



## The Relation Between Topological Ordering and Adjacency Matrix in Digraphs

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**Abstract.** In this paper the properties of node-node adjacency matrix in acyclic digraphs are considered. It is shown that topological ordering and node-node adjacency matrix are closely related. In fact, first the one to one correspondence between upper triangularity of node-node adjacency matrix and existence of directed cycles in digraphs is proved and then with this correspondence other properties of adjacency matrix in acyclic digraphs are presented.

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**Keywords:** adjacency matrix, topological ordering, acyclic digraph.

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### 1. Introduction

Let the nodes of a loopless digraph  $G = (N, A)$  be labeled by distinct numbers from 1 through  $n$  and the array order contains the labels [i.e., gives the label of node  $i$ ]. A labeling scheme is a topological ordering of nodes if every arc of  $G$  joins a lower-labeled node to a higher-labeled node. That is, for every arc  $(i, j) \in A$ ,  $order(i) < order(j)$ .

A digraph without any directed cycle is called an acyclic digraph. It has been shown that a digraph is acyclic if and only if it possesses a topological ordering [1]. We will show that topological ordering and node-node adjacency matrix are closely related.

### 2. Adjacency Matrix in Acyclic Digraphs

**DEFINITION 2.1** *A matrix is said strongly upper triangular if it is upper triangular with zero diagonal elements.*

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**THEOREM 2.2** *A digraph  $G = (N, A)$  has a topological ordering if and only if its node-node adjacency matrix can be transformed to a strongly upper triangular matrix.*

*Proof* To show sufficient condition, suppose  $\mathbf{H} = \{h_{ij}\}_{n \times n}$  is the node-node adjacency matrix of  $G$  and it is strongly upper triangular, that is:

$$h_{ij} = \begin{cases} 1 & (i, j) \in A \text{ \& } i < j \\ 0 & \text{otherwise} \end{cases}$$

Set  $order(i) = i$  ( $\forall i \in N$ ). This labeling is a topological ordering, because according to the hypothesis, for every  $(i, j) \in A$ ,  $i < j$  hence  $order(i) < order(j)$ .

To prove necessary condition, assume that the digraph has a topological ordering. It is sufficient to arrange the rows (and corresponding columns) of the node-node adjacency matrix according to this ordering. That is, if  $order(i) = k$  then the  $k^{th}$  row and column of the new matrix is corresponding to node  $i$ . The resulting matrix is called  $\bar{\mathbf{H}} = \{\bar{h}_{ij}\}_{n \times n}$ . In fact  $\bar{H}$  is the transformed form of  $\mathbf{H}$ .  $\bar{\mathbf{H}}$  is a strongly upper triangular matrix:

$$\forall (i, j) \in A, \quad h_{ij} \text{ \& } order(i) < order(j) \Rightarrow \bar{h}_{order(i), order(j)} = 1$$

According to the above mentioned expression if  $\bar{h}_{ij} = 1$  then  $i < j$ , so  $\bar{H}$  is an upper triangular matrix. In addition, the graph is loopless, hence:

$$\forall i \in N, \quad h_{ii} = 0 \Rightarrow \bar{h}_{order(i), order(j)} = 0.$$

■

Now the following corollaries can be concluded:

**COROLLARY 2.3** *A digraph is acyclic if and only if its node-node adjacency matrix can be transformed to a strongly upper triangular matrix.*

**COROLLARY 2.4** *In an acyclic digraph if  $order(i) = k$  then:*

- a.  $indgree(i) \leq k - 1$
- b.  $outdegree(i) \leq n - k$

Proof

$$\begin{aligned} \text{indegree}(i) &= \sum_{j=1}^n h_{jk} = \sum_{j=1}^{k-1} h_{jk} + \sum_{j=k}^n h_{jk} \\ &= \sum_{j=1}^{k-1} h_{jk} + 0 \leq k - 1 \\ \text{outdegree}(i) &= \sum_{j=1}^n h_{kj} = \sum_{j=1}^k h_{kj} + \sum_{j=k+1}^n h_{kj} \\ &= 0 + \sum_{j=k+1}^n h_{kj} \leq n - k. \end{aligned}$$

■

**COROLLARY 2.5** *If a digraph is acyclic then there exists at least one node  $i$  with  $\text{indegree}(i) = 0$  and one node  $j$  with  $\text{outdegree}(j) = 0$ .*

**COROLLARY 2.6** *An upper bound on the number of arcs in acyclic digraphs is  $\frac{n(n-1)}{2}$ .*

*Proof* The number of nonzero elements in the adjacency matrix that are above the diagonal shows the number of arcs that an acyclic digraphs can have. The maximum of this number is:

$$\sum_{i=1}^n \sum_{j=1}^n h_{ij} = \sum_{i=1}^{n-1} \sum_{j=i+1}^n h_{ij} \leq \frac{n(n-1)}{2}.$$

■

Now, the necessary and sufficient condition for uniqueness of a topological ordering is presented. Suppose that the digraph  $G = (N, A)$  is acyclic and topological ordering is given as  $\text{order}(i) = i, \forall i \in N$ . In addition assume that  $\mathbf{H} = \{h_{ij}\}_{n \times n}$  is the strongly upper triangular node-node adjacency matrix corresponding to the current topological ordering. For making a new topological ordering it is sufficient to interchange the rows and corresponding columns of this matrix such that the resulting matrix become strongly upper triangular.

$c_j$  and  $r_j$  are defined as:

$$c_j = \begin{cases} \max \left\{ k : \sum_{i=1}^k h_{j-1,j} = 0 \right\} & , \text{if } h_{j-1,j} = 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$r_j = \begin{cases} \max \left\{ k : \sum_{i=1}^k h_{j,j+1} = 0 \right\} & , \text{if } h_{j,j+1} = 0 \\ 0 & , \text{otherwise} \end{cases}$$

It is clear that for each node  $j, 0 \leq c_j \leq j - 1$  and  $0 \leq r_j \leq n - j$ .

**LEMMA 2.7** *Suppose  $\mathbf{H} = \{h_{ij}\}_{n \times n}$  is the adjacency matrix for acyclic digraph  $G = (N, A)$  and  $i, j \in N, (i < j)$ :*

- a) The necessary and sufficient condition for the matrix which is resulted from interchanging  $i$  and  $j$  columns of  $\mathbf{H}$  to be strongly upper triangular is:  $c_j \geq j - i$ .
- b) The necessary and sufficient condition for the matrix which is resulted from interchanging  $i$  and  $j$  rows of  $\mathbf{H}$  to be strongly upper triangular is:  $r_j \geq j - i$ .

Proof

- a) It is clear that  $h_{li} = 0$  for  $i + 1 \leq l \leq j$  (since this components are under diagonal). Hence the necessary and sufficient condition for the matrix which is resulted from interchanging  $i$  and  $j$  columns of  $\mathbf{H}$  to be strongly upper triangular is  $h_{kj} = 0$  for  $i \leq k \leq j - 1$ . In other words, according to the definition of  $c_j$  the following inequality should hold:  $c_j \geq j - i$ .
- b) It is clear that  $h_{jl} = 0$  for  $i \leq l \leq j - 1$  (since this components are under diagonal). Hence the necessary and sufficient condition for the matrix which is resulted from interchanging  $i$  and  $j$  rows of  $\mathbf{H}$  to be strongly upper triangular is  $h_{ik} = 0$  for  $i + 1 \leq k \leq j$ . In other words, according to the definition of  $r_j$  the following inequality should hold:  $r_j \geq j - i$ . ■

Since in adjacency matrix  $j^{\text{th}}$  row and column are corresponding to node  $j$  (or  $order(j) \dots$ ), if the  $j^{\text{th}}$  row is interchanged with  $i^{\text{th}}$  row, the  $j^{\text{th}}$  column should be interchanged with  $i^{\text{th}}$  column, too. In this case if the resulting matrix is strongly upper triangular, it is sufficient to interchange  $order(j)$  with  $order(i)$  to produce another topological ordering.

Let  $P = \{j : j \in N, c_j > 0\}$  &  $Q = \{j : j \in N, r_j > 0\}$ .

LEMMA 2.8  $P = \emptyset \Rightarrow Q = \emptyset$ .

Proof Suppose  $P = \emptyset$ , in other words for each node  $j$ ,  $c_j = 0$  that means  $\forall j \in N, h_{j-1} = 1$ . Now assume that  $Q = \emptyset$ , hence there is a node  $i$  such that  $r_i \geq 1$ . So which shows  $c_{i+1} \geq 1$  that is a contradiction.

In a similar way it can be shown that if  $q = \emptyset$  then  $P = \emptyset$ . ■

THEOREM 2.9 The necessary and sufficient condition for uniqueness of a topological ordering is  $P = \emptyset$  (or  $Q = \emptyset$ ).

Proof Suppose  $P = \emptyset$  ( $Q = \emptyset$ ). Hence for each node  $j$ ,  $c_j = 0$  ( $r_j = 0$ ). According to Lemma 1 no columns (rows) can be interchanged with another one such that the resulting matrix becomes strongly upper triangular. Hence the topological ordering of this digraph is unique. Conversely, suppose  $P \neq \emptyset$  then there exist  $j \in P$  such that  $c_j \geq 1$ . It is clear that  $h_{j-1,j} = 0$  hence  $r_{j-1} \geq 1$ . According to Lemma 1 the columns (rows)  $j$  and  $j - 1$  can be interchanged such that the resulting matrix becomes strongly upper triangular. The new topological ordering is produced by interchanging  $order(j)$  with  $order(j - 1)$ . ■

### 3. conclusion

In this paper the properties of node-node adjacency matrix in acyclic digraphs were considered. It was shown that topological ordering and node-node adjacency matrix were closely related. In fact, first the one to one correspondence between upper triangularity of node-node adjacency matrix and existence of directed cycles in digraphs was proved and then with this correspondence other properties of adjacency matrix in acyclic digraphs were presented.

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