of S^{k-1} in the case $k = 1$ is at the origin of various substantial differences with the case $k > 1$.

2. Symmetry of isoperimetric sets

After a brief review of some basic facts from geometric measure theory (section 2.1), we introduce two notions of symmetrization for sets in the product space $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ (sections 2.2 and 2.3). We next use these tools to prove the main result of this section, namely that every isoperimetric set E is associated to an increasing and non negative function τ as in (1.5) (Theorem 2.7 in section 2.4).

2.1. **Basic notation and preliminaries from geometric measure theory.** We will always denote the generic point of $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ as $z = (x, y)$, and the integration with respect to the Lebesgue measure over $\mathbb{R}^n$, $\mathbb{R}^h$ or $\mathbb{R}^k$ will be denoted respectively by dz, dx and dy. Moreover, expressions like "for a.e. $(x, y) \in \mathbb{R}^n$", "for a.e. $x \in \mathbb{R}^h$" and "for a.e. $y \in \mathbb{R}^k$" are meant with respect to the suitable Lebesgue measures. Finally, given $E \subseteq \mathbb{R}^n$ we define its vertical and horizontal sections respectively as

$$\begin{aligned} E_x &= \{y \in \mathbb{R}^k : (x,y) \in E\} \subseteq \mathbb{R}^k\,, \qquad x \in \mathbb{R}^h\,,\\ E^y &= \{x \in \mathbb{R}^h : (x,y) \in E\} \subseteq \mathbb{R}^h\,, \qquad y \in \mathbb{R}^k\,. \end{aligned}$$

Given a Borel set $E \subset \mathbb{R}^n$ and $\lambda \in [0,1]$ we denote by $E^{(\lambda)}$ the set of points having density λ with respect to E, i.e.

$$E^{(\lambda)} = \left\{ (x,y) \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{\mathcal{H}^n(E \cap B(z,r))}{\omega_n r^n} = \lambda \right\}.$$

The *essential boundary* $\partial^M E$ of E is defined as

$$\partial^M E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})\,.$$

The *Euclidean perimeter* $P(E)$ and the *mixed perimeter* $P_{mix}(E)$ of E are

$$\begin{aligned} P(E) &= \mathcal{H}^{n-1}(\partial^M E)\,,\\ P_{mix}(E) &= \int_{\partial^M E} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}(z)\,, \end{aligned}$$

whether these quantities are finite or not. We say that E is a *set of locally finite perimeter* if $\mathcal{H}^{n-1}(K \cap \partial^M E) < \infty$ for every compact set $K \subset \mathbb{R}^n$. We notice that if $P_{mix}(E) < \infty$ then E is a set of locally finite perimeter. If E is a set of locally finite perimeter, denoting by $\partial^{1/2} E$ the set $E^{1/2}$ of points having density $1/2$ with respect to E, we have (see e.g. [1, Theorem 3.61])

$$\partial^{1/2} E \subset \partial^M E, \qquad \mathcal{H}^{n-1}\left(\partial^M E \setminus \partial^{1/2} E\right) = 0\,.$$

If $v \in BV_{loc}(\mathbb{R}^n)$ then we denote by Dv the distributional derivative of v, that is a $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$. We denote by $Dv = \nabla v\, dx + D_S v$ the Lebesgue-Nikodým decomposition of Dv with respect to the Lebesgue measure on $\mathbb{R}^n$. The singular part $D_S v$ of Dv can be further decomposed into a jump part $D_J v$ and into a Cantor part, denoted by $D_C v$. If $\tau \in BV_{loc}(\mathbb{R})$ then we define two Borel functions $\tau^+, \tau^- : \mathbb{R} \to \mathbb{R}$ by setting

$$\tau^+(s) = \max\{\tau(s^+), \tau(s^-)\}\,, \qquad \tau^-(s) = \min\{\tau(s^+), \tau(s^-)\}\,,$$

where $\tau(s^+)$ and $\tau(s^-)$ denote respectively the right and the left limit, which always exist for a BV real function. In the special case when τ is increasing, then $\tau^+(s) = \tau(s^+)$ and $\tau^-(s) = \tau(s^-)$ for every $s \in \mathbb{R}$.

2.2. **Steiner symmetrization (vertical symmetrization).** We define here the Steiner symmetrization $\mathbf{S}E$ of a Borel set $E \subseteq \mathbb{R}^n$. Let us start by defining the two Borel measurable, non-negative functions v_E and p_E on $\mathbb{R}^h$ by setting

$$v_E(x) = \mathcal{H}^k\big(E_x\big)\,, \qquad\qquad p_E(x) = \mathcal{H}^{k-1}\big(\partial^M(E_x)\big)\,, \qquad x \in \mathbb{R}^h\,.$$

If we let ω_k denote the Lebesgue measure of the unit ball of $\mathbb{R}^k$, then for every $x \in \mathbb{R}^h$, the set

$$\Big\{y \in \mathbb{R}^k : \omega_k |y|^k < v_E(x)\Big\}\,, \quad x \in \mathbb{R}^h\,,$$

is a $k-$dimensional ball with center at the origin and k-dimensional measure equal to $\mathcal{H}^k(E_x)$. We define now the *Steiner symmetrization* $\mathbf{S}E$ *of* E as

$$\mathbf{S}E = \Big\{(x,y) \in \mathbb{R}^n : \ \omega_k |y|^k < v_E(x)\Big\}\,.$$

Notice that, since by construction one has $\mathcal{H}^k\big((\mathbf{S}E)_x\big) = \mathcal{H}^k(E_x)$ for every $x \in \mathbb{R}^h$, by Fubini Theorem one has

$$V_{mix}(\mathbf{S}E) = V_{mix}(E)\,.$$

The behavior of the mixed perimeter under the Euclidean symmetrization is described in the following result. We omit the proof which can be found for instance in [2].

Lemma 2.1. *If E is a set of locally finite perimeter, then $v_E \in BV_{\rm loc}(\mathbb{R}^h)$, and*

$$P_{mix}(E) \geq \frac{1}{(2\pi)^{h/2}} \int_{\mathbb{R}^h} \sqrt{p_E(x)^2 + \big|\nabla v_E(x)\big|^2}\, e^{-\frac{|x|^2}{2}}\, dx + \frac{1}{(2\pi)^{h/2}} \int_{\mathbb{R}^h} e^{-\frac{|x|^2}{2}}\, d\big|D_S v_E\big|\,. \tag{2.1}$$

If $E = \mathbf{S}E$ then equality holds in (2.1)*. Conversely, if equality holds in* (2.1)*, then for a.e. $x \in \mathbb{R}^h$ the section E_x is equivalent to a $k-$dimensional ball.*

Corollary 2.2. *If $E \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, then*

$$P_{mix}(\mathbf{S}E) \leq P_{mix}(E)\,. \tag{2.2}$$

If equality holds in (2.2)*, then for a.e. $x \in \mathbb{R}^h$ the vertical section E_x is a ball in $\mathbb{R}^k$.*

Proof. It is enough to apply Lemma 2.1 twice, to the sets $F = E$ and $F = \mathbf{S}E$ respectively, and to keep in mind that $v_{\mathbf{S}E} \equiv v_E$ by definition, while $p_{\mathbf{S}E} \leq p_E$ since balls are isoperimetric sets in the Euclidean setting. Hence, one has

$$\begin{aligned}
\big(2\pi\big)^{h/2} P_{mix}(E) &\geq \int_{\mathbb{R}^h} \sqrt{p_E(x)^2 + \big|\nabla v_E(x)\big|^2}\, e^{-\frac{|x|^2}{2}}\, dx + \int_{\mathbb{R}^h} e^{-\frac{|x|^2}{2}}\, d\big|D_s v_E\big| \\
&\geq \int_{\mathbb{R}^h} \sqrt{p_{\mathbf{S}E}(x)^2 + \big|\nabla v_{\mathbf{S}E}(x)\big|^2}\, e^{-\frac{|x|^2}{2}}\, dx + \int_{\mathbb{R}^h} e^{-\frac{|x|^2}{2}}\, d\big|D_s v_{\mathbf{S}E}\big| \\
&= \big(2\pi\big)^{h/2} P_{mix}\big(\mathbf{S}E\big)\,.
\end{aligned} \tag{2.3}$$

This gives inequality (2.2); moreover, if equality holds, then in particular the second inequality in (2.3) is an equality, and this implies that for almost all x the set E_x is a ball. □

Remark 2.3. We briefly underline two things: first of all, the opposite implication in Corollary 2.2 does not hold: in general, if all the sections E_x of a set E are balls but with different centers, one may easily have $P_{mix}\big(\mathbf{S}E\big) < P_{mix}(E)$. On the other hand, it is not even true that if the equality $P_{mix}\big(\mathbf{S}E\big) = P_{mix}(E)$ holds, then $E = \mathbf{S}E$ up to a translation in the y variable (or, equivalently, that the centers of all the balls E_x coincide).

2.3. **Ehrhard symmetrization (horizontal symmetrization).** We define now the Ehrhard symmetrization $\mathbf{G}E$ of a Borel set $E \subseteq \mathbb{R}^n$ [8]. This time, we consider the horizontal sections E^y of E, and define the two Borel measurable, non-negative functions v_E and p_E on $\mathbb{R}^k$ as

$$v_E(y) = \frac{1}{(2\pi)^{\frac{h}{2}}} \int_{E^y} e^{-\frac{|x|^2}{2}}\, dx\,, \qquad p_E(y) = \frac{1}{(2\pi)^{h/2}} \int_{\partial^M(E^y)} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{h-1}(x)\,, \qquad y \in \mathbb{R}^k\,.$$

Now, exactly as for the Euclidean symmetrization we replaced each vertical section E_x with a $k-$dimensional ball (i.e., the Euclidean isoperimetric set) with the same measure as E_x, this time we will replace each horizontal section E^y with a $h-$dimensional half-space (i.e., the Gaussian isoperimetric set) with the same measure as E^y. To do so, notice that for each $s \in \mathbb{R}$ the Gaussian measure of the half-space

$$\Big\{x \in \mathbb{R}^h : x_1 > s\Big\} \subseteq \mathbb{R}^h$$

is given by

$$\frac{1}{(2\pi)^{h/2}} \int_{\{x:\, x_1 > s\}} e^{-\frac{|x|^2}{2}}\, dx = \Psi(s)\,,$$

where we have defined a strictly decreasing smooth function $\Psi : \mathbb{R} \to [0,1]$ on setting

$$\Psi(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-\frac{t^2}{2}}\, dt\,, \qquad s \in \mathbb{R}\,.$$

Of course Ψ agrees with a suitable re-scaling of the standard error function. We shall set (by continuity) $\Psi(-\infty) = 1$ and $\Psi(\infty) = 0$. We can then define the *Gaussian symmetrization* $\mathbf{G}E$ *of* E as

$$\mathbf{G}E = \Big\{(x,y) \in \mathbb{R}^n :\ x_1 > \Psi^{-1}\Big(v_E(y)\Big)\Big\}\,.$$

Notice that, as for the Euclidean symmetrization we arbitrarily decided to put all the balls centered at $0 \in \mathbb{R}^k$, in this case we are arbitrarily deciding to put all the half-spaces orthogonal to the direction x_1. Moreover, since by construction for any $y \in \mathbb{R}^k$ one has $v_E(y) = v_{\mathbf{G}E}(y)$, again by Fubini's Theorem we have that

$$V_{mix}(\mathbf{G}E) = V_{mix}(E)\,.$$

We can now prove the Gaussian version of Corollary 2.2, that in turn is based on the analogue of Lemma 2.1. The proof of the following lemma can be easily derived by adapting the argument from [7, Section 4].

Lemma 2.4. *If E is a set of locally finite perimeter, then $v_E \in BV_{\rm loc}(\mathbb{R}^k)$ and*

$$P_{mix}(E) \geq \int_{\mathbb{R}^k} \sqrt{p_E(y)^2 + \big|\nabla v_E(y)\big|^2}\, dy + \big|D_S v_E\big|(\mathbb{R}^k)\,. \tag{2.4}$$

If $E = \mathbf{G}E$, then equality holds in (2.4). Conversely, if equality holds in (2.4), then for a.e. $y \in \mathbb{R}^k$ the section E^y is equivalent to a $h-$dimensional half-space.

Corollary 2.5. *For any set $E \subseteq \mathbb{R}^n$ of locally finite perimeter, it is*

$$P_{mix}(\mathbf{G}E) \leq P_{mix}(E)\,. \tag{2.5}$$

Moreover, if the above inequality is an equality, then for a.e. $y \in \mathbb{R}^k$ the horizontal section E^y is a half-space in $\mathbb{R}^h$.

Proof. Since by construction $v_{\mathbf{G}E} \equiv v_E$, while $p_{\mathbf{G}E} \leq p_E$ by the Gaussian Isoperimetric Theorem, applying Lemma 2.4 to E and $\mathbf{G}E$ we get

$$\begin{aligned} P_{mix}(E) &\geq \int_{\mathbb{R}^k} \sqrt{p_E(y)^2 + \left|\nabla v_E(y)\right|^2}\, d\mathcal{H}^k(y) + \left|D_s v_E\right|(\mathbb{R}^k) \\ &\geq \int_{\mathbb{R}^k} \sqrt{p_{\mathbf{G}E}(x)^2 + \left|\nabla v_{\mathbf{S}E}(x)\right|^2}\, d\mathcal{H}^k(y) + \left|D_s v_{\mathbf{G}E}\right|(\mathbb{R}^k) = P_{mix}\big(\mathbf{G}E\big)\,. \end{aligned} \tag{2.6}$$

This gives inequality (2.5); moreover, if equality holds, then in particular the second inequality in (2.6) is an equality, thus for almost each y the section E^y is a half-space. □

Remark 2.6. Exactly as noticed in Remark 2.3, we again have that the other implication in Corollary 2.5 is false, since the inequality can be strict even if all the sections E^y are half-spaces, provided they are not all parallel. On the other hand, if the equality $P_{mix}(\mathbf{G}E) = P_{mix}(E)$ holds, this does not necessarily imply that $\mathbf{G}E = E$ up to a rotation in y (or, in other words, that all the half-spaces E^y are parallel).

2.4. **Proof of the symmetry of isoperimetric sets.** In this section we prove that every isoperimetric set is associated to a non-negative increasing function as in (1.5). The exposition of this theorem is greatly simplified by the introduction of the following notation. Given $m > 0$, we let $Z_0(m)$ be the family of those sets of locally finite perimeter $E \subset \mathbb{R}^n$ with mixed volume $V_{mix}(E) = m$. Next we define sub-families $\{Z_i(m)\}_{i=1}^k$, $Y(m)$ and $X(m)$ of $Z_0(m)$, satisfying the inclusions,

$$X(m) \subset Y(m) \subset Z_k(m) \subset Z_{k-1}(m) \subset \cdots \subset Z_1(m) \subset Z_0(m)\,,$$

as follows:

(a) We say that $E \in Z_i(m)$, $1 \leq i \leq k$, if $E \in Z_0(m)$ and there are i orthogonal affine hyperplanes $H_1, \ldots, H_i$ in $\mathbb{R}^k$ such that, for every $x \in \mathbb{R}^h$, the vertical section $E_x \subset \mathbb{R}^k$ is symmetric by reflection with respect to each of the H_j's;

(b) We say that $E \in Y(m)$, if $E \in Z_0(m)$ and there exist $y_E \in \mathbb{R}^k$ and a measurable function $u : \mathbb{R}^h \to [0, \infty)$ such that
$$E = \Big\{(x, y) \in \mathbb{R}^n : \, |y - y_E| < u(x)\Big\}\,. \tag{2.7}$$

(c) We say that $E \in X(m)$, if $E \in Z_0(m)$ and there exist $y_E \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{h-1}$ and an increasing function $\tau : \mathbb{R} \to [0, \infty)$ such that
$$E = \Big\{(x, y) \in \mathbb{R}^n : \, |y - y_E| < \tau(x \cdot \nu)\Big\}\,.$$

With these definitions in force, we can state the main result of this section as follows.

Theorem 2.7. *Let $m > 0$. If E is an isoperimetric set with $V_{mix}(E) = m$, then $E \in X(m)$.*

As already said, to prove this theorem we shall make use of the symmetrization tools established in sections 2.2 and 2.3. We shall also rely on the remarks about symmetrization by reflection contained in the following lemma.

Lemma 2.8 (Some properties of symmetrization by (vertical) reflection)**.** *If E^+, E^- are sets of locally finite perimeter in $\mathbb{R}^n$ that are symmetric by reflection with respect to the hyperplane $\{y_k = 0\}$, and if we define*

$$E = \big\{z \in E^+ : y_k > 0\big\} \cup \big\{z \in E^- : y_k < 0\big\}\,,$$

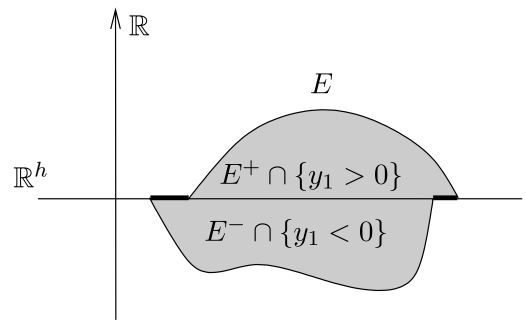

FIGURE 4. Lemma 2.8. When equality holds in (2.8), then the essential projections of E^+ and E^- are equivalent. In the case $k = 1$, shown in the picture, this condition just means that the bold segments in the picture collapse to have zero length, and in particular it does not force the profiles u^+ and u^- to be equal.

then

$$P_{mix}(E) \geq \frac{P_{mix}(E^+) + P_{mix}(E^-)}{2} \,. \tag{2.8}$$

If, moreover, there exist two Borel measurable functions $u_+, u_- : \mathbb{R}^h \to [0, \infty)$, such that

$$E^+ = \{(x, y) \in \mathbb{R}^n : |y| < u_+(x)\} \,, \qquad E^- = \{(x, y) \in \mathbb{R}^n : |y| < u_-(x)\} \,, \tag{2.9}$$

then equality holds in (2.8) if and only if

$$\begin{aligned} &E^+ = E^- \subseteq \mathbb{R}^n \text{ (when } k > 1\text{);} \\ &pE^+ = pE^- \subseteq \mathbb{R}^h \text{ (when } k = 1\text{),} \end{aligned}$$

where pE^+ and pE^- denote the essential projections of E^+ and E^- over $\mathbb{R}^h$.

Remark 2.9. It is important to remark the peculiarity of the case $k = 1$ in the above lemma. In fact, as soon as the projections of E^+ and E^- on $\mathbb{R}^h$ coincide, the fact that E^+ and E^- are different does not give any horizontal part of the boundary of E on $\{y = 0\}$. This is different from what happens for the case $k > 1$, where the two different parts of the boundary would meet giving rise to some boundary on $\{y_k = 0\}$. The reason of this difference is basically that $\mathbb{S}^{k-1}$ is connected for $k > 1$ and disconnected for $k = 1$.

As a last tool to be used in the proof of Theorem 2.7 we present the following lemma, providing the formulas for the mixed-volume and the mixed-perimeter of a set E satisfying (1.5) in terms of the corresponding function τ. In particular, we shall need the linearity of $\mathcal{V}$ and the convexity of $\mathcal{P}$ with respect to τ that are characteristic of the case $k = 1$.

Lemma 2.10. *If $\tau \in BV_{loc}(\mathbb{R}; [0, \infty))$ and if*

$$E = \{(x, y) \in \mathbb{R}^n : |y| < \tau(x_1)\} \,,$$

then $V_{mix}(E) = \mathcal{V}(\tau)$ and $P_{mix}(E) = \mathcal{P}(\tau)$, where

$$\mathcal{V}(\tau) = \frac{\omega_k}{\sqrt{2\pi}} \int_{\mathbb{R}} \tau(s)^k e^{-\frac{s^2}{2}} \, ds \,, \tag{2.10}$$

$$\mathcal{P}(\tau) = \frac{k\omega_k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}} \tau(s)^{k-1} \sqrt{1 + \tau'(s)^2} \, ds + \frac{k\omega_k}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}} \tau(s)^{k-1} \, d|D_S\tau|(s) \,. \tag{2.11}$$

Remark 2.11. When $k=1$, in the definition of $\mathcal{P}(\tau)$ we have adopted the convention $0^0=0$ to define the expression $\tau(s)^{k-1}$ for those $s\in\mathbb{R}$ such that $\tau(s)=0$. When $k\geq 2$ and $s\in\operatorname{spt}(D_S\tau)$ we have set for brevity

$$\tau(s)^{k-1}=\begin{cases}\tau^+(s)^{k-1}\,, & \text{if } s\in\operatorname{spt}(D_C\tau)\,,\\ \frac{1}{\tau^+(s)-\tau^-(s)}\int_{\tau^-(s)}^{\tau^+(s)}t^{k-1}dt\,, & \text{if } s\in\operatorname{spt}(D_J\tau)\,.\end{cases}$$

We now come to the proofs of Lemma 2.8, Lemma 2.10 and Theorem 2.7.

Proof of Lemma 2.8. By construction E^+ is symmetric by reflection with respect to hyperplane $\{y_k=0\}$. Moreover $\mathcal{H}^{n-1}\big(\partial^{1/2}E^+\cap\{y_k=0\}\big)=0$, and thus we easily find

$$P_{mix}(E^+)=\frac{2}{(2\pi)^{h/2}}\int_{\partial^M E^+\cap\{y_k>0\}}e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}=\frac{2}{(2\pi)^{h/2}}\int_{\partial^M E^+\cap\{y_k<0\}}e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}\,.$$

Of course, analogous identities hold for E^-. Taking into account that

$$\partial^{1/2}E^+\cap\{y_k>0\}=\partial^{1/2}E\cap\{y_k>0\}\,,\qquad \partial^{1/2}E^-\cap\{y_k<0\}=\partial^{1/2}E\cap\{y_k<0\}\,,$$

we conclude

$$\begin{aligned}P_{mix}(E)\ &=\ \frac{1}{(2\pi)^{h/2}}\int_{\partial^M E^+\cap\{y_k>0\}}e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}+\frac{1}{(2\pi)^{h/2}}\int_{\partial^M E^-\cap\{y_k<0\}}e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}\\ &\quad+\frac{1}{(2\pi)^{h/2}}\int_{\partial^M E\cap\{y_k=0\}}e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}\\ &\geq\ \frac{P_{mix}(E^+)+P_{mix}(E^-)}{2}\,,\end{aligned}$$

that is, (2.8). Moreover, we infer from this argument that equality holds in (2.8) if and only if

$$\mathcal{H}^{n-1}(\partial^{1/2}E\cap\{y_k=0\})=0\,. \tag{2.12}$$

We now pass to discuss separately the cases $k=1$ and $k>1$, under the assumption that (2.9) holds true.

Case I. $k=1$.
For all $x\in pE$, the essential projection of E over $\mathbb{R}^h$, $E_x=(-u_-(x),u_+(x))$ and thus by Vol'pert theorem (see [10, Theorem 3.21]) we have $(\partial^{1/2}E)_x=\{-u_-(x),u_+(x)\}$, for a.e. $x\in pE$. Therefore, recalling (2.12), we may conclude that for a.e. $x\in pE$

$$u_+(x)>0\,,\quad u_-(x)>0\,,$$

thus proving that $pE^+=pE^-$.

Case II. $k>1$.
From the assumption (2.9), using Vol'pert theorem again, we get that for $\mathcal{H}^h$-a.e. $x\in pE$

$$\begin{aligned}(\partial^{1/2}E)_x=&\Big\{y\in\mathbb{R}^k:\,y_k>0,\,|y|=u_+(x)\Big\}\cup\Big\{\{y\in\mathbb{R}^k:\,y_k<0,\,|y|=u_-(x)\Big\}\\ &\bigcup\Big\{y\in\mathbb{R}^k:\,y_k=0,\,\min\big\{u_-(x),u_+(x)\big\}\leq|y|\leq\max\big\{u_-(x),u_+(x)\big\}\Big\}\,,\end{aligned}$$

up to a set of zero $\mathcal{H}^{k-1}$-measure. Therefore, from (2.12), using Fubini's theorem we have

$$\begin{aligned}0=\mathcal{H}^{n-1}(\partial^{1/2}E\cap\{y_k=0\})&=\int_{pE}\mathcal{H}^{k-1}\big((\partial^{1/2}E)_x\cap\{y_k=0\}\big)dx\\ &=\omega_{k-1}\int_{pE}\big|u_+(x)^k-u_-(x)^k\big|\,dx\,,\end{aligned}$$

thus proving that $u_+(x)=u_-(x)$ for a.e. $x\in pE$. $\square$

Proof of Lemma 2.10: By repeatedly applying Fubini's theorem we find

$$\begin{aligned}V_{mix}(E) &= \frac{1}{(2\pi)^{h/2}}\int_{\mathbb{R}^h} e^{-\frac{x^2}{2}}\,dx\int_{\{y\in\mathbb{R}^k:|y|<\tau(x_1)\}} dy\\ &= \frac{\omega_k}{(2\pi)^{h/2}}\int_{\mathbb{R}^h}\tau(x_1)^k e^{-\frac{x^2}{2}}\,dx=\frac{\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}}\tau(x_1)^k e^{-\frac{x_1^2}{2}}\,dx_1\,,\end{aligned}$$

i.e. $V_{mix}(E)=\mathcal{V}(\tau)$, as required. On the other hand from Lemma 2.1 we have

$$P_{mix}(E)=\frac{1}{(2\pi)^{h/2}}\int_{\mathbb{R}^h}\sqrt{p_E(x)^2+\big|\nabla v_E(x)\big|^2}e^{-\frac{|x|^2}{2}}\,dx+\frac{1}{(2\pi)^{h/2}}\int_{\mathbb{R}^h}e^{-\frac{|x|^2}{2}}\,d\big|D_S v_E\big|\,,$$

where $v_E(x)=\omega_k\tau(x_1)^k$, $p_E(x)=k\omega_k\tau(x_1)^{k-1}$ for a.e. x. Then (2.11) follows immediately from the equality above and from the chain rule formula for BV functions (see [1, Theorem 3.96]). $\square$

Proof of Theorem 2.7. We divide the proof in four steps.

<u>Step I.</u> *If $E\in Y(m)$ is an isoperimetric set, then $E\in X(m)$.*
Since $E\in Y(m)$, by (2.7) and up to a vertical translation we have

$$E=\Big\{(x,y)\in\mathbb{R}^n:\,|y|<u(x)\Big\}\,,\tag{2.13}$$

for some measurable function $u:\mathbb{R}^h\to[0,\infty)$. Since E is an isoperimetric set, we have $P(\mathbf{G}E)=P(E)$. By Corollary 2.5, for a.e. $y\in\mathbb{R}^k$ the horizontal section E^y of E is a half-space in $\mathbb{R}^h$. More precisely, there exist functions $\nu:\mathbb{R}^k\to\mathbb{S}^{h-1}$ and $\xi:\mathbb{R}^k\to[-\infty,\infty]$ such that

$$E^y=\Big\{x\in\mathbb{R}^h:\,x\cdot\nu(y)>\xi(y)\Big\}\,,\tag{2.14}$$

for a.e. $y\in\mathbb{R}^k$. By (2.13) we have

$$(x,y)\in E\Longrightarrow(x,\tilde y)\in E\,,\quad\forall\big|\tilde y\big|\le|y|\,,$$

i.e.

$$\big|\tilde y\big|\le|y|\Longrightarrow E^y\subseteq E^{\tilde y}\,.\tag{2.15}$$

Since an inclusion between two non-empty half-spaces can hold if and only if the two half-spaces are parallel, by combining (2.14) with (2.15) we deduce the existence of $\nu\in\mathbb{S}^{h-1}$ such that $\nu(y)=\nu$ for a.e. $y\in\mathbb{R}^k$. Thus,

$$E^y=\Big\{x\in\mathbb{R}^h:\,x\cdot\nu>\xi(y)\Big\}\,,\tag{2.16}$$

for a.e. $y\in\mathbb{R}^k$. By combining (2.16) with (2.13) we deduce that $u(x)=\tau(x\cdot\nu)$ for some measurable function $\tau:\mathbb{R}\to[0,\infty)$. To show that τ is increasing it suffices to notice that, if $x,\tilde x\in\mathbb{R}^h$ are such that $\tilde x\cdot\nu\ge x\cdot\nu$, then for a.e. $y\in\mathbb{R}^k$ we have

$$(x,y)\in E\Longleftrightarrow x\in E^y\Longleftrightarrow x\cdot\nu>\xi_y\Longrightarrow\tilde x\cdot\nu>\xi_y\;\Longleftrightarrow(\tilde x,y)\in E\,.$$

Thus $E\in X(m)$, as required.

<u>Step II.</u> *If $E\in Z_k(m)$ is an isoperimetric set, then $E\in X(m)$.*
Since $E\in Z_k(m)$ we may assume that, up to a vertical translation,

$$(x,y)\in E\Longleftrightarrow(x,-y)\in E\,.\tag{2.17}$$

Since E is an isoperimetric set, we have $P_{mix}(E)=P_{mix}(\mathbf{S}E)$. Applying Corollary 2.2 to E, for a.e. $x\in\mathbb{R}^h$ we find that the vertical section E_x of E is a ball $\mathbb{R}^k$. If E_x is such a section, then by (2.17) we see that the point $(x, 0)$ is the center of the ball E_x. If $u(x)$ denotes the radius of this ball, we have just proved that

$$y \in E_x \Longleftrightarrow |y| < u(x)\,,$$

for some measurable function $u : \mathbb{R}^h \to [0, \infty)$. Thus $E \in Y(m)$ and, by step one, $E \in X(m)$.

Step III. *Proof for the case $k > 1$.*
Let $0 \leq i \leq k$. It suffices to show that if $E \in Z_i(m)$ is an isoperimetric set, then $E \in X(m)$. We will argue inductively on i, the case $i = k$ having already be solved in Step II. Let now $0 \leq i \leq k-1$, assume the claim for every j with $i < j \leq k$, and let $E \in Z_i(m)$ be an isoperimetric set. We denote by $H_1, \dots H_i$ the orthogonal affine hyperplanes with respect to which E is symmetric by reflection. Since $i < k$, there exist $\nu \in \mathbb{S}^{k-1}$ and $\xi \in \mathbb{R}$ such that the affine hyperplane

$$H_{i+1} = \{y \in \mathbb{R}^k : \, y \cdot \nu = \xi\}\,,$$

is orthogonal to the hyperplanes $H_1, \dots H_i$, and divides E in two parts of equal mixed volume, i.e. if we set

$$E_1 = \Big\{(x, y) \in E : \, y \cdot \nu > \xi\Big\}\,, \qquad\qquad E_2 = \Big\{(x, y) \in E : \, y \cdot \nu < \xi\Big\}\,,$$

then $V_{mix}(E_1) = V_{mix}(E_2) = m/2$. The reflection of $\mathbb{R}^n$ with respect to H_{i+1} is given by the linear map $R : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$R(x, y) = \Big(x, y - 2\nu\Big(y \cdot \nu - \xi\Big)\Big)\,, \qquad (x, y) \in \mathbb{R}^n\,.$$

Finally, let us consider the two sets E^+ and E^- defined as

$$E^+ = E_1 \cup R\big(E_1\big)\,, \qquad\qquad E^- = E_2 \cup R\big(E_2\big)\,.$$

By construction $V_{mix}(E^+) = V_{mix}(E^-) = m$, and both sets are symmetric by reflection with respect to the hyperplanes $H_1, \dots H_i, H_{i+1}$. In particular, $E^+, E^- \in Z_{i+1}(m)$. Since E^+ and E^- are symmetric with respect to H_{i+1}, then by the first part of Lemma 2.8 we find that

$$P_{mix}(E) \geq \frac{P_{mix}(E^+) + P_{mix}(E^-)}{2}\,. \tag{2.18}$$

Since E is an isoperimetric set and $V_{mix}(E^+) = V_{mix}(E^-) = m$, we deduce that equality holds in (2.18). In particular, both E^+ and E^- are isoperimetric sets in $Z_{i+1}(m)$. By inductive assumption, $E^+, E^- \in X(m)$. In particular, $E^+ E^- \in Y(m)$ and, since equality holds in (2.18), we can apply the second part of Lemma 2.8 to deduce, as $k > 1$, that E^+ is equivalent to E^-. This ensures that E is equivalent to $E^+ \in X(m)$, so $E \in X(m)$ as required.

Step IV. *Proof for the case $k = 1$.*
In this case, the argument of Step III guarantees the existence of two increasing functions $\tau_1, \tau_2 : \mathbb{R} \to [0, \infty)$ such that, up to a horizontal rotation, for some $s_0 \in [-\infty, \infty)$ one has

$$E = \Big\{(x, y) \in \mathbb{R}^n : -\tau_1(x_1) < y < \tau_2(x_1)\Big\}\,,$$

being

$$(s_0, \infty) = \{s \in \mathbb{R} : \tau_1(s) > 0\} = \{s \in \mathbb{R} : \tau_2(s) > 0\}\,, \tag{2.19}$$

$$V_{mix}(E \cap \{y > 0\}) = V_{mix}(E \cap \{y < 0\})\,. \tag{2.20}$$

By (2.20) we have $\mathcal{V}(\tau_1) = \mathcal{V}(\tau_2)$. Since $k = 1$, from (2.10) we see that $\tau \mapsto \mathcal{V}(\tau)$ is linear. Hence, if we set $\tau_0 = (\tau_1 + \tau_2)/2$ and define,

$$E' = \{(x, y) \in \mathbb{R}^n : |y| < \tau_0(x_1)\},$$

then we conclude that $V_{mix}(E) = V_{mix}(E')$. By (2.19), (2.11) and the assumption $k = 1$, we find that

$$P_{mix}(E') = \mathcal{P}(\tau_0) \leq \frac{\mathcal{P}(\tau_1) + \mathcal{P}(\tau_2)}{2} = P_{mix}(E). \tag{2.21}$$

By Corollary 3.4 (which is proved in the next section without relying on Theorem 2.7), τ_1, τ_2 are locally absolutely continuous, therefore strict sign holds in (2.21) unless unless $\tau_1' = \tau_2'$. Since equality holds in (2.21), we conclude from $\tau_1(s_0) = \tau_2(s_0) = 0$ that $\tau_1 = \tau_2$, i.e. $E \in X(m)$. $\square$

3. Existence and regularity of isoperimetric sets

In section 3.1 we prove the existence of isoperimetric sets (Theorem 3.1), whose regularity is addressed in section 3.2, Theorem 3.3. Finally, we remark that Theorem 1.1 will follow as an immediate corollary of these results and of Theorem 2.7 from the previous section.

3.1. **Existence of isoperimetric sets.** We begin with the basic existence result.

Theorem 3.1 (Existence of isoperimetric sets)**.** *For every $m > 0$, there exist isoperimetric sets with mixed-volume m. They necessarily belong to $X(m)$.*

Remark 3.2. In the proof of Theorem 3.1 (as well as in Section 4.2), we shall use the elementary estimate,

$$\int_s^\infty e^{-\frac{t^2}{2}}\, dt < \frac{e^{-\frac{s^2}{2}}}{s}, \qquad \forall t > 0, \tag{3.1}$$

that is valid since

$$\int_s^\infty e^{-\frac{t^2}{2}}\, dt < \int_s^\infty \frac{t}{s} e^{-\frac{t^2}{2}}\, dt = \frac{(-e^{-\frac{t^2}{2}})|_s^\infty}{s},$$

whenever $s > 0$.

Proof of Theorem 3.1. We divide the proof in two steps.

Step I. Reduction to the sets in $X(m)$.
We start proving that it is enough to restrict our attention to the elements of $X(m)$. In other words, we are claiming that if a set $E \in X(m)$ minimizes the perimeter among the elements of $X(m)$, then it is an isoperimetric set. Notice that this is not already ensured by Theorem 2.7, since that result does not prevent, in principle, the possibility that

$$\inf\Big\{P_{mix}(E) :\, E \in X(m)\Big\} > \inf\Big\{P_{mix}(F) :\, V_{mix}(F) = m\Big\},$$

being only the first infimum attained. On the other hand, by Theorem 2.7 it is of course enough to check that

$$\inf\Big\{P_{mix}(E) :\, E \in X(m)\Big\} \leq \inf\Big\{P_{mix}(F) :\, V_{mix}(F) = m\Big\}.$$

To show this inequality, just take a set F of locally finite perimeter in $\mathbb{R}^n$, with $V_{mix}(F) = m$, and let $E = \mathbf{SG}F$. Clearly, $V_{mix}(E) = m$, and by Corollary 2.2 and Corollary 2.5, we have

$P_{mix}(F) \geq P_{mix}(E)$. Hence, to conclude we only need to check that $E \in X(m)$. By definition of $\mathbf{G}F$, the vertical sections $(\mathbf{G}F)_x$ satisfy

$$(\mathbf{G}F)_x = \{y \in \mathbb{R}^k : x_1 > \Psi^{-1}(v_F(y))\}\,.$$

In particular, if $x, \tilde{x} \in \mathbb{R}^h$ with $x_1 \leq \tilde{x}_1$, then $(\mathbf{G}F)_x \subset (\mathbf{G}F)_{\tilde{x}}$. Therefore the function $\tau : \mathbb{R} \to [0,\infty]$ defined as

$$\tau(s) = \left(\frac{\mathcal{H}^k((\mathbf{G}F)_{s\,e_1})}{\omega_k}\right)^{1/k}, \qquad s \in \mathbb{R}\,,$$

turns out to be increasing. Since, by definition,

$$E = \mathbf{SG}F = \Big\{(x,y) \in \mathbb{R}^n : \omega_k |y|^k < \mathcal{H}^k((\mathbf{G}F)_x)\Big\} = \Big\{(x,y) \in \mathbb{R}^n : |y| < \tau(x_1)\Big\}\,,$$

we conclude that $E \in X(m)$.

Step II. *Isoperimetric sets in $X(m)$ exist.*
Thanks to the first Step, we only have to show that there are minimizers of the mixed perimeter $P_{mix}(E)$ within the class $X(m)$. By Lemma 2.10, it is enough to prove that the variational problem

$$\inf\left\{\mathcal{P}(\tau) : \tau \text{ is increasing},\ \tau \geq 0,\ \mathcal{V}(\tau) = m\right\}\,, \tag{3.2}$$

admits a minimizer τ_0. Let us consider a minimizing sequence $\{\tau_i\}_{i\in\mathbb{N}}$ in (3.2). By an approximation argument we may directly assume that each τ_i is smooth and strictly increasing on the half-line $(t_i,\infty) = \{\tau_i > 0\}$. For every $M > 0$ we have

$$\sup_{i\in\mathbb{N}} |D\tau_i|(-M,M) \leq e^{M^2/2}\frac{\sqrt{2\pi}}{k\omega_k}\sup_{i\in\mathbb{N}} \mathcal{P}(\tau_i)\,,$$

therefore there exists an increasing function $\tau_0 : \mathbb{R} \to [0,\infty)$ such that, up to extracting a sub-sequence, $\tau_i \to \tau_0$ in $L^1_{loc}(\mathbb{R})$ and a.e. on $\mathbb{R}$. By lower semicontinuity we have $\mathcal{P}(\tau_0) \leq \liminf_{i\to\infty}\mathcal{P}(\tau_i)$. By Fatou's lemma $\mathcal{V}(\tau_0) \leq m$. We are thus left to prove that $\mathcal{V}(\tau_0) \geq m$. To this end, we assume that

$$\mathcal{V}(\tau_0) = m - 2\varepsilon\,,$$

for some $\varepsilon > 0$, and then derive a contradiction. Let us consider a sequence $\{r_i\}_{i\in\mathbb{N}} \subset (0,\infty)$ with the property that

$$\frac{\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}} (\min\{\tau_i(s), r_i\})^k e^{-\frac{s^2}{2}}\,ds = m - \varepsilon\,, \qquad \forall i \in \mathbb{N}\,.$$

Such a sequence exists as $\mathcal{V}(\tau_i) = m$ for every $i \in \mathbb{N}$. We claim that $r_i \to \infty$. Indeed, if

$$r = \sup_{i\in\mathbb{N}} r_i < \infty\,,$$

then we could apply the dominated convergence theorem to find that

$$\begin{aligned} m - 2\varepsilon &\geq \frac{\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}} (\min\{\tau_0(s), r\})^k e^{-\frac{s^2}{2}}\,ds \\ &= \lim_{i\to\infty}\frac{\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}} (\min\{\tau_i(s), r_i\})^k e^{-\frac{s^2}{2}}\,ds = m - \varepsilon\,, \end{aligned}$$

a contradiction. Moreover $\tau_i^{-1}(r_i) \to \infty$: indeed,

$$m - \varepsilon = \frac{\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}} (\min\{\tau_i(s), r_i\})^k e^{-\frac{s^2}{2}}\,ds \geq \frac{\omega_k\, r_i^k}{\sqrt{2\pi}}\int_{\tau_i^{-1}(r_i)}^{\infty} e^{-\frac{s^2}{2}}\,ds\,.$$

Since $r_i \to \infty$, it must be

$$\lim_{i\to\infty} \int_{\tau_i^{-1}(r_i)}^{\infty} e^{-\frac{s^2}{2}}\, ds = 0\,,$$

and thus $\tau_i^{-1}(r_i) \to \infty$, as claimed. We now conclude by the following argument. If we set $M_i = \sup_{\mathbb{R}} \tau_i$, then by the change of variable $w = \tau_i(s)$ and by (3.1) we find that

$$\begin{aligned}\mathcal{P}(\tau_i) &\geq \frac{k\omega_k}{\sqrt{2\pi}} \int_{\tau_i^{-1}(r_i)}^{\infty} \tau_i'(s)\, \tau_i(s)^{k-1} e^{-\frac{s^2}{2}} ds \geq \frac{k\omega_k}{\sqrt{2\pi}} \int_{r_i}^{M_i} w^{k-1} e^{-\frac{\tau_i^{-1}(w)^2}{2}} dw \\ &\geq \frac{k\omega_k}{\sqrt{2\pi}} \int_{r_i}^{M_i} w^{k-1} \tau_i^{-1}(w) \int_{\tau_i^{-1}(w)}^{\infty} e^{-\frac{t^2}{2}} dt\, dw \geq \tau_i^{-1}(r_i) \frac{k\omega_k}{\sqrt{2\pi}} \int_{r_i}^{M_i} w^{k-1} \int_{\tau_i^{-1}(w)}^{\infty} e^{-\frac{t^2}{2}} dt\,.\end{aligned}$$

By definition of r_i,

$$\frac{k\omega_k}{\sqrt{2\pi}} \int_{r_i}^{M_i} w^{k-1} \int_{\tau_i^{-1}(w)}^{\infty} e^{-\frac{t^2}{2}} dt = \varepsilon\,,$$

and recalling that $\tau_i^{-1}(r_i) \to \infty$ this leads to $\mathcal{P}(\tau_i) \to \infty$, a contradiction. □

3.2. **Regularity of isoperimetric sets.** We now combine the basic regularity theory for almost-minimizers of the perimeter with the symmetry properties that are characteristic of the elements of $X(m)$.

Theorem 3.3. *If E is an isoperimetric set, then $\partial E \setminus \{(x,y) \in \mathbb{R}^n : y = 0\}$ is an analytic manifold. Moreover, if $k < 7$, then ∂E is an analytic manifold.*

Proof. By the regularity theory of isoperimetric hypersurfaces (see, e.g. [13, Section 3.10]), there exists a (possibly empty) closed set $\Sigma \subset \partial E$ such that $\partial E \setminus \Sigma$ is an analytic manifold, Σ is empty if $2 \leq n \leq 7$ and Σ has Hausdorff dimension bounded above by $n-8$ if $n \geq 8$. Moreover there exists a positive constant $\varepsilon_0 = \varepsilon_0(n)$ such that the singular set Σ can be characterized as follows:

$$\Sigma = \left\{ z \in \partial E : \inf_{\nu \in S^{n-1}} \limsup_{r\to 0^+} \frac{1}{r^{n-1}} \int_{B(z,r)\cap\partial E} |\nu_E - \nu|^2 d\mathcal{H}^{n-1} \geq \varepsilon_0 \right\}. \tag{3.3}$$

We can therefore assume that $n \geq 8$. Since $E \in X(m)$, up to a vertical translation and a horizontal rotation, we know that E has the following symmetries: (i) first, when $h \geq 2$,

$$(x,y) \in E \qquad \Leftrightarrow \qquad (x + te_i, y) \in E\,, \tag{3.4}$$

for every $i = 2, ..., k$, $t \in \mathbb{R}$; (ii) second,

$$(x,y) \in E\,, y \neq 0 \qquad \Leftrightarrow \qquad (x, Qy) \in E\,, y \neq 0\,. \tag{3.5}$$

for every $Q \in \mathbf{O}(k)$. Since ∂E has the same symmetries as E, by the integral characterization (3.3), we find that Σ has the same symmetries as E too. We can now argue as follows.

Let us assume that $k < 7$. In this case $n \geq 8$ forces $h \geq 2$. Hence, if $z \in \Sigma$, then by (3.4) (casted with Σ in place of E) we find that Σ contains a $(h-1)$-dimensional plane (passing through z). In particular the Hausdorff dimension of Σ is at least $h-1$, i.e. $h - 1 \leq n - 8 = h + k - 8$. Since this would force $k \geq 7$, we conclude that if $k < 7$ then ∂E is an analytic manifold.

Let us now show that in any case $\partial E \setminus \{z : y = 0\}$ is an analytic manifold. Since $n \geq 8$ we have that either $k \geq 2$ or $h \geq 2$. If now $z \in \Sigma$ with $y \neq 0$ then by (3.4) and by (3.5) we find that Σ contains a $[(k-1)+(h-1)]$-dimensional cylinder (passing through z). Therefore $(k-1)+(h-1) = n-2 > n-8$, a contradiction. □

Theorem 3.3 has an interesting consequence about the regularity of the functions τ associated to isoperimetric sets E.

Corollary 3.4. *If $\tau:\mathbb{R}\to[0,\infty)$ is an increasing function such that*

$$E=\{(x,y)\in\mathbb{R}^n:|y|<\tau(x_1)\}\,,$$

is an isoperimetric set in $X(m)$, then τ is locally absolutely continuous on $\mathbb{R}$.

Proof. Let τ^+ and τ^- denote the right continuous and the left continuous representatives of τ. By Theorem 3.3, the set

$$\partial E\setminus\{(x,y)\in\mathbb{R}^n:y=0\}=\{(x,y)\in\mathbb{R}^n:\,\tau^-(x_1)\le|y|\le\tau^+(x_2)\}\setminus\{y=0\}\,,$$

is an analytic $(n-1)$-dimensional manifold in $\mathbb{R}^n$. Hence

$$M=\{(s,t)\in\mathbb{R}^2:\,t>0\,,\tau^-(s)\le t\le\tau^+(s)\}\,,\tag{3.6}$$

is a connected, analytic 1-dimensional manifold in $\mathbb{R}^2$ (the coordinates (s,t) of $\mathbb{R}^2$ refer to the canonical basis $\{e_1,e_2\}$ of $\mathbb{R}^2$). It is immediately seen that τ is continuous. Indeed if $\tau^-(s)<\tau^+(s)$ for some $s\in\mathbb{R}$ then M would contain a relatively open vertical segment passing through $(s,\tau(s))$. The analyticity and connectedness of M would then force M to be a (possibly larger) vertical segment, against the fact that, by (3.6), the horizontal projection of M agrees with the non-empty, open half-line $\{s\in\mathbb{R}:\tau^+(s)>0\}$. Thus, τ is continuous and

$$M=\{(s,t)\in\mathbb{R}^2:0<\tau(s)=t\}\,.$$

Let us now prove that the distributional derivative $D\tau$ of τ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. By analyticity we known that for every $(s,t)\in M$ there exists an orthonormal basis $\{v_1,v_2\}$ of $\mathbb{R}^2$, $r>0$, and an analytic function $g:(-r,r)\to\mathbb{R}$ such that the curve $\gamma:(-r,r)\to\mathbb{R}^2$ defined by

$$\gamma(u)=(s,t)+u\,v_1+g(u)\,v_2\,,\qquad |u|<r\,,$$

gives a bijection between $(-r,r)$ and a neighborhood of (s,t) in M. By repeating the argument used in showing the continuity of τ we see that the horizontal projection $\{\gamma(u)-(\gamma(u)\cdot e_2)e_2:|u|<r\}$ of the curve $\{\gamma(u):|u|<r\}$ coincides with a neighborhood of s, that we denote by $(s-\varepsilon,s+\varepsilon)$. We are now going to prove that τ is absolutely continuous on $(s-\varepsilon,s+\varepsilon)$. If $e_2=\pm v_2$, then τ is analytic, and there is nothing to prove. Otherwise, there exists $\kappa\in\mathbb{R}$ such that

$$e_2\text{ is parallel to }v_1+\kappa v_2\,.$$

Since g' is analytic on $(-r,r)$, the set $I=(g')^{-1}\{\kappa\}\subset(-r,r)$ is finite (again, if this were not the case, then the whole M would be a vertical segment). Therefore

$$\gamma'(u)=v_1+g'(u)v_2\,,$$

is parallel to e_2 if and only if $u\in I$, with I finite. The horizontal projection of $\{\gamma(u):u\in I\}$ is a finite subset J of $(s-\varepsilon,s+\varepsilon)$, with the property that τ is (classically) differentiable at every point in $(s-\varepsilon,s+\varepsilon)\setminus J$. As a consequence the singular part $D_S\tau$ of $D\tau$ is concentrated in the finite set J. Hence, by [1, Theorem 3.28], $D_S\tau$ is purely atomic. Since atoms in $D\tau$ correspond to jumps discontinuities of τ, and the presence of the latter has been already ruled out, we conclude that $D_S\tau=0$ on $(s-\varepsilon,s+\varepsilon)$, as required. □

We are finally ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. The theorem is an immediate corollary of Theorem 3.1, Theorem 3.3 and Corollary 3.4. □

4. Stationarity and stability

Given a set of locally finite perimeter E, we can consider a *volume-preserving variation of E* $\{\Phi_t\}_{|t|<\varepsilon}$, i.e., a one-parameter family of smooth diffeomorphisms of $\mathbb{R}^n$ such that $\Phi_0(z) = z$ for every $z \in \mathbb{R}^n$ and $V_{mix}(\Phi_t(E)) = V_{mix}(E)$ whenever $|t| < \varepsilon$. By the area formula the function $t \mapsto P_{mix}(\Phi_t(E))$ is smooth in a neighborhood of $t = 0$. We say that E is stationary (with respect to volume-preserving variations) if

$$\frac{d}{dt} P_{mix}(\Phi_t(E))\Big|_{t=0} = 0\,, \tag{4.1}$$

and we say that E is stable (with respect to volume-preserving variations) if it is stationary and

$$\frac{d^2}{dt^2} P_{mix}(\Phi_t(E))\Big|_{t=0} \geq 0\,. \tag{4.2}$$

Since the sets $\Phi_t(E)$ are sets of locally finite perimeter with $V_{mix}(\Phi_t(E)) = V_{mix}(E)$, it turns out that stability (and, in particular, stationarity) is a necessary condition for a set E to be an isoperimetric set.

4.1. **Stationary sets.**

We now turn to the study of the stationarity condition (4.1). As recalled in the introduction, if E is an open set with C^2-boundary, then this condition is equivalent to the Euler-Lagrange equation

$$H_E(z) - (x, 0) \cdot \nu_E(z) = \text{constant}\,, \qquad \forall z \in \partial E\,, \tag{4.3}$$

(see, e.g. [14, Proposition 3.2]). In light of Theorem 1.1, we are interested to stationary sets E satisfying (1.5), namely

$$E = \{(x, y) \in \mathbb{R}^n : |y| < \tau(x_1)\}\,, \tag{4.4}$$

for a non-negative, increasing, locally absolutely continuous function $\tau : \mathbb{R} \to [0, \infty)$. In this case (4.3) can be seen as a second order ODE that is solved by τ in the distributional sense. We begin our analysis with a detailed derivation of (4.3) formulated in terms of τ, in order to derive an explicit formula for the Lagrange multiplier appearing on the right hand side of (4.3).

Lemma 4.1 (Euler-Lagrange equation)**.** *Let $m > 0$ and let E be an isoperimetric set with $V_{mix}(E) = m$ satisfying* (4.4)*, for a non-negative, increasing, locally absolutely continuous function $\tau : \mathbb{R} \to [0, \infty)$. Let*

$$s_0 = \inf\{s \in \mathbb{R} : \tau(s) > 0\} \in [-\infty, \infty)\,,$$

so that $\{\tau > 0\} = (s_0, \infty)$, and define two Borel functions $\sigma : (s_0, \infty) \to [0, 1]$ and $\kappa : (s_0, \infty) \to (0, \infty)$ by setting

$$\sigma(s) = \frac{\tau'(s)}{\sqrt{1 + \tau'(s)^2}}\,, \qquad s > s_0\,, \tag{4.5}$$

$$\kappa(s) = \frac{k - 1}{\tau(s)\sqrt{1 + \tau'(s)^2}}\,, \qquad s > s_0\,. \tag{4.6}$$

Then there exists a positive constant λ such that σ is a weak solution of the ODE

$$-\sigma' + \kappa + s\sigma = \lambda\,, \tag{4.7}$$

on (s_0,∞). *Moreover, the Lagrange multiplier* λ *is characterized as*

$$\lambda = \frac{1}{m}\left\{\left(1-\frac{1}{k}\right) P_{mix}(E) + \frac{1}{k}\,\frac{1}{(2\pi)^{h/2}}\int_{\partial E} e^{-\frac{|x|^2}{2}}(\nu_E\cdot e_1)^2\,d\mathcal{H}^{n-1}\right\}. \tag{4.8}$$

In particular, we always have

$$\frac{k-1}{k}\,\frac{\Lambda(m)}{m} \le \lambda \le \frac{\Lambda(m)}{m}. \tag{4.9}$$

Remark 4.2. If $\tau \in C^2(\mathbb{R})$ and it is positive on an interval $I \subset \mathbb{R}$, then we can define a k-dimensional C^2-manifold M in $\mathbb{R}\times\mathbb{R}^k$ by setting

$$M = \{(s,y)\in\mathbb{R}\times\mathbb{R}^k : s\in I\,, |y| = \tau(s)\}\,.$$

Denoting by $\kappa_1,\dots,\kappa_k$ the principal curvatures of M, it is easily seen that

$$\kappa_1 = -\left(\frac{\tau'(s)}{\sqrt{1+\tau'(s)^2}}\right)' = -\frac{\tau''(s)}{(1+\tau'(s)^2)^{3/2}}\,,$$

$$\kappa_2 = \dots = \kappa_k = \frac{1}{\tau(s)\sqrt{1+\tau'(s)^2}}\,, \qquad \text{(when } k\ge 2\text{)}.$$

In particular, if H_M denotes the mean curvature of M, then we have

$$H_M = -\left(\frac{\tau'(s)}{\sqrt{1+\tau'(s)^2}}\right)' + \frac{k-1}{\tau(s)\sqrt{1+\tau'(s)^2}}\,.$$

Therefore we recognize in (4.7) the Euler-Lagrange equation (4.3) in cylindrical coordinates.

Proof of Lemma 4.1. By Lemma 2.10 and the claim appearing in the proof of Theorem 1.1, we see that τ is in turn a minimizer in the one-dimensional variational problem

$$\inf\left\{\mathcal{P}(\tau) : \tau\in BV_{loc}(\mathbb{R};[0,\infty)),\ \mathcal{V}(\tau) = m\right\}\,,$$

where $\mathcal{V}(\tau)$ and $\mathcal{P}(\tau)$ are defined as in (2.10) and (2.11). We now proceed as follows.

Step I. Derivation of (4.7).
Let $\psi\in C^\infty_c(\mathbb{R})$ with spt$\psi \subset\subset (s_0,\infty)$. Since $\{\tau>0\} = (s_0,\infty)$, we can define a bounded test function $\varphi\in W^{1,1}(\mathbb{R};[0,\infty))$ with spt$\varphi\subset\subset(s_0,\infty)$ by setting

$$\varphi(s) = \frac{e^{\frac{s^2}{2}}\psi(s)}{\tau(s)^{k-1}}\,, \qquad \text{if } s>s_0\,,$$

and $\varphi(s) = 0$ otherwise. Moreover the existence of $\varepsilon>0$ such that $\tau+t\varphi\ge 0$ on $\mathbb{R}$ for every $|t|<\varepsilon$ is easily proved. For every $|t|<\varepsilon$ we define $\alpha(t)>0$ by solving

$$m = \mathcal{V}(\alpha(t)(\tau+t\varphi))\,,$$

namely,

$$\alpha(t)^k\int_{\mathbb{R}} e^{-\frac{s^2}{2}}(\tau+t\varphi)^k\,ds = \int_{\mathbb{R}} e^{-\frac{s^2}{2}}\tau^k\,ds\,.$$

In particular $\alpha(t)$ is a smooth function of t, with

$$\alpha(0) = 1\,, \qquad \alpha'(0) = -\frac{\int_{\mathbb{R}} e^{-\frac{s^2}{2}}\tau^{k-1}\varphi}{\int_{\mathbb{R}} e^{-\frac{s^2}{2}}\tau^k}\,. \tag{4.10}$$

The minimality of τ implies that the function

$$\beta(t)=\mathcal{P}\big(\alpha(t)(\tau+t\varphi)\big)=\int_{\mathbb{R}} e^{-\frac{s^2}{2}}\alpha(t)^{k-1}(\tau+t\,\varphi)^{k-1}\sqrt{1+\alpha(t)^2(\tau'+t\varphi')^2}\,,\qquad |t|<\varepsilon\,,$$

has a minimum at $t=0$. By taking into account that $\alpha(0)=1$ we thus find

$$\begin{aligned}0=\beta'(0)=&\int_{\mathbb{R}}(k-1)\alpha'(0)e^{-\frac{s^2}{2}}\tau^{k-1}\sqrt{1+(\tau')^2}\\ &+\int_{\mathbb{R}}(k-1)e^{-\frac{s^2}{2}}\tau^{k-2}\varphi\sqrt{1+(\tau')^2}+\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\frac{\alpha'(0)(\tau')^2}{\sqrt{1+(\tau')^2}}+\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\varphi'\sigma\,,\end{aligned}$$

where σ has been defined in (4.5). By (4.10), we can gather the first and the third integral and introduce a positive factor $\lambda(\tau)$ such that

$$0=-\lambda(\tau)\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\varphi+(k-1)\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-2}\varphi\sqrt{1+(\tau')^2}+\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\varphi'\sigma\,.$$

Since $\psi=e^{-\frac{s^2}{2}}\tau^{k-1}\varphi$ and

$$\psi'=-s\psi+e^{-\frac{s^2}{2}}\tau^{k-1}\varphi'+\frac{(k-1)\tau'}{\tau}\,\psi\,,$$

we conclude that

$$0=\int_{\mathbb{R}}\left((k-1)\frac{\sqrt{1+(\tau')^2}}{\tau}-\lambda(\tau)\right)\psi+\int_{\mathbb{R}}\sigma\left(\psi'+s\psi-\frac{(k-1)\tau'}{\tau}\,\psi\right)\,,$$

i.e.

$$\int_{\mathbb{R}}\sigma\psi'=\int_{\mathbb{R}}\psi\left(\lambda-s\sigma-\frac{(k-1)}{\tau\sqrt{1+(\tau')^2}}\right)\,,$$

which, recalling (4.6), corresponds to (4.7).

Step II. Derivation of (4.8).
A quick inspection of the above argument shows that λ was defined so to satisfy

$$-\lambda(\tau)\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\varphi=(k-1)\alpha'(0)\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\sqrt{1+(\tau')^2}+\alpha'(0)\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\frac{(\tau')^2}{\sqrt{1+(\tau')^2}}\,.$$

By (4.10), (2.10) and (2.11), we thus find

$$\begin{aligned}\lambda(\tau)&=(k-1)\frac{\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\sqrt{1+(\tau')^2}}{\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^k}+\frac{\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\frac{(\tau')^2}{\sqrt{1+(\tau')^2}}}{\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^k}\\&=\left(1-\frac{1}{k}\right)\frac{\mathcal{P}(\tau)}{\mathcal{V}(\tau)}+\frac{\frac{k\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\frac{(\tau')^2}{\sqrt{1+(\tau')^2}}}{k\mathcal{V}(\tau)}\,.\end{aligned}$$

Since

$$(\nu_E\cdot e_1)^2=\frac{(\tau')^2}{1+(\tau')^2}\,,$$

by an application of the coarea formula we finally get that

$$\frac{k\omega_k}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\tau^{k-1}\frac{(\tau')^2}{\sqrt{1+(\tau')^2}}=\frac{1}{(2\pi)^{h/2}}\int_{\partial E}e^{-\frac{|x|^2}{2}}(\nu_E\cdot e_1)^2d\mathcal{H}^{n-1}\,.$$

Hence (4.8) is proved, and (4.9) follows simply by noticing that

$$0 \leq \frac{1}{(2\pi)^{h/2}} \int_{\partial E} e^{-\frac{|x|^2}{2}} (\nu_E \cdot e_1)^2 d\mathcal{H}^{n-1} \leq \mathcal{P}(\tau) .$$

□

Remark 4.3. Assume now to know that for every $m \in (0, m_0)$ there exists a unique (up to vertical translations and horizontal rotations) isoperimetric set E_m with $V_{mix}(E_m) = m$, and that $\Lambda(m) = P_{mix}(E_m)$ is absolutely continuous on $(0, m_0)$. Then we would have $\lambda(E_m) = \Lambda'(m)$ for a.e. $m \in (0, m_0)$. Correspondingly we would deduce from (4.9) that

$$\left(\frac{m}{m_0}\right) \Lambda(m_0) \leq \Lambda(m) \leq \left(\frac{m}{m_0}\right)^{(k-1)/k} \Lambda(m_0) ,$$

for every $m \in (0, m_0)$.

4.2. **Isoperimetric sets in the case $k = 1$.** We now turn to a more detailed study of the case $k = 1$, in which the Euler-Lagrange equation (4.7) can be explicitly solved. Let us recall that a family of functions $\{\tau_{s_0}\}_{s_0 \in \mathbb{R}}$ was introduced in Remark 1.5 by setting

$$\begin{aligned} \tau_{s_0}(s) &= 0 & s \leq s_0 , \\ \tau'_{s_0}(s) &= \frac{\zeta(s)}{\sqrt{\zeta(s_0)^2 - \zeta(s)^2}} & s > s_0 , \end{aligned} \tag{4.11}$$

where $\zeta : \mathbb{R} \to (0, \infty)$ is given by

$$\zeta(s) = e^{\frac{s^2}{2}} \int_s^\infty e^{-\frac{t^2}{2}} dt , \quad s \in \mathbb{R} .$$

The role of the family $\{\tau_{s_0}\}_{s_0 \in \mathbb{R}}$ is clarified by the following lemma.

Lemma 4.4 (An alternative for isoperimetric sets)**.** *Let $k = 1$, and let E be an isoperimetric set with*

$$E = \{(x, y) \in \mathbb{R}^n : |y| < \tau(x_1)\} ,$$

for a non-negative, increasing, locally absolutely continuous function $\tau : \mathbb{R} \to (0, \infty)$. Let s_0 and σ be defined starting from τ as in Lemma 4.1. Then the following holds:

(i) *if $s_0 = -\infty$, then τ is constant and E is a cylinder, i.e. $E = K_r$ for some $r > 0$;*
(ii) *if $s_0 \in \mathbb{R}$, then $\tau = \tau_{s_0}$ and E solves* (4.3) *with the Lagrange multiplier*

$$\lambda = \frac{1}{\zeta(s_0)} . \tag{4.12}$$

Proof. <u>*Step I.*</u> *The case $s_0 = -\infty$.*
If $s_0 = -\infty$, then

$$\mathcal{P}(\tau) = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}} \sqrt{1 + \tau'(s)^2} ds \geq \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}} ds = 2 = \mathcal{P}(r) ,$$

where $r > 0$ is the positive constant such that $\mathcal{V}(r) = \mathcal{V}(\tau)$. Since the inequality is strict unless $\tau' = 0$ a.e. on $\mathbb{R}$, we deduce that if $s_0 = -\infty$ then $\tau = r$ on $\mathbb{R}$, hence E is a cylinder.

<u>*Step II.*</u> *The case $s_0 > -\infty$.*
If $s_0 > -\infty$, then $(s_0\, e_1, 0) \in \partial E$. Since $k = 1 < 7$, ∂E is analytic. In particular, ∂E admits a tangent plane at $(s_0\, e_1, 0)$, that, by symmetry, must be orthogonal to e_1. Thus it must be

$\tau'(s_0^+) = +\infty$, and in particular by (4.5) we find $\sigma(s_0) = 1$. By Lemma 4.1, and the fact that $\kappa = 0$ if $k = 1$, we find that σ is weak solution to the Cauchy problem

$$\begin{cases} -\sigma' + s\,\sigma = \lambda\,, & \text{on } (s_0,\infty)\,, \\ \sigma(s_0) = 1\,. \end{cases}$$

Solving the linear ODE we find that

$$\sigma(s) = Ce^{\frac{s^2}{2}} + \lambda\zeta(s)\,, \qquad s > s_0\,,$$

for some $C \in \mathbb{R}$. Since we know a priori that $0 \le \sigma \le 1$ and since, as a consequence of (3.1), $\zeta(s) \to 0$ as $s \to +\infty$, we deduce from this identity that it must be $C = 0$. Thus

$$\sigma(s) = \lambda\zeta(s)\,, \qquad s > s_0\,.$$

From the boundary condition $\sigma(s_0) = 1$ we find

$$\sigma(s) = \frac{\zeta(s)}{\zeta(s_0)}\,, \qquad s > s_0\,,$$

and immediately deduce that

$$\tau'(s) = \frac{\zeta(s)}{\sqrt{\zeta(s_0)^2 - \zeta(s)^2}}\,, \qquad \forall s > s_0\,,$$

so that $\tau = \tau_{s_0}$ by definition of τ_{s_0}. □

We now collect some basic properties of the functions $\{\tau_{s_0}\}_{s_0\in\mathbb{R}}$. For the sake of brevity it is convenient at this point to define $M : \mathbb{R} \to (0, \sqrt{2\pi})$, by setting

$$M(s) = \int_s^\infty e^{-\frac{t^2}{2}}\,dt\,, \qquad s \in \mathbb{R}\,.$$

Clearly M is strictly decreasing, with $M(-\infty) = \sqrt{2\pi}$ and $M(+\infty) = 0$. The upper bound (3.1) takes the form

$$M(s) < \frac{e^{-\frac{s^2}{2}}}{s}\,, \qquad \forall s > 0\,. \tag{4.13}$$

We shall also use the lower bound

$$M(s) > \frac{e^{-\frac{s^2}{2}}}{s + (1/s)}\,, \qquad \forall s > 0\,. \tag{4.14}$$

To prove (4.14), let $F(s)$ denote the difference between the left and the right hand side of (4.14). Then it is easily seen that $F(0) > 0$, $F(+\infty) = 0$ and that $F'(s) < 0$ for $s > 0$. Therefore it must be $F > 0$ on $(0,\infty)$, as claimed.

Lemma 4.5 (Properties of τ_{s_0})**.** *For every $s_0 \in \mathbb{R}$, the function τ_{s_0} is strictly increasing and strictly concave on $[s_0,\infty)$ with $\tau'_{s_0}(s_0^+) = +\infty$ and with*

$$\lim_{s\to s_0^+} \frac{\tau_{s_0}(s)}{\sqrt{s - s_0}} = \sqrt{\frac{2\zeta(s_0)}{|\zeta'(s_0)|}}\,, \tag{4.15}$$

$$\lim_{s\to+\infty} s\zeta(s_0)\tau'_{s_0}(s) = 1\,. \tag{4.16}$$

In particular, for any $\varepsilon > 0$ one has

$$\frac{1-\varepsilon}{\zeta(s_0)}\log(s) \le \tau_{s_0}(s) \le \frac{1+\varepsilon}{\zeta(s_0)}\log(s)\,,$$

for s large enough (depending only on s_0 and on ε).

Proof. Step I. *Some properties of the function* ζ.
From (4.13) and (4.14) we see that

$$\frac{1}{s+(1/s)} < \zeta(s) < \frac{1}{s}, \qquad \forall s > 0, \tag{4.17}$$

while on the other hand we have

$$\sqrt{\frac{\pi}{2}}\, e^{\frac{s^2}{2}} < \zeta(s) < \sqrt{2\pi}\, e^{\frac{s^2}{2}}, \qquad \forall s < 0. \tag{4.18}$$

From (4.17) and (4.18) we clearly deduce

$$\lim_{s\to+\infty} \zeta(s) = 0, \qquad \lim_{s\to-\infty} \zeta(s) = +\infty,$$

that in fact are easily turned in the more precise form

$$\lim_{s\to+\infty} s\,\zeta(s) = 1, \tag{4.19}$$

$$\lim_{s\to-\infty} \sqrt{2\pi}\, e^{\frac{s^2}{2}} - \zeta(s) = 0,$$

with the aid of (4.17). Since

$$\zeta'(s) = -1 + s\zeta(s), \qquad \forall s \in \mathbb{R},$$

we see that $\zeta'(s) < 0$ by the upper bound in (4.17) if $s > 0$, and trivially if $s \le 0$. Similarly, as

$$\zeta''(s) = \zeta(s) + s\zeta'(s) = (1+s^2)\zeta(s) - s, \qquad \forall s \in \mathbb{R},$$

we find that $\zeta''(s) > 0$ by the lower bound in (4.17) if $s > 0$, and trivially if $s \le 0$. In conclusion, ζ is strictly decreasing and strictly convex on $\mathbb{R}$.

Step II. *Conclusions.*
By (4.11) we immediately see that τ_{s_0} is strictly increasing on $[s_0, \infty)$ with $\tau'_{s_0}(s_0^+) = +\infty$. Differentiating (4.11) we find

$$\tau''_{s_0}(s) = \zeta'(s)\,\frac{\zeta(s_0)^2}{(\zeta(s_0)^2 - \zeta(s)^2)^{3/2}}, \qquad \forall s > s_0.$$

Since ζ is strictly decreasing on $\mathbb{R}$, it turns out that τ_{s_0} is strictly concave on $[s_0, \infty)$. From

$$\left(\frac{\zeta(s_0)}{\zeta(s_0+t)}\right)^2 = 1 - \frac{2\zeta'(s_0)}{\zeta(s_0)}t + o(t),$$

we immediately find

$$\tau_{s_0}(s) = \sqrt{\frac{\zeta(s_0)}{2|\zeta'(s_0)|}} \int_0^{s-s_0} \frac{dt}{\sqrt{t+o(t)}},$$

by which we prove (4.15). Since,

$$s\zeta(s_0)\tau'_{s_0}(s) = \frac{s\zeta(s)}{\sqrt{1-(\zeta(s)/\zeta(s_0))^2}}$$

we immediately deduce (4.16) from (4.19).

□

Let us now define two functions $v, p : \mathbb{R} \to [0, \infty)$, by setting

$$v(s_0) = V_{mix}(E(s_0)), \qquad p(s_0) = P_{mix}(E(s_0)), \qquad s_0 \in \mathbb{R}.$$

In the next lemma we establish some crucial properties of these functions.

Lemma 4.6 (Properties of v and p)**.** *The functions v and p are analytic on $\mathbb{R}$, with*

$$p(s_0) \geq \sqrt{\frac{2}{\pi}} M(s_0)\,, \qquad \forall s_0 \in \mathbb{R}\,, \tag{4.20}$$

$$\lim_{s_0 \to +\infty} p(s_0) = 0\,, \tag{4.21}$$

$$\lim_{s_0 \to -\infty} p(s_0) = 2\,, \tag{4.22}$$

$$\lim_{s_0 \to +\infty} v(s_0) = 0\,, \tag{4.23}$$

$$\lim_{s_0 \to -\infty} v(s_0) = 0\,. \tag{4.24}$$

Moreover, p is strictly decreasing on the half-line $[\sqrt{3/2}, \infty)$.

Proof. *Step I. A formula for v and p.*
In this first step we prove that, for every $s_0 \in \mathbb{R}$,

$$v(s_0) = \sqrt{\frac{2}{\pi}} \int_{s_0}^{\infty} \frac{\zeta(s)\, M(s)}{\sqrt{\zeta(s_0)^2 - \zeta(s)^2}}\, ds\,, \tag{4.25}$$

$$p(s_0) = \sqrt{\frac{2}{\pi}} \int_{s_0}^{\infty} \frac{\zeta(s_0)\, e^{-\frac{s^2}{2}}}{\sqrt{\zeta(s_0)^2 - \zeta(s)^2}}\, ds\,. \tag{4.26}$$

We notice that (4.20) follows from (4.26), and that (4.26) is in turn an immediate consequence of (2.11) and (4.11). From (2.10) we see that

$$\begin{aligned} v(s_0) = \mathcal{V}(\tau_{s_0}) &= \sqrt{\frac{2}{\pi}} \int_{s_0}^{\infty} \tau_{s_0}(s) e^{-\frac{s^2}{2}}\, ds \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left(-\tau_{s_0} M\right)\Big|_{s_0}^{+\infty} + \int_{s_0}^{\infty} \tau'_{s_0}(s) M(s)\, ds \right\}. \end{aligned}$$

Since $\tau_{s_0}(s)$ behaves like $\log(s)$ as $s \to \infty$ and since by (3.1) we have

$$0 \leq \tau_{s_0}(s)\, M(s) \leq \tau_{s_0}(s) \frac{e^{-\frac{s^2}{2}}}{s}\,,$$

we conclude that $\tau_{s_0}(s) M(s) \to 0$ as $s \to \infty$, and thus we prove (4.25).

Step II. The estimate (4.27) for ζ'.
As a direct consequence of (4.17) and of the equality $\zeta'(s) = -1 + s\zeta(s)$ we know that

$$|\zeta'(s)| \leq \frac{1}{s^2}\,, \qquad \forall s > 0\,.$$

Let us now show that, in fact, $|\zeta'(s)|$ does not tend to zero too quickly. More precisely, we prove that for every $\lambda \in (0,1)$ there exists $\varepsilon(\lambda) > 0$ such that

$$|\zeta'(s)| \geq \frac{\lambda}{6s^2}\,, \qquad \forall s > \frac{1}{\varepsilon(\lambda)}\,. \tag{4.27}$$

It suffices to chose $\varepsilon(\lambda)$ such that

$$e^{-\frac{w^2}{2}} \leq 1 - \frac{\lambda}{2} w^2\,, \qquad \forall |w| < \varepsilon(\lambda)\,. \tag{4.28}$$

Indeed, starting from the identity $e^{-\frac{s^2}{2}} = \int_s^\infty te^{-\frac{t^2}{2}}dt$, we find that

$$\begin{aligned}|\zeta'(s)| = 1 - s\zeta(s) &= e^{\frac{s^2}{2}} \int_s^\infty (t-s)e^{-\frac{t^2}{2}}\,dt \\ &= e^{\frac{s^2}{2}} \int_0^\infty we^{-\frac{(w+s)^2}{2}}\,dw = \int_0^\infty e^{-sw}\,we^{-\frac{w^2}{2}}\,dw \\ &\geq \int_0^{1/s} (1-sw)\,we^{-\frac{w^2}{2}}\,dw = 1 - e^{\frac{1}{2s^2}} - s\int_0^{1/s} w^2e^{-\frac{w^2}{2}}\,dw\,.\end{aligned} \tag{4.29}$$

An integration by parts reveals that

$$\int_0^{1/s} w^2e^{-\frac{w^2}{2}}\,dw = (-we^{-\frac{w^2}{2}})|_0^{1/s} + \int_0^{1/s} e^{-\frac{w^2}{2}}\,dw = -\frac{e^{\frac{1}{2s^2}}}{s} + \int_0^{1/s} e^{-\frac{w^2}{2}}\,dw \tag{4.30}$$

so that, by (4.29) and (4.30), we conclude

$$|\zeta'(s)| \geq 1 - s\int_0^{1/s} e^{-\frac{w^2}{2}}\,dw\,. \tag{4.31}$$

Combining (4.28) with (4.31) we come to (4.27).

Step III. Proof of (4.21) *and* (4.23).
Since ζ is strictly decreasing, for every $s_0 \in \mathbb{R}$ and $t \in (0,1)$ there exists a unique $F(s_0,t) > s_0$ such that

$$\zeta(F(s_0,t)) = t\,\zeta(s_0)\,.$$

Since ζ is analytic, with $\zeta' < 0$ everywhere, the Lagrange inversion theorem ensures that F is an analytic function of (s_0,t) on $\mathbb{R}\times(0,1)$, with

$$\frac{\partial F(s_0,t)}{\partial t} = \frac{\zeta(s_0)}{\zeta'(F(s_0,t))}\,, \tag{4.32}$$

$$\frac{\partial F(s_0,t)}{\partial s_0} = \frac{t\,\zeta'(s_0)}{\zeta'(F(s_0,t))}\,. \tag{4.33}$$

By the change of variable $s = F(s_0,t)$, by (4.32), (4.25) and (4.26), we find that

$$\sqrt{\frac{\pi}{2}}\,v(s_0) = \zeta(s_0)\int_0^1 \frac{M(F(s_0,t))}{|\zeta'(F(s_0,t))|}\frac{t\,dt}{\sqrt{1-t^2}}\,, \tag{4.34}$$

$$\sqrt{\frac{\pi}{2}}\,p(s_0) = \zeta(s_0)\int_0^1 \frac{e^{-\frac{F(s_0,t)^2}{2}}}{|\zeta'(F(s_0,t))|}\frac{dt}{\sqrt{1-t^2}}\,, \tag{4.35}$$

(note that the analyticity of v and p follows immediately from (4.34) and (4.35)). Let us fix $\lambda \in (0,1)$, and let $\varepsilon(\lambda) > 0$ be such that (4.27) holds true. Up to decrease the value of $\varepsilon(\lambda)$, we can also assume that the functions $s \mapsto s^2e^{-\frac{s^2}{2}}$ and $s \mapsto se^{-\frac{s^2}{2}}$ are decreasing on the half-line $(\varepsilon(\lambda)^{-1},\infty)$. If $s_0 > \varepsilon(\lambda)^{-1}$, then, by $F(s_0,t) > s_0$, (4.13) and (4.27), we find that

$$\frac{M(F(s_0,t))}{|\zeta'(F(s_0,t))|} \leq \frac{6F(s_0,t)e^{-\frac{F(s_0,t)^2}{2}}}{\lambda} \leq \frac{6s_0e^{-\frac{s_0^2}{2}}}{\lambda}\,,$$

$$\frac{e^{-\frac{F(s_0,t)^2}{2}}}{|\zeta'(6F(s_0,t))|} \leq \frac{F(s_0,t)^2e^{-\frac{F(s_0,t)^2}{2}}}{\lambda} \leq \frac{6s_0^2e^{-\frac{s_0^2}{2}}}{\lambda}\,,$$

so that, by (4.17),

$$\sqrt{\frac{\pi}{2}}\,v(s_0)\le\frac{6e^{-\frac{s_0^2}{2}}}{\lambda}\int_0^1\frac{t\,dt}{\sqrt{1-t^2}}\,,\qquad\sqrt{\frac{\pi}{2}}\,p(s_0)\le\frac{6s_0e^{-\frac{s_0^2}{2}}}{\lambda}\int_0^1\frac{dt}{\sqrt{1-t^2}}\,.$$

We let $s_0\to+\infty$ in these inequalities to prove (4.21) and (4.23).

<u>*Step IV.*</u> *Proof of* (4.22).
Since $M(-\infty)=\sqrt{2\pi}$, by (4.20) it will suffice to show that

$$\limsup_{s_0\to-\infty}p(s_0)\le 2\,. \tag{4.36}$$

To this end we notice that for every $\lambda\in(0,1)$ we have

$$\sqrt{\frac{\pi}{2}}\,p(s_0)\le\int_{s_0}^{F(s_0,\lambda)}\frac{e^{-\frac{s^2}{2}}}{\sqrt{1-(\zeta(s)/\zeta(s_0))^2}}\,ds+\frac{M(s_0)}{\sqrt{1-\lambda^2}}\,. \tag{4.37}$$

Using again the change of variables $s=F(s_0,t)$, and taking into account that $|\zeta'|=-\zeta'$ is decreasing, we find that

$$\begin{aligned}\int_{s_0}^{F(s_0,\lambda)}\frac{e^{-\frac{s^2}{2}}}{\sqrt{1-(\zeta(s)/\zeta(s_0))^2}}\,ds\;&=\;\zeta(s_0)\int_\lambda^1\frac{e^{-\frac{F(s_0,t)^2}{2}}}{|\zeta'(F(s_0,t))|\sqrt{1-t^2}}\,dt\\&\le\;M(s_0)\int_\lambda^1\frac{dt}{|\zeta'(F(s_0,t))|\sqrt{1-t^2}}\le\frac{M(s_0)}{|\zeta'(F(s_0,\lambda))|}\int_0^1\frac{dt}{\sqrt{1-t^2}}\,.\end{aligned}$$

As $M(-\infty)=\sqrt{2\pi}$, $\zeta'(-\infty)=+\infty$, and $s_0(\lambda)\to-\infty$ as $s_0\to-\infty$, we conclude that

$$\lim_{s_0\to-\infty}\int_{s_0}^{F(s_0,\lambda)}\frac{e^{-\frac{s^2}{2}}}{\sqrt{1-(\zeta(s)/\zeta(s_0))^2}}\,ds=0\,,\qquad\forall\lambda\in(0,1)\,.$$

Hence (4.36) follows by letting first $s_0\to-\infty$ and then $\lambda\to0^+$ in (4.37).

<u>*Step V.*</u> *Proof of* (4.24).
Since $x\mapsto(x^2-1)^{-1/2}$ is decreasing on $x>1$, we find that

$$\int_{s_0+1}^\infty\frac{\zeta(s)M(s)}{\sqrt{\zeta(s_0)^2-\zeta(s)^2}}\,ds\le\frac{\zeta(s_0+1)}{\sqrt{\zeta(s_0)^2-\zeta(s_0+1)^2}}\int_{s_0+1}^\infty M(s)\,ds\,. \tag{4.38}$$

If s_0 is negative and large enough, then by (4.13) we have

$$\int_{s_0+1}^\infty M(s)\,ds\le(1+|s_0|)\sqrt{2\pi}+\int_1^\infty M(s)ds\le(1+|s_0|)\sqrt{2\pi}+\int_1^\infty\frac{e^{-\frac{s^2}{2}}}{s}\,ds\le C\,|s_0|\,,$$

for a suitable constant C independent of s_0. Moreover, by (4.18),

$$\frac{\zeta(s_0+1)}{\zeta(s_0)}\le\frac{\sqrt{2\pi}e^{\frac{(s_0+1)^2}{2}}}{\sqrt{\pi/2}e^{\frac{s_0^2}{2}}}\le2e^{s_0+\frac12}\le C\,e^{s_0}\,,$$

at least up to increase the value of C, and again for s_0 negative and large enough. We combine the last two estimates with (4.38) to conclude that

$$\limsup_{s_0\to-\infty}\int_{s_0+1}^\infty\frac{\zeta(s)M(s)}{\sqrt{\zeta(s_0)^2-\zeta(s)^2}}\,ds\le C\lim_{s_0\to-\infty}|s_0|e^{s_0}=0\,.$$

Hence, taking (4.25) into account, in order to prove (4.24) we are left to show that

$$\lim_{s_0\to-\infty}\int_{s_0}^{s_0+1}\frac{\zeta(s)M(s)}{\sqrt{\zeta(s_0)^2-\zeta(s)^2}}\,ds=0\,. \tag{4.39}$$

From the very definition of $\zeta(s)$ we notice that, if $0>s>s_0$, then

$$\left(\frac{\zeta(s_0)}{\zeta(s)}\right)^2-1=e^{s_0^2-s^2}\left(\frac{M(s_0)}{M(s)}\right)^2-1\geq e^{s_0^2-s^2}-1\geq s_0^2-s^2\geq|s_0|(s-s_0)\,.$$

Thus

$$\int_{s_0}^{s_0+1}\frac{\zeta(s)M(s)}{\sqrt{\zeta(s_0)^2-\zeta(s)^2}}\,ds\leq\sqrt{\frac{2\pi}{|s_0|}}\int_{s_0}^{s_0+1}\frac{ds}{\sqrt{s-s_0}}=2\sqrt{\frac{2\pi}{|s_0|}}\,,$$

and (4.39) is proved.

Step VI. Conclusion.

We finally have only to prove that p is strictly decreasing on the half-line $[\sqrt{3/2},\infty)$. We first need to notice the following improvement of (4.17),

$$\zeta(s)<\frac{1}{s+(1/2s)}\,,\qquad\forall s>\sqrt{\frac{3}{2}}\,. \tag{4.40}$$

Indeed, let us define $f:\mathbb{R}\to\mathbb{R}$ by setting

$$f(s)=M(s)-\frac{2s}{2s^2+1}\,e^{-\frac{s^2}{2}}\,,\qquad s\in\mathbb{R}\,.$$

A simple computation shows that

$$f'(s)=\frac{e^{-\frac{s^2}{2}}}{(1+2s^2)^2}\,(2s^2-3)\,,\qquad s\in\mathbb{R}\,.$$

Thus $f'>0$ on $(\sqrt{3/2},\infty)$. Since $f(+\infty)=0$, we conclude that $f<0$ on $(\sqrt{3/2},\infty)$, thus proving (4.40). We now compute $p'(s_0)$ from (4.35), thus finding, also thanks to (4.33),

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\,p'(s_0)&=\zeta'(s_0)\int_0^1\frac{e^{-\frac{F^2}{2}}}{|\zeta'(F)|}\frac{dt}{\sqrt{1-t^2}}+\zeta(s_0)\int_0^1\frac{d}{dr}\left(\frac{e^{-\frac{r^2}{2}}}{|\zeta'(r)|}\right)\Big|_{r=F}\frac{\partial F}{\partial s_0}\frac{dt}{\sqrt{1-t^2}}\\&=\frac{\zeta'(s_0)}{\zeta(s_0)}\sqrt{\frac{\pi}{2}}p(s_0)+\zeta(s_0)\zeta'(s_0)\int_0^1\frac{d}{dr}\left(\frac{e^{-\frac{r^2}{2}}}{|\zeta'(r)|}\right)\Big|_{r=F}\frac{1}{\zeta'(F)}\frac{t\,dt}{\sqrt{1-t^2}}\,.\end{aligned}$$

Since ζ is strictly decreasing, the first term in the above sum is strictly negative for every $s_0\in\mathbb{R}$. Taking into account that $F=F(s_0,t)>s_0$ for every $s_0\in\mathbb{R}$ and $t\in(0,1)$, we are going to conclude that p is strictly decreasing on $[\sqrt{3/2},\infty)$ by showing that

$$\frac{d}{dr}\left(\frac{e^{-\frac{r^2}{2}}}{|\zeta'(r)|}\right)\leq0\,,\qquad\forall r>\sqrt{\frac{3}{2}}\,. \tag{4.41}$$

Indeed, since $\zeta'(r)=-1+r\zeta(r)$ and $\zeta''(r)=(1+r^2)\zeta(r)-r$, we find that

$$\frac{d}{dr}\left(\frac{e^{-\frac{r^2}{2}}}{|\zeta'(r)|}\right)=\frac{e^{-\frac{r^2}{2}}}{\zeta'(r)^2}\left(r\zeta'(r)+\zeta''(r)\right)=\frac{e^{-\frac{r^2}{2}}}{\zeta'(r)^2}\left((1+2r^2)\zeta(r)-2r\right)\,.$$

We thus deduce (4.41) from (4.40), and conclude the proof of the lemma. $\square$

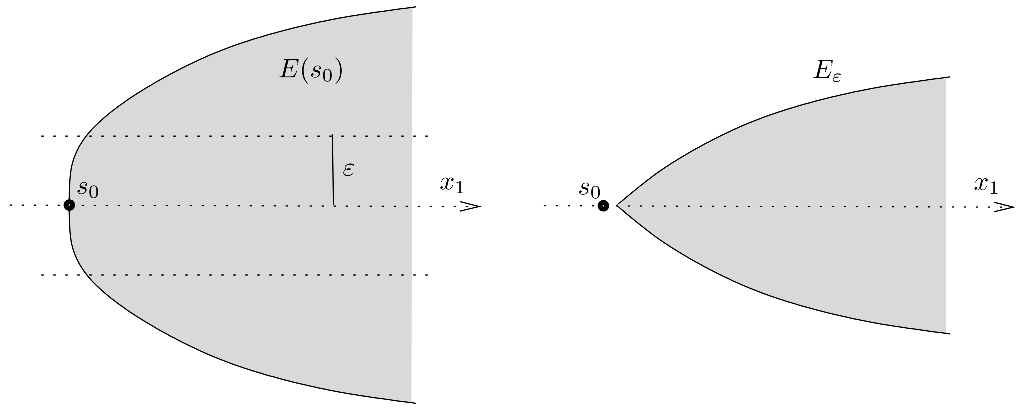

FIGURE 5. The set E_ε obtained from $E(s_0)$.

We can now conclude with the proof of Theorem 1.6.

Proof of Theorem 1.6. *Step I.* *Characterization of isoperimetric sets and some properties of* Λ. The isoperimetric function Λ defined in (1.4) is increasing and continuous, with

$$\lim_{m\to 0^+} \Lambda(m) = 0\,, \tag{4.42}$$

and with $0 \le \Lambda(m) \le 2$ for every $m > 0$ (indeed, there are cylinders of any given mixed-volume, and they all have mixed-perimeter equal to 2).

We now *claim* that, if for some $s_0 \in \mathbb{R}$ the set $E(s_0)$ is an isoperimetric set, then there exists $\delta \in (0, v(s_0))$ such that Λ is strictly increasing on $(v(s_0) - \delta, v(s_0))$. Indeed in this case we may define a comparison set

$$E_\varepsilon = \{(x, y) \in \mathbb{R}^h \times \mathbb{R} : \, |y| < \max\{\tau_{s_0}(x_1) - \varepsilon, 0\}\}\,, \qquad \varepsilon > 0\,,$$

which is obtained first by "cutting" a tiny horizontal slice from $E(s_0)$, and then by gluing together the two remaining pieces, see Figure 5. It is immediate to observe that

$$V_{mix}(E_\varepsilon) < v(s_0)\,, \qquad\qquad P_{mix}(E_\varepsilon) < p(s_0)\,,$$

with

$$\lim_{\varepsilon\to 0^+} V_{mix}(E_\varepsilon) = v(s_0) \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} P_{mix}(E_\varepsilon) = p(s_0)\,.$$

Therefore there exists $\delta \in (0, v(s_0))$ such that $\Lambda(m) < p(s_0) = \Lambda(v(s_0))$ for every $m \in (v(s_0) - \delta, v(s_0))$, i.e. Λ is strictly increasing on $(v(s_0) - \delta, v(s_0))$.

We now argue as follows. Let E be an isoperimetric set with $V_{mix}(E) = m$. By Theorem 1.1 and by Lemma 4.4, up to a vertical translation and a horizontal rotation we may assume that

$$E = \{(x, y) \in \mathbb{R}^h \times \mathbb{R} : |y| < \tau(x_1)\}\,, \tag{4.43}$$

where either τ is constant (and hence E is a cylinder) or $\tau = \tau_{s_0}$ for some $s_0 \in \mathbb{R}$. In the former case $\Lambda(m) = 2$, and this is excluded by (4.42) whenever m is small enough. Hence

$$m_0 = \sup\{m > 0 : \text{isoperimetric sets of mixed-volume } m \text{ are not cylinders}\} \in (0, \infty]\,.$$

By (4.23) and (4.24), the set $\{v(s_0) : s_0 \in \mathbb{R}\}$ is a bounded interval, therefore we have in fact $m_0 < \infty$. By construction, $\Lambda(m) < 2$ for every $m < m_0$, and statement (iii) is proved. In particular, by our claim, Λ is strictly increasing on $(0, m_0)$. By continuity, $\Lambda(m_0) = 2$. Since Λ is increasing and $0 \le \Lambda \le 2$, we conclude that

$$[m_0, \infty) = \{m > 0 : \Lambda(m) = 2\} . \tag{4.44}$$

By (4.44), and again by our claim, we see that if $m > m_0$ then the only isoperimetric sets are cylinders, and thus prove statement (i). In order to prove statement (ii) we are left to show the existence of $s_0 \in \mathbb{R}$ such that $E(s_0)$ is an isoperimetric set with $v(s_0) = m_0$. Indeed, let $\{m_h\}_{h\in\mathbb{N}}$ be a sequence with $m_h \to m_0^-$. By the above arguments, there exists a sequence $\{s_h\}_{h\in\mathbb{N}}$ such that $m_h = v(s_h)$ and $p(s_h) = \Lambda(m_h) \to \Lambda(m_0) = 2$. Since $p(s_h) \to 2$, by (4.21), s_h is bounded from above. Since $v(s_h) \to m_0 > 0$, by (4.24), s_h is bounded from below. Hence, up to extract a not-relabeled subsequence, we may assume that $s_h \to s_0$ for some $s_0 \in \mathbb{R}$. By continuity of Λ,

$$\Lambda(m_0) = \lim_{h\to\infty} \Lambda(m_h) = \lim_{h\to\infty} p(s_h) = p(s_0) ,$$

and thus $E(s_0)$ is an isoperimetric set with mass m_0.

<u>*Step II.*</u> *A strict concavity property of* Λ.
We start showing that if I is an open interval such that $\Lambda(v(s)) = p(s)$ for $s \in I$, and p is strictly decreasing on I, then v is strictly decreasing on I, and Λ is analytic, strictly increasing and strictly concave on $J = \{v(s) : s \in I\}$, with

$$\Lambda'(v(s)) = \frac{1}{\zeta(s)} , \qquad \forall s \in I . \tag{4.45}$$

Indeed, if $(s_1, s_2) \subset I$, then $\Lambda(v(s_1)) = p(s_1) > p(s_2) = \Lambda(v(s_2))$. Since Λ is increasing, it must be $v(s_1) > v(s_2)$, i.e. v is strictly decreasing on I and, in particular, by the Lagrange inversion theorem, $\Lambda = p \circ v^{-1}$ is analytic on J. Let $s \in I$ and let us define a one-parameter family of diffeomorphisms by setting

$$\Phi_t(x) = x + t\,\varphi(x)N(x) , \qquad x \in \mathbb{R}^n ,$$

where N is a smooth extension of $\nu_{E(s)}$ to an open neighborhood A of $\partial E(s)$, and where $\varphi \in C^\infty(\mathbb{R}^n; [0, 1])$ with $\varphi = 1$ on $\partial E(s)$ and with $\mathrm{spt}\varphi \subset A$. By a standard argument and by (4.12) we see that

$$V_{mix}(\Phi_t(E(s))) = v(s) + t\,p(s) + o(t) , \qquad P_{mix}(\Phi_t(E(s))) = p(s) + t\frac{p(s)}{\zeta(s)} + o(t) . \tag{4.46}$$

If we now set

$$f(t) = \Lambda(V_{mix}(\Phi_t(E(s)))) , \qquad g(t) = P_{mix}(\Phi_t(E(s))) ,$$

then we have $f(t) \le g(t)$ in a neighborhood of $t = 0$, with $f(0) = g(0)$. Since both f and g are smooth in a neighborhood of $t = 0$, we conclude that $f'(0) = g'(0)$, where, by (4.46),

$$f'(0) = \Lambda'(v(s))p(s) , \qquad g'(0) = \frac{p(s)}{\zeta(s)} .$$

This proves (4.45), from which we deduce

$$\Lambda''(v(s))v'(s) = -\frac{\zeta'(s)}{\zeta(s)^2} > 0 , \qquad \forall s \in I ,$$

so that, in particular, $\Lambda'' < 0$ on J.

Step III. Conclusion.
By Lemma 4.6, we know that p is strictly decreasing on the half-line $(\sqrt{3/2},\infty)$, and that $p(s)\ge\sqrt{2/\pi}\,M(s)$ for every $s\in\mathbb{R}$. Hence there exists $\varepsilon_*\in(0,2)$ such that,

$$I=\{s\in\mathbb{R}:p(s)<\varepsilon_*\}\,,$$

is a half-line, contained in $(\sqrt{3/2},\infty)$. Since Λ is increasing and continuous, with $\Lambda(m)=2$ if and only if $m\ge m_0$, and with $\Lambda(0^+)=0$, we see that

$$\{m>0:\Lambda(m)<\varepsilon_*\}=(0,m_1)\,,$$

for some $m_1\le m_0$. We infer from Lemma 4.4 that for every $m\in(0,m_1)$ there exists $s\in I$ such that $v(s)=m$ and $\Lambda(m)=p(s)$. By Step II, Λ is strictly concave on $(0,m_1)$. □

4.3. **Stability of cylinders.** As said, a necessary condition for E to be an isoperimetric set is that it satisfies the stability condition (4.2). When E is an open set with C^2-boundary, a standard argument (see for example [14, Lemma 3.8]) shows that E is stable if and only if the following Poincaré-type inequality holds true on the boundary of ∂E, namely

$$\int_{\partial E}|\nabla_{\partial E}u|^2\,e^v\,d\mathcal{H}^{n-1}\ge\int_{\partial E}\big(A_E^2+\nabla^2v(\nu_E,\nu_E)\big)\,u^2\,e^v\,d\mathcal{H}^{n-1}\,,$$

for every test function $u\in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\partial E}u\,e^v d\mathcal{H}^{n-1}=0$. Here $\nabla_{\partial E}u$ denotes the tangential gradient of u with respect to ∂E, and A_E^2 denotes the sum of the squares of the principal curvatures κ_i of ∂E, i.e. $A_E^2=\sum_{i=1}^{n-1}\kappa_i^2$. If we denote by $\mathbf{p}\,\nu_E$ the horizontal projection of ν_E (i.e. $\mathbf{p}(x,y)=x\in\mathbb{R}^h$ for every $(x,y)\in\mathbb{R}^n$), then in the mixed Euclidean-Gaussian case we see that this condition takes the form

$$\int_{\partial E}|\nabla_{\partial E}u|^2e^{-\frac{|x|^2}{2}}\,d\mathcal{H}^{n-1}\ge\int_{\partial E}\big(A_E^2+(\mathbf{p}\,\nu_E)^2\big)\,u^2\,e^{-\frac{|x|^2}{2}}\,d\mathcal{H}^{n-1}\,,\tag{4.47}$$

for every $u\in C_c^\infty(\partial E)$ such that

$$\int_{\partial E}u\,e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}=0\,.\tag{4.48}$$

Starting from (4.47) we can prove Theorem 1.8.

Proof of Theorem 1.8. If $\varphi\in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\varphi(s)ds=0\,,\tag{4.49}$$

and if we define $u\in C_c^\infty(K_r)$ by setting $u(z)=\varphi(x_1)$, $z\in\mathbb{R}^n$, then we find

$$\int_{\partial K_r}u\,e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}=(2\pi)^{(h-1)/2}\omega_{k-1}r^{k-1}\int_{\mathbb{R}}\varphi(s)e^{-\frac{s^2}{2}}ds=0\,,$$

i.e. u satisfies (4.48) with $E=K_r$. By taking into account that $A_{K_r}^2=(k-1)r^{-2}$, that $\mathbf{p}\,\nu_{K_r}=0$ on ∂K_r, and that

$$|\nabla_{\partial K_r}u(z)|^2=\varphi'(x_1)^2\,,\qquad\forall z\in\partial K_r\,,$$

we conclude that the cylinder K_r is stable if and only if

$$\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\varphi'(s)^2\,ds\ge\frac{k-1}{r^2}\int_{\mathbb{R}}e^{-\frac{s^2}{2}}\varphi(s)^2\,ds\,,\tag{4.50}$$

whenever φ satisfies (4.49). It is known that

$$\inf\left\{\frac{\int_{\mathbb{R}} e^{-\frac{s^2}{2}}\varphi'(s)^2\,ds}{\int_{\mathbb{R}} e^{-\frac{|s|^2}{2}}\varphi(s)^2\,ds} : \varphi \neq 0\,, \int_{\mathbb{R}} e^{-\frac{s^2}{2}}\varphi(s)\,ds = 0\right\} = 1\,.$$

Therefore we deduce from (4.50) that K_r is stable if and only if $(k-1) \leq r^2$, as required. □

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Dipartimento di Matematica ed Applicazioni, via Cintia, 80126 Napoli, Italy
E-mail address: `nicola.fusco@unina.it`

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy
E-mail address: `maggi@math.unifi.it`

Dipartimento di Matematica, Via Ferrata 1, 27100 Pavia, Italy
E-mail address: `pratelli@unipv.it`