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Optimal Cantilever Dynamic Vibration Absorbers

This work considers the use of a double-ended cantilever beam as a distributed parameter dynamic vibration absorber applied to a single-degree-of-freedom system in the presence of sinusoidal forces. The problem is analyzed exactly and by an energy approach using a single mode approximation for the cantilever beam. The results for both techniques compare favorably and damping is introduced in the form of a complex beam modulus. Optimal tuning and optimal damping parameters are found for a given ratio of absorber mass to main mass.

Introduction

In the past three-quarters of a century the dynamic vibration absorber has proven to be a useful device to limit undesirable vibration in hundreds of diverse applications in machine design and structural dynamics. The dynamic vibration absorber was first applied to damp the rolling motions of ships by means of a tuned U-shaped column of water aboard the ship [1].[1] Another early application was the Stockbridge absorber, which was employed to damp wind-induced oscillations which cause fatigue in electrical power transmission lines [2].

The analysis of the dynamic absorber was first presented in the now classic paper of Ormondroyd and Den Hartog [3]. Optimal damping in dynamic absorbers was the topic of a later paper by Brock [4]. Subsequently this device has been used successfully to damp building oscillations caused by reciprocating textile weaving equipment [5].

Young considered the application of dynamic absorbers to beams [6], while later the application to elastic plates was considered [7]. Several investigators have considered the design of optimal vibration absorbers to minimize some measure of system response in the case of randomly excited vibratory systems [8, 9]. Another paper considers the time domain optimization of a dynamic absorber which minimizes transmitted force or time for energy dissipation [10]. Recently Karnopp investigated the use of auxiliary distributed systems as dynamic vibration absorbers for distributed parameter main systems [11]. In a current paper Snowdon treats the application of circular plates with massive rims to act as dynamic absorbers for force- and displacement-excited single-degree-of-freedom lumped parameter systems [12]. It is this paper which stimulated the authors to recall some earlier investigations in which a double-ended cantilever beam was employed as a dynamic vibration absorber [7].

This problem is analyzed by employing the principle of superposition and the solution of the Bernoulli-Euler beam for the absorber force in terms of the vibratory amplitude of the mass where vibration control is sought. An assumed mode approximate analysis is also given and proves to be very accurate when compared to the exact analysis. Optimal tuning and structural damping in the absorber beam are derived in the context of the approximate analysis and are shown to hold also for the exact analysis for frequencies well below the second natural frequency of the distributed parameter absorber. Design curves for optimal beam-type absorbers are presented and should be useful to designers of machine elements and structural systems.

Theory

Exact Analysis. Consider the undamped system shown in Fig. 1, which is composed of a spring-supported lumped mass which is free to move only vertically. Attached to that mass is the double-ended cantilever beam as shown. For purposes of analysis, separate the system into three parts as shown in Fig. 2 and employ the method of superposition to examine the coupling between the subsystems. The equation of motion for the main mass M is

$$M\frac{d^2w}{dt^2} + Kw = P_0e^{j\omega t} + 2V_0e^{j\omega t} \quad (1)$$

where the $V_0e^{j\omega t}$ term is the yet unknown vibration absorber force for a single beam on the main mass. The solution to (1) would be but a simple matter if the complex transmitted force amplitude V_0 were known.

This force can be obtained as the shear force at the root of a displacement-excited cantilever beam. The governing equation for the cantilever beam under the assumption of Bernoulli-Euler bending theory is

$$EI\frac{\partial^4y}{\partial x^4} + \rho A\frac{\partial^2y}{\partial t^2} = 0 \quad (2)$$

with the following boundary conditions:

[1] Numbers in brackets designate References at end of paper.

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$$y(0, t) = W_0 e^{j\omega t} \tag{3}$$

$$\frac{\partial y}{\partial x}(0, t) = 0 \tag{4}$$

$$\frac{\partial^2 y}{\partial x^2}(L, t) = 0 \tag{5}$$

$$\frac{\partial^3 y}{\partial x^3}(L, t) = 0 \tag{6}$$

In order to obtain a steady-state solution to (2), assume a solution of the form

$$Y(x, t) = Y(x)e^{j\omega t} \tag{7}$$

which will yield the spatial complex amplitude distribution $Y(x)$. The shear force amplitude at the root of the cantilever is then

$$V_0 = -EI\frac{d^3Y(0)}{dx^3} \tag{8}$$

Solving (2) subject to boundary conditions (3)–(6) and then using expression (8) gives a shear force amplitude of

$$V_0 = EIW_0\beta^3\left[\frac{\sinh \beta L \cos \beta L + \cosh \beta L \sin \beta L}{1 + \cosh \beta L \cos \beta L}\right] \tag{9}$$

where the parameter β is defined by

$$\beta^4 = \frac{\rho A \omega^2}{EI} \tag{10}$$

The steady-state solution to (1) is

$$w(t) = W_0 e^{j\omega t} \tag{11}$$

where W_0 is the complex amplitude, so

$$(-M\omega^2 + K)W_0 = P_0 + 2EIW_0\beta^3\left[\frac{\sinh \beta L \cosh \beta L + \cosh \beta L \sin \beta L}{1 + \cosh \beta L \cos \beta L}\right] \tag{12}$$

Note that the second forcing term on the right side of (12) is a function of the complex vibratory amplitude of the mass W_0. This presents no problem in that (12) can still be solved for W_0 as a function of the external forcing function amplitude P_0 or

$$W_0 = \frac{P_0(1 + \cosh \beta L \cos \beta L)}{(K - M\omega^2)(1 + \cosh \beta L \cos \beta L) - 2EI\beta^3(\sinh \beta L \cos \beta L + \cosh \beta L \sin \beta L)} \tag{13}$$

In order to get maximum benefit from the foregoing analysis it is expedient to introduce some nondimensional quantities. Define the tuning ratio, T, to be the ratio of the first natural frequency of the cantilever to the natural frequency of the main lumped parameter system

$$T = \frac{\omega_a}{\Omega_1} = \frac{1.875^2}{L}\sqrt{\frac{EIM}{\rho AK}} \tag{14}$$

The mass ratio μ is the ratio of the total absorber mass to that of mass M

$$\mu = \frac{2\rho AL}{M} \tag{15}$$

Define the frequency ratio λ as the ratio of the forcing frequency to the natural frequency of the $K - M$ combination of

$$\lambda = \frac{\omega}{\Omega_1} = \omega\sqrt{\frac{M}{K}} \tag{16}$$

while the static deflection of the main system is defined to be

$$W_{st} = \frac{P_0}{K} \tag{17}$$

The frequency response function in dimensionless form is then

$$\frac{W_0}{W_{st}} = \frac{1 + \cos \beta L \cosh \beta L}{(1 - \lambda^2)(1 + \cos \beta L \cosh \beta L) - \dfrac{\mu T^{1/2}\lambda^{3/2}}{1.875}(\sinh \beta L \cos \beta L + \sin \beta L \cosh \beta L)} \tag{18}$$

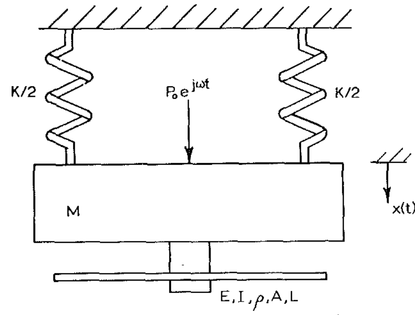

Fig. 1 Single-degree-of-freedom system with beam-type dynamic absorber

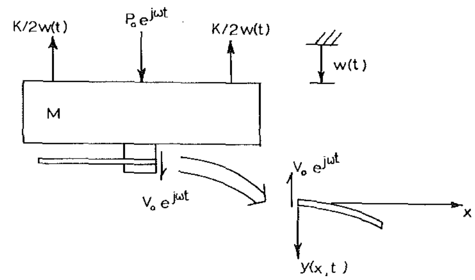

Fig. 2 Freebody diagram of the absorber system

where

$$\beta L = 1.875\sqrt{\frac{\lambda}{T}} \tag{19}$$

Approximate Analysis. This analysis will be accomplished in order to see whether or not the results of such analysis compare favorably with the exact answers given in the previous section. The technique to be employed will be the method of assumed modes [13] and, hence, there is a need to formulate the kinetic and potential energies, which are respectively

$$KE = \frac{1}{2}M\dot{w}^2 + \rho A\int_0^L [\dot{w} + \dot{q}\phi_1(x)]^2\, dx \tag{20}$$

and

$$V = \frac{1}{2}Kw^2 + EI\int_0^L q^2\left(\frac{d^2\phi_1}{dx^2}\right)^2 dx \tag{21}$$

where $\phi_1(x)$ is the first cantilever mode as given by Felgar and Young [14]. The integrals involving $\phi_1(x)$ and its second derivative in (20) and (21) have been tabulated by Felgar [15]. Application of Lagrange's equations to these energies yields the following differential equations, which are written in nondimensional form using the previously defined dimensionless quantities:

$$\begin{bmatrix} 1+\mu & 0.7829 \\ 0.7829\mu & 1 \end{bmatrix}\begin{bmatrix} \ddot{w} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} \Omega_1{}^2 & 0 \\ 0 & T^2\Omega_1{}^2 \end{bmatrix}\begin{bmatrix} w \\ q \end{bmatrix} = \begin{bmatrix} \dfrac{P_0}{M}\sin \omega t \\ 0 \end{bmatrix} \tag{22}$$

Solving for the system frequency response function yields

$$\frac{W_0}{W_{st}} = \frac{T^2 - \lambda^2}{\lambda^4(1 + 0.3871\mu) - \lambda^2[1 + T^2(1 + \mu)] + T^2} \tag{23}$$

where the nondimensional parameters are the same as defined previously. The nondimensional result given by Den Hartog [1] for the case of a lumped parameter absorber is

$$\frac{W_0}{W_{st}} = \frac{T^2 - \lambda^2}{\lambda^4 - \lambda^2[1 + T^2(1 + \mu)] + T^2} \tag{24}$$

It is interesting to note the only difference in expressions (23) and (24) is the modification of the first term of the denominator.

Evaluation of these expressions for the undamped frequency response yields the frequency response curves of Fig. 3 for a tuning ratio of unity and a mass ratio of 0.2. It is apparent that the lumped parameter equation (24) does not accurately predict the new induced natural frequencies but that the approximate equation predicts very accurately the behavior given by the transcendental harmonic response function (18) as long as the forcing frequency is less than half the second natural frequency of the cantilever, which is 6.27 times the first natural frequency [14]. The two lowest natural frequencies as a function of the mass ratio are shown in Fig. 4 for a tuning ratio of unity. It is clear from these results that for a distributed absorber of a given mass ratio, the frequency spread will not be as wide as that for a lumped absorber of the same mass ratio. This occurs because those portions of the absorber near the absorber root do not vibrate with the relative amplitude as those portions further outboard and, hence, the shear forces developed are not so great.

Optimization of the Damped Absorber. Damping may be incorporated into the absorber by treating the beam as having a complex elastic modulus as first discussed by Soroka [16] and Myklestad [17]. This type of damping is considered in the excellent text by Snowdon [18]. Since the beam's elastic modulus directly controls the tuning ratio, as indicated by equation (14), it is appropriate to replace the square of the tuning ratio, T^2, in the undamped analyses by the square of a complex tuning ratio, T^{*2}, where

$$T^{*2} = T^2(1 + j\delta) \tag{25}$$

This type of modulus represents damping forces which for harmonic motion are in phase with velocity and proportional to displacement.

The approximate absorber will now be optimized to make the frequency response as flat as possible with the result being peaks of very nearly equal height.

This procedure for absorber optimization was derived by Brock [4] and is outlined by Snowdon [18] and yields an optimal tuning ratio

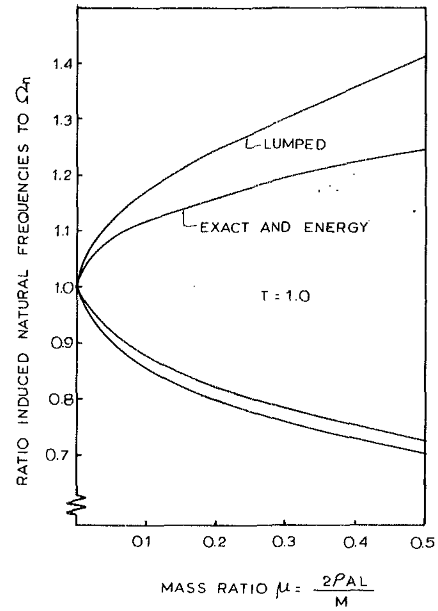

Fig. 4 Frequency spread of induced frequencies due to absorber system

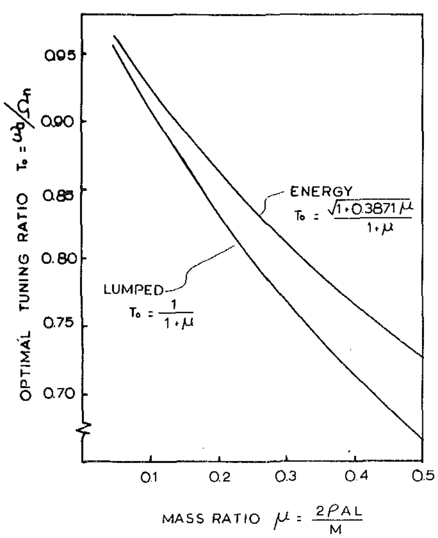

Fig. 5 Optimal tuning ratio for beam-type dynamic absorber

$$T_0 = \frac{\sqrt{1 + 0.3871\mu}}{1 + \mu} \tag{26}$$

which is illustrated in Fig. 5 as a function of the mass ratio. With this tuning ratio then the optimal damping coefficient δ_0 is found and is illustrated in Fig. 6 for various mass ratios μ. For comparison, the optimal lumped structural damping coefficient computed from a frequency response function similar to (24) except for the inclusion of complex structural damping is also given in Fig. 6. To evaluate the incorporation of the optimal values evaluated for the approximate system in the transcendental (distributed) system, the transcendental frequency response function (18) was evaluated using those values and compared to the results of equation (23) and again there was negligible difference for frequencies well below the second natural frequency of the cantilevers, as illustrated in Fig. 7. The damped absorber was shown to also suppress the higher resonances of the system, as shown in Fig. 8 for the third natural frequency of the composite system.

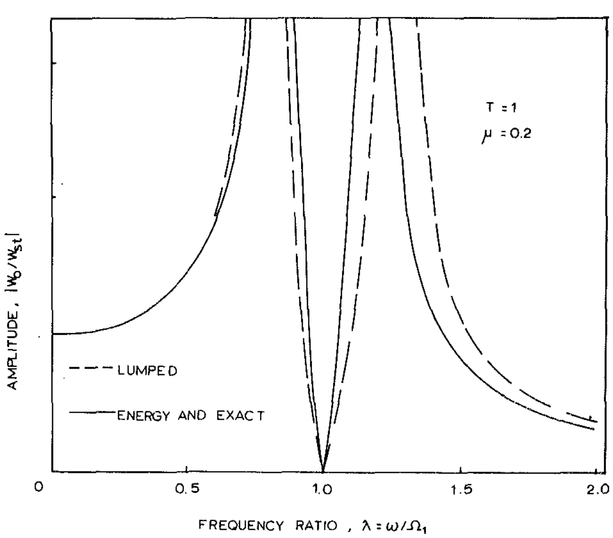

Fig. 3 Typical frequency response for undamped beam absorber

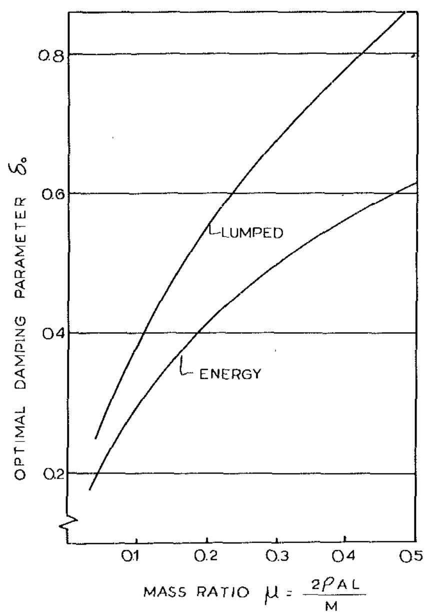

Fig. 6 Optimal complex damping parameter for various mass ratios

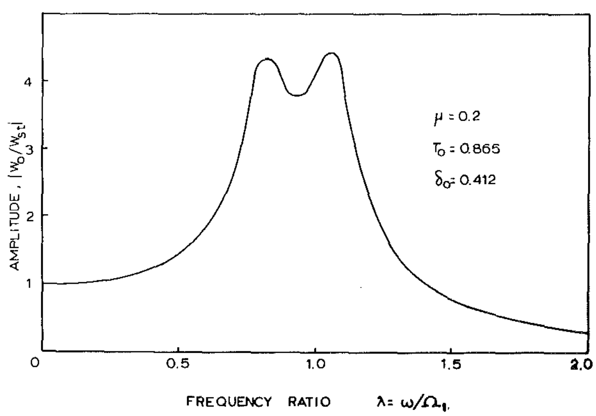

Fig. 7 Frequency response optimally tuned and damped beam-type absorber

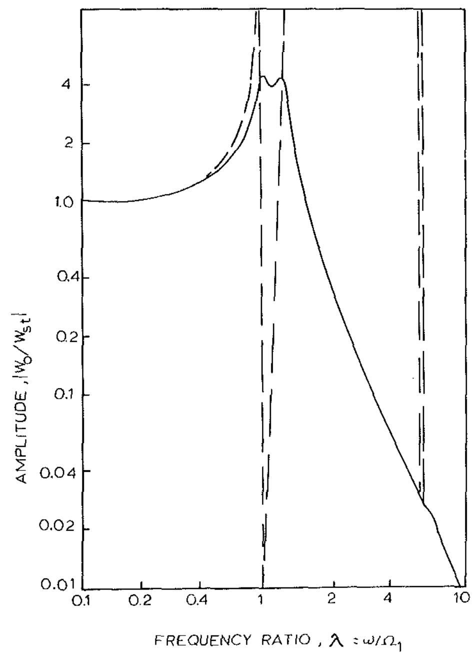

Fig. 8 Optimal and undamped frequency response over a large range of frequencies

Summary and Conclusions

The authors have investigated the possibility of the use of a double-ended cantilever beam as a dynamic vibration absorber for a lumped-parameter single-degree-of-freedom vibration system but its application would not be limited to such a system. It is found that treating the system as a lumped system yields significant errors in prediction of system dynamics. It was also found that for frequencies well below the second natural frequency of the beam, an assumed mode energy technique yields answers which agree quite favorably with those of an exact analysis treating the cantilevers exactly. Based on these results, the authors would be hesitant to attack a similar problem with a tedious exact analysis when an assumed mode energy approach yields answers which are not distinctly different. The availability of the tables of references [14] and [15] makes the formulation of potential and kinetic energy expressions a relatively simple task.

Structural damping is incorporated into the beam model by choosing a complex elastic modulus and the parameters of optimal absorber beams of a given mass are given and presented in the form of design curves.

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