A Brief Survey of Just-In-Time Sequencing for Mixed-Model Systems *

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Abstract

The concept of penalizing jobs both for being tardy and for being early has proven one of most important and fertile research topics in Operations Research. In this survey, we consider just-in-time mixed-model, multi-level supply chains. Obtaining an optimal sequence in a multi-level chain is a challenging nonlinear integer programming problem. Problems for two or more levels are strongly NP-hard. The problem of minimizing deviations between actual and desired production for single-level can be solved efficiently. Also the multi-level problems with pegging assumption are solvable by reducing them to the single-level. Cyclic schedules are optimal for single-level problem. We present various ways of dealing with these problems such as the elegant concept of balanced words and different optimization techniques. We provide a short review of different mathematical models, discuss their complexity and compare them. The research results obtained in past several years are presented along with open problems and possible extensions.

Keywords: nonlinear integer programming, scheduling theory, just-in-time systems, balanced (level) schedules, balanced words, computational complexity, polynomial algorithms.

1 Introduction

The central goal of mixed-model or flexible assembly processes is to increase profit by reducing costs. The just-in-time (JIT) production systems, which require producing only the necessary product in the necessary quantities at the necessary time, have been used for controlling such flexible assembly systems. The intention of these methods is to satisfy the customer demands for a variety of models without holding large inventories or incurring large shortages of the products. We assume a flow line manufacturing, called flexible transfer lines, where negligible switch over costs from one model to another allow for diversified

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small-lot production avoiding production of each model in large-lots. The most important optimization problem that has to be solved for the mixed-models, just-in-time systems is to determine the sequence in which different models are produced. This solution impacts the entire supply chain.

There has been growing interest in JIT systems research since Monden [32]. Miltenburg [30] considers the problem of determining the sequence for producing different products on the line that keeps a constant rate of usage of every part used by the line. In other words, the quantity of each part used by the mixed-model assembly line per unit of time should be kept as constant as possible. This allows very little variability in the usage of each part from one time horizon to the other. Monden [32] states this as the most important goal of a JIT production system implemented by the Toyota company. Toyota's so-called *Goal Chasing Method*, a local search heuristic, has been most popular for solving the problem. The sequences refereed to as level, balanced or fair sequences always keep the actual production level and the desired production level as close to each other as possible all the times. The other production issues studied are cycle times, lead times, work-in-process and loading [26, 30, 28, 32, 33, 17].

Multi-level production systems, where components required for different models may or may not be distinct, make the problem more challenging than the single-level production systems where different models require the same number and mix of components. Because of the *pull* nature of the JIT systems, the production sequences at all other lower levels are also inherently fixed as soon as the final level production sequence is fixed. That is why the determination of the sequence of different products at final assembly level is crucial. Miltenburg [30] provides a nonlinear integer programming formulation for the minimization of total deviation for mixed-model JIT production systems under the assumption that the products require approximately the same number and mix of parts. As optimal sequence at the final assembly level would simultaneously achieve an even rate of parts usage at the feeder production levels, this formulation can be considered as a single-level problem. An exact exponential time algorithm and two heuristics are also presented in [30]. Miltenburg and Goldstein [28] and Miltenburg and Sinnamon [31] extend the formulation to multi-level assembly systems. Most of these optimization problems would require enumerative or exponential algorithms. Miltenburg, Steiner and Yeomans [29], Yeomans [43] and Kubiak, Steiner and Yeomans [24] present dynamic programming approaches to the multi-level problems. We refer the reader to Groeflin, Luss, Rosenwein and Wahls [14], Inman and Bulfin [16], Ding and Cheng [9, 10], Sumichrast, Russell and Taylor [39], Sumichrast and Russell [40] and [28, 31] for several heuristics for the problem.

Kubiak and Sethi [25, 27] reduce the minimization of total deviation JIT problem to an assignment problem and thereby present an efficient optimization algorithm for this problem. The algorithm works for more general sum objective functions consisting of nonnegative convex functions of deviations between cumulative average demand and cumulative production of various models over time.

Steiner and Yeomans [37], following the optimization algorithm for the total deviation given in [25, 27], give a graph theoretic optimization algorithm for minimizing maximum deviation JIT single-level sequencing problem. They also give an algorithm for minimizing multi-level maximum deviation JIT assembly systems under the pegging assumption [35]. If outputs at production levels which feed the final assembly level are dedicated to the final product into which they will be assembled, then the problem with pegging is equivalent to a weighted single-level of problem which can then be minimized by modified algorithm for un-weighted single-level problem.

For both maximum and total deviations, there are always cyclic schedules which are optimal, see Steiner and Yeomans [35] and Kubiak [21], which significantly reduces the computational requirements.

Brauner and Crama [6] prove that the minimization of maximum deviation or bottleneck for a single-level is Co-NP, but in general, the complexity of the single level problems remains open for the binary encoding. The multi-level problem for two or more production levels is strongly NP-hard, Kubiak [26].

Brauner and Crama [6] present an algebraic approach to the results of [37] and formulate the small deviation conjecture. Kubiak [22] presents a geometric proof of the conjecture and later Brauner, Jost and Kubiak [7] exploit the concept of balanced words to give another proof of the conjecture. Kubiak [19, 20] presents properties of JIT sequences obtained through mathematically elegant concept of balanced words. We refer the interested readers to Vuillon [42] for a survey and the references about balanced words.

Bautista, Companys and Corominas [3], Kubiak [26] and Palli [34] present efficient algorithms for maximum deviation problem based on the reduction to the *bottleneck assignment problem*.

Bautista, Companys and Corominas [5] establish an interesting link between the JIT sequencing and the apportionment problem. An apportionment problem deals with the allocation of seats of a legislature among the states or provinces of a nation. Balinski and Shahidi [1] consider the JIT sequencing problem as the quota method of apportionment.

Corominas and Moreno [8] investigate relationships between the solution spaces of different objective functions.

The plan of the paper is as follows. In Section 2, we review optimization models of JIT sequences. In Sections 3 and 4 we survey the efficient algorithms for total-deviation and maximum-deviation objective functions, respectively. Section 5 summarizes the balance properties of min-max sequences. The cyclic schedules are discussed in Section 6. Section 7 relates the optimality conditions between different objective functions. Section 8 is devoted to the study of computational complexity of the problems, heuristic solutions and a dynamic programming approach. The final Section 9 includes conclusions with possible directions and open questions for further research.

2 The Mathematical Programming Formulation

2.1 Multi-Level Formulation

A mixed-model multi-level assembly chain consists of a hierarchy of several distinct production levels (for example, products $\leftarrow$ subassemblies $\leftarrow$ components $\leftarrow$ raw-materials). In these supply chains, the multiple copies of different models are produced at the final assembly level. The assembly system also contains several other lower production levels where subassemblies, component parts and raw materials are either fabricated or purchased for use in the products.

Let there be L different production levels l, where $l = 1, 2, \dots, L$ with the highest level, the product level 1. We denote the number of different part types of level l by n_l and the demand for part i, where $i = 1, 2, \ldots, n_l$, of level l by d_{il}. Denoting by t_{ilp} the number of total units of output i at level l required to produce one unit of product p, we have $d_{il} = \sum_{p=1}^{n_1} t_{ilp} d_{p1}$, the *dependent demand* for part i of level l determined by the final product demands d_{p1}, $p = 1, 2, \ldots, n_1$. Note that $t_{i1p} = 1$ if $i = p$ and 0 otherwise. Let $D_l = \sum_{i=1}^{n_l} d_{il}$ be the total output demand of level l. The demand ratio for part i of level l is $r_{il} = \frac{d_{il}}{D_l}$ and $\sum_{i=1}^{n_l} r_{il} = 1$ at each level $l = 1, 2, \ldots, L$.

Under the assumption of non-preemptive schedule, a schedule is completely defined by the sequence of product copies of the product level. A copy is said to be in stage k, $k = 1, 2, \ldots, D_1$, if k units of product have been produced at level 1. The total horizon will be of D_1 time units and there will be k complete units of the various products p at level 1 during the first k stages. Due to the pull nature of the JIT system along with the fact that the lower level outputs are drawn as needed by the final product level, the particular combination of the products produced at the product level during the first k stages determines the necessary cumulative part production at every other level. Let x_{ilk} be the necessary cumulative production of output i at level l during stages 1 through k and $y_{lk} = \sum_{i=1}^{n_l} x_{ilk}$ be the total output of level l during stages 1 through k. Clearly, the cumulative production of level 1 through the first k stages is $y_{1k} = \sum_{i=1}^{n_1} x_{i1k}$. The required cumulative production for part i at level l, where $l \geq 2$, through k stages will be $x_{ilk} = \sum_{p=1}^{n_1} t_{ilp} x_{p1k}$. Finally, we could impose a weight w_{il} because of relative importance of balancing the schedule for part i at level l. For simplicity, we take $w_{il} = 1$ for all i and for all l.

The feasible solution region is denoted by $\mathcal{X}_{\mathcal{M}} = \{X \mid X = (x_{p1k})_{n_1 \times D_1}\}$, where the variables satisfy the following constraints:

$$
\begin{aligned}
x_{ilk} &= \sum_{p=1}^{n_1} t_{ilp} x_{p1k} && i = 1, \ldots, n_l,\ l = 1, \ldots, L,\ k = 1, \ldots, D_1, && (1)\\
y_{lk} &= \sum_{i=1}^{n_l} x_{ilk} && l = 2, \ldots, L,\ k = 1, \ldots, D_1, && (2)\\
y_{1k} &= \sum_{p=1}^{n_1} x_{p1k} && k = 1, \ldots, D_1, && (3)\\
k &= \sum_{p=1}^{n_1} x_{p1k} && k = 1, \ldots, D_1, && (4)\\
x_{p1k} &\geq x_{p1(k-1)} && p = 1, \ldots, n_1,\ k = 1, \ldots, D_1, && (5)\\
x_{p1D_1} &= d_{p1}\ ,\ x_{p10} = 0 && p = 1, \ldots, n_1, && (6)\\
x_{ilk} &\in \mathcal{N} && i = 1, \ldots, n_l,\ l = 1, \ldots, L,\ k = 1, \ldots, D_1. && (7)
\end{aligned}
$$

In this paper, $\mathcal{N}$ denotes the set of all nonnegative integers. Constraint (1) ensures that the necessary cumulative production of output i of level l by the end of stage k is determined explicitly by the quantity of products produced at production level 1. Constraints (2) and (3) calculate the total cumulative production of level l and 1, respectively, through stages 1 to k. Constraint (5) is to ensure that the total production of every product over k stages is a non-decreasing function of k. Constraint (6) guarantees that the production requirements for each product are met exactly. Constraints (4), (5) and (7) ensure that exactly one product is scheduled for final assembly during each stage.

Then the mixed-model, multi-level schedule problem is to select $X = (x_{p1k})_{n_1 \times D_1}$ that minimizes one of the following min-max/min-sum objective function(s)

$$
\begin{aligned}
G_{AMD}(X) &= \max_{i,l,k} |x_{ilk} - y_{lk} r_{il}| && (8)\\
G_{QMD}(X) &= \max_{i,l,k} (x_{ilk} - y_{lk} r_{il})^2 && (9)\\
G_{ASD}(X) &= \sum_{i=1}^{n_l}\sum_{l=1}^{L}\sum_{k=1}^{D} |x_{ilk} - y_{lk} r_{il}| && (10)\\
G_{QSD}(X) &= \sum_{i=1}^{n_l}\sum_{l=1}^{L}\sum_{k=1}^{D} (x_{ilk} - y_{lk} r_{il})^2 && (11)
\end{aligned}
$$

With this notation, the multi-level JIT sequencing problem is equivalent to

$$\min\{G(X) \mid X \in \mathcal{X}_{\mathcal{M}}\}, \text{ where } G \in \{G_{AMD}, G_{QMD}, G_{ASD}, G_{QSD}\}.$$

By [24], $w_{il}|x_{ilk} - y_{lk} r_{il}| = |\sum_{p=1}^{n_1} \gamma_{ilp} x_{p1k}|$, where $\gamma_{ilp} = w_{il}\delta_{ilp}$ and $\delta_{ilp} = t_{ilp} - r_{il}\sum_{h=1}^{n_l} t_{hlp}$. We define a matrix $\Gamma = [\gamma_{ilp}]_{n \times n_1}$ with $n = \sum_{l=1}^{L} n_l$, where γ_{ilp} representing the $(\sum_{m=1}^{l-1} n_m + i)$th row and pth column element. Let $\mathbf{X}_k = (x_{11k}, \ldots, x_{n_1 1k})^T$ be a vector representing the cumulative production at level 1 through the first k stages. Then

$$G_{AMD}(X) = \max_k ||\Gamma \mathbf{X_k}||_1, \text{ where } ||\Gamma \mathbf{X_k}||_1 = \max_{il}\{w_{il}|x_{ilk} - y_{lk} r_{il}|\}.$$

Here, $||\Gamma \mathbf{X_k}||_1$ represents the maximum deviation at stage k over all i and j. Notice from the matrix representation that at any particular stage, the deviation of any part of any level is determined by the level 1 sequence.

Likewise, sum deviation objective $G_{QSD}(X) = \sum_{k=1}^{D_1}(||\Omega \mathbf{X}_k||_2)^2$ with deviation matrix $\Omega = [\sqrt{w_{il}}\delta_{ilp}]$ and the Euclidean norm $||\mathbf{a}||_2 = \sqrt{\sum_{i=1}^{m} a_i^2}$ of a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m)$.

The sequencing problems, *maximum-deviation JIT* and *total-deviation JIT*, are denoted by *MDJIT* and *SDJIT* problems, respectively. The problem is one of the most fundamental problems in flexible just-in-time mixed-model production systems, referred to as *JIT* sequences. In these formulations, the *min-sum* and *min-max* objectives are similar to Miltenburg and Sinnamon [31], Steiner and Yeomans [35] and Kubiak, Steiner, and Yeomans [24], respectively. Note that the min-max objectives seek to minimize the deviations for each output at each stage, whereas the min-sum objectives are concerned for finding the lowest possible total deviation which may result in relatively large deviation for a certain product.

The effects of weights in single-level as well as other multi-level problems are considered in [43, 24, 35, 29, 28, 32].

2.2 Single-Level Formulation

For $i = 1, \ldots, n$, given n *products* (*models*) i, n positive integers (*demands*) d_i and n convex-symmetric functions f_i of a single variable, called deviation, all assuming minimum 0 at 0. The following optimization problem have been considered to model single-level system.

Find a sequence $s = s_1 s_2 \dots s_D$ with total demand $D = \sum_i^n d_i$ of products $1, \dots, i, \dots, n$, where product i occurs exactly d_i times that minimizes the following objective function(s)

$$F_{MD}(s) = \max_{i,k} f_i(x_{ik} - kr_i) \quad (12)$$

$$F_{SD}(s) = \sum_{i=1}^{n} \sum_{k=1}^{D} f_i(x_{ik} - kr_i) \quad (13)$$

where x_{ik} represents the number of product i occurrences (copies) in the prefix $s_1 s_2 \dots s_k$, $k = 1, \dots, D$, and $r_i = \frac{d_i}{D}$, $i = 1, \dots, n$. The following two measures of deviations have been studied in the literature.

$$f_i(x_{ik} - r_i k) = \begin{cases} |x_{ik} - kr_i| & \text{the absolute-deviation objective,} \\ (x_{ik} - kr_i)^2 & \text{the squared-deviation objective.} \end{cases}$$

The whole feasible solution region $\mathcal{X} = \{X \mid X = (x_{ik})_{n \times D}\}$ in single-level problem is constrained as

$$\begin{array}{ll} \sum_{i=1}^n x_{i,k} = k & k = 1, \dots, D, \\ x_{i,k} \le x_{i,k+1} & i = 1, \dots, n, \ k = 1, \dots, D-1, \\ x_{i,D} = d_i \ , \ x_{i0} = 0 & i = 1, \dots, n, \\ x_{i,k} \in \mathcal{N} & i = 1, \dots, n, \ k = 1, \dots, D. \end{array}$$

This problem is refereed to as the *Product Rate Variation Problem (PRV)* in the literature Kubiak [26]. A solution of this problem always keeps the *actual production level* x_{ik} and the *desired production level* $r_i k$ as close to each other as possible all the times. A solution $s = s_1 s_2 \dots s_D$ of the single-level MDJIT problem for n models is called *B-feasible* (or *B-bounded*) if $\max_{i,k} f_i(x_{ik} - r_i k) \le B$ holds for the $n \times D$ matrix variables $X = (x_{ik})$. We denote the sets of all single-level B-feasible solutions by $\mathcal{X}_B$.

Note that the above formulation gives the following number-theoretic interpretation of JIT sequencing problem: given n rational numbers r_i, $i = 1, 2, \dots, n$, with common denominator D, the problem is to find nD integers x_{ik} which optimally approximate the sequence (kr_i) under the cardinality and monotonicity restrictions defined above (see also [6], for the references).

A multi-level, min-max problem under the pegging assumption has been reduced to a weighted single-level problem [12] (see also [35]). Similarly, the min-sum, multi-level problem with pegging can be reduced to a weighted single level problem considered by Yeomans [43]. Goldstein and Miltenburg [12] were the first to provide mathematical formulation of pegging in JIT systems (see also [35]).

Under the pegging assumption, parts of output i at level l are dedicated to the particular product at level 1 into which they will be assembled. This assumption decomposes the lower level parts that will be assembled into different level 1 products into disjoint sets. With this assumption, the multi-level AMDJIT sequencing problem subject to the constraint set $\mathcal{X}_{\mathcal{M}}$ with $p = 1, \dots, n_1$, $i = 1, \dots, n_l$, $l = 1, \dots, L$ and $k = 1, \dots, D_1$ can be formulated as

$$\min \max_{p,i,l,k} \{w_{p1}|x_{p1k} - kr_{p1}|, \ w_{il} t_{ilp} |x_{p1k} - kr_{p1}|\}.$$

Since $t_{i1p} = 1$ if $i = p$ and 0 otherwise, the above problem is equivalent to

$$\min \max_{p,k} \{v_{p1}|x_{p1k} - kr_{p1}|\}, \text{ where } v_{p1} = \max_{i,l} \{w_{il} t_{ilp}\}, \ l = 1, \dots, L.$$

By dropping the superfluous subscript 1, we obtain the following weighted single-level AMD-JIT problem

$$\min \max_{i,k} \{v_i|x_{ik} - kr_i| \; : \; X \in \mathcal{X}\}, \; i = 1, \ldots n, \; k = 1, \ldots, D. \tag{14}$$

3 Efficiently Solvable SDJIT Sequencing

In this section, we study the single-level, min-sum problems with the objective defined in (13). Unless otherwise specified, single level, min-sum problems will be denoted by SDJIT. These results are valid for convex, symmetric, nonnegative functions which take value 0 at 0.

Let $\mathcal{Y} = \{(i, j, k) : i = 1, \ldots, n; j = 1, \ldots, d_i; k = 1, \ldots, D\}$. Define cost $C^i_{jk} \geq 0$ for $(i, j, k) \in \mathcal{Y}$ with respect to the *ideal position* $Z^i_j = \lceil \frac{2j-1}{2r_i} \rceil$, for the j-th copy of model i as follows

$$C^i_{jk} = \begin{cases} \sum_{l=k}^{Z^i_j - 1} \psi^i_{jl} & \text{if } k < Z^i_j, \\ 0 & \text{if } k = Z^i_j, \\ \sum_{l=Z^i_j}^{k-1} \psi^i_{jl} & \text{if } k > Z^i_j, \end{cases}$$

where Z^j_i uniquely solves $f_i(j - kr_i) = f_i(j - 1 - kr_i)$ and $f_i(x) = |x|$,

$$\psi^i_{jl} = \begin{cases} f_i(j - lr_i) - f_i(j - 1 - lr_i) & \text{if } l < Z^i_j, \\ f_i(j - 1 - lr_i) - f_i(j - lr_i) & \text{if } l \geq Z^i_j. \end{cases}$$

A subset Y of $\mathcal{Y}$ is called *feasible* if it satisfies the following constraints

- C_1. For each $k = 1, \ldots, D$, there is exactly one (i, j), $i = 1, \ldots, n; j = 1, \ldots, d_i$ s.t. $(i, j, k) \in Y$, i.e., exactly one copy product at each time.
- C_2. For each (i, j), $i = 1, \ldots, n$; $j = 1, \ldots, d_i$, there is exactly one k, $k = 1, \ldots, D$ s.t. $(i, j, k) \in Y$, i.e., each copy is produced exactly once.
- C_3. If $(i, j, k), (i, j', k') \in Y$ and $k < k'$, then j, j', i.e., lower indices copies are produced earlier.

Consider any set S of D triples (i, j, k) satisfying C_1, C_2, C_3 and define the sequence $s = s_1 s_2 \ldots s_D$ with $s_k = i$ if $(i, j, k) \in S$ for some $j = 1, \ldots d_i$ corresponding to the set S. Then the sequence s is feasible for any given instance $(d_1, d_2, \ldots d_n)$ and following results hold, Kubiak and Sethi [25].

Theorem 1 *Let $c(S) = \sum_{(i,j,k)} C^i_{jk}$ for any $S \subseteq \mathcal{Y}$. Then,*

- *a. For any feasible S, it holds $F_{SD}(s) = c(S) + \sum_{i=1}^{n} \sum_{k=1}^{D} \inf_j f_i(j - kr_i)$.*
- *b. If S satisfies C_1 and C_2, then S^* satisfying C_1, C_2 and C_3 with $c(S) \geq c(S^*)$ can be determined in $O(D)$ steps. Moreover, each product copies preserve the order in the sequence s^* as it does in the sequence s.*

As the term $\sum_{i=1}^{n}\sum_{k=1}^{D}\inf_j f_i(j-kr_i)$ is independent of the set S, that is constant, an optimal solution to SDJIT would be an immediate consequence if an optimal set S is found. But an optimal set cannot be obtained by simply solving the Assignment Problem 15 with constraints C_1 and C_2 and the costs C^i_{jk} with $(i,j,k)\in S$, as the constraint C_3 is not of the assignment type. Notice that the latter constraint is essential as it ties up the copy j of a product with the j-th ideal position for the product. The main idea of the proof is to show that there exists at least one optimal sequence for the assignment problem such that copy $(i,j+1)$ of product i should appear after the copy (i,j). The proof is done by mathematical induction. With these costs the corresponding assignment problem has been formulated as follows Kubiak and Sethi [25]:

$$\min[F(s)=\sum_{(i,j)}^{(n,d_i)}\sum_{k=1}^{D}C^i_{jk}x^i_{jk}] \tag{15}$$

subject to the constraints

$$\begin{aligned}\sum_{k=1}^{D}x^i_{jk} &= 1,\ \text{for } i=1,\ldots,n;\ j=1,\ldots d_i,\\ \sum_{(i,j)}^{(n,d_i)}x^i_{jk} &= 1,\ \text{for } k=1,\ldots,D,\end{aligned}$$

where, $x^i_{jk}=\begin{cases}1, & \text{if } (i,j) \text{ is assigned at position } k,\\ 0, & \text{otherwise.}\end{cases}$

Observe that an obvious optimal solution could be obtained if sequencing all copies in their ideal positions were possible without competition for these positions. As this is not the case in general, we need to resolve competition to minimize the given objective. This is done efficiently by solving the assignment problem [25, 26]. Recall that the assignment problem with $2m$ nodes can be solved in $O(m^3)$ time (see [25, 26], for the references).

The approach proposed by [25, 27] for the total deviation product rate variation problem is applicable to any l_p norm with $F_{SD}=l_p$, and in particular to l_∞-norm. In the latter case the approach minimizes maximum deviation objective.

Consequently, solution to multi-level min-sum problem with pegging assumption could be obtained as in Kubiak and Sethi [27].

Steiner and Yeomans [36] look at the min-sum problem as a weighted matching problem in a complete bipartite graph $G=(V,E)$, where weights of the edges equal penalty costs C^i_{jk}. Then the problem is to find a perfect matching with the minimum sum of the weights. An incomplete bipartite graph is defined by introducing the earliest and latest completion times possible for a copy (i,j) of product i (see Section 4 for the definition). Moreover, a 1-bounded solution that is optimal (if such solution exists) could be obtained in $O(nD^2\log D)$ time, since for $B\leq 1$ implies $|E|\leq(n+2)D$. A Pareto optimal solution can be found in $O(nD^2\log D)$ time. But the existence of 1-bounded solutions optimal for min-sum problems is not always the case (see Section 7). The following question remains open. What is minimum B such that optimal solution for min-sum problem is B-bounded? It is known that an upper bound on the optimal min-sum-absolute and min-sum-squared objectives is nD though the bound is not tight [36].

For the sake of completeness, we mention that several heuristics for single-level problem have already been investigated in [9, 10, 12, 16, 28, 29, 30, 31, 39, 40]

4 Efficiently Solvable MDJIT Sequencing

In this and in Section 5 we study the min-max problems that are either single-level with the objective defined in (12). Unless otherwise specified, single level absolute deviation min-max problems will be denoted by AMDJIT.

Steiner and Yeomans [37] study AMDJIT problem reducing it to a single machine scheduling decision problem with release times and due dates. They represent the problem as a perfect matching problem in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ where the set $V_1 = \{1, \ldots, D\}$ represents positions and the set $V_2 = \{(i,j) \mid i = 1, \ldots, n \; ; \; j = 1, \ldots, d_i\}$ represents the copies of the products. Here, for $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$, the notation (i,j) denotes the j-th copy of product model i. There exists an edge $\{k, (i,j)\} \in E$ if and only if k lies in the permissible interval $[E(i,j),\ L(i,j)] \subseteq V_1$ of release time and due date for the j-th copy of the product i. They prove the following result (see also Brauner and Crama [6]).

Lemma 1 *Let $d_1, \ldots, d_n$ be any instance of AMDJIT problem. A sequence $s = s_1 s_2 \ldots s_D$ is B-feasible if and only if for all $i = 1, \ldots, n$ and $j = 1, \ldots d_i$, this sequence assigns the copy (i,j) to the interval $[E(i,j),\ L(i,j)]$, where*

$$E(i,j) = \lceil \frac{j-B}{r_i} \rceil \ \text{and} \ L(i,j) = \lfloor \frac{j-1+B}{r_i} + 1 \rfloor$$

denote the release date and the due date of the copy (i,j) for given upper bound B.

An interesting question would be to show similar closed form formula for other measure of deviation, for instance squared deviation.

Amongst various versions of the earliest due date algorithms for scheduling unit time jobs with release times and due dates on a single machine (see Steiner and Yeomans [37] for the references), they apply a modified version of Glover's [11] $O(|E|)$ earliest due date (EDD) algorithm for finding a maximum matching in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ such that each ascending $k \in V_1$ is matched to the unmatched copy (i,j) with smallest due date value of $L(i,j)$ as defined in Lemma 1. They conclude the following.

Theorem 2 *The AMDJIT sequence s is 1-feasible if and only if the V_1-convex bipartite graph G with bound $B = 1$ has a perfect matching. Moreover, an optimal solution can be determined by an exact pseudo-polynomial algorithm with complexity $O(D \log D)$.*

Steiner and Yeomans [35] consider weighted AMDJIT problem and show that a binary search finds an optimal solution for the weighted AMDJIT in $O(D \log(D\phi G_{\max}))$ time, where ϕ is a positive integer constant depending upon problem data. The maximum weight $G_{\max} = \max_i G_i$ gives an upper bound and $LB_W = \min_i G_i(1 - r_i)$ gives a lower bound for the optimal objective value of the considered problem.

Theorem 3 *An optimal solution to the pegging multi-level AMDJIT can be determined by an exact pseudo-polynomial algorithm in $O(D\log(D\phi G_{\max}))$ time.*

Let B^* be the optimal value of the AMDJIT problem. Then for any instance d_i, $i=1,\dots,n$ of the AMDJIT problem, it holds that $1-\frac{1}{D} \geq B^* \geq \frac{1}{\Delta_i}\lfloor\frac{\Delta_i}{2}\rfloor$ for $i=1,\dots,n$ where $\Delta_i = \frac{D}{\gcd(d_i,D)}$, Brauner and Crama [6]. A stronger upper bound has been obtained by Tijdeman [41], $B \leq 1-\frac{1}{2(n-1)}$. Thus we have

Theorem 4 *For any instance d_i, $i=1,\dots,n$ $(n>1)$ of the AMDJIT problem, the optimal value B^* satisfies the inequality $B^* \leq 1-\max\{\frac{1}{D},\frac{1}{2(n-1)}\}$.*

As $D<2(n-1)$ when $d_i=1$ for all i with $n>2$, and $D\geq 2(n-1)$ in most practical cases, both possibilities have to be taken into account. Obviously, $B^*=0$ for $n=1$.

An instance of the AMDJIT sequencing problem is defined as *standard* if $\gcd(d_1,\dots,d_n)=1$. We call the corresponding sequence *standard*. The *small deviation conjecture* states that for $n\geq 3$, a standard instance $(d_1,\dots,d_n)$ of the AMDJIT problem has $B^*<\frac{1}{2}$ if and only if $d_i=2^{i-1}$ for $i=1,\dots n$, Brauner and Crama [6].

Brauner and Crama [6] prove the conjecture for $n\leq 6$ and conjectured it "true" for all positive n. Kubiak [22] presents a geometric proof that the conjecture holds true for any $n>2$. His proof exploits a natural symmetry of regular polygons inscribed in a circle of circumference D are described. Subsequently, Brauner, Jost and Kubiak [7] exploit the concept of balanced words to give another proof of the conjecture (see Section 5). Thus, we can state the following theorem.

Theorem 5 *For $n\geq 3$, a standard instance $(d_1,d_2,\dots,d_n)$ of the AMDJIT problem has optimal value $B^*<\frac{1}{2}$ if and only if $d_i=2^{i-1}$ for $i=1,\dots,n$, and $B^*=\frac{2^{n-1}-1}{2^n-1}$.*

This result can be restated as follows. For given rational numbers $r_1\leq r_2,\dots,\leq r_n$ with $n\geq 3$, it holds $\sum_i^n [kr_i]=k$ for any integer k if and only if $r_i=\frac{2^{i-1}}{2^n-1}$ for $i=1,\dots,n$. The statement observes that $|x_{i,k}-kr_i|<\frac{1}{2}$ implies $x_{i,k}=[kr_i]$, where $[x]$ denotes the rounding of x to the closest integer.

The structure of instances with $B\leq\frac{1}{2}$ becomes more complex as x_{ik} may then be equal either to $[kr_i]=kr_i-\frac{1}{2}$ or to $[kr_i]+1=kr_i+\frac{1}{2}$ for half-integer kr_i [6].

5 Balanced Words and AMDJIT Sequences

Brauner and Crama [6], Brauner, Jost and Kubiak [7], Jost [13], Kubiak [19, 22] study the AMDJIT sequences as balanced words. One of the main problems of balanced words in practice is to construct an infinite periodic sequence over a finite set of letters where each letter is distributed as "evenly" throughout the sequence as possible and each letter occurs with a given rate. Unfortunately, the existence of balanced sequences for most rates is unlikely.

We write an *infinite word* as $w=a_1a_2\dots$ such that $a_i\in\mathcal{A}=\{a_1,a_2,\dots,a_n\}$ for all $i\in\{1,\dots,n\}$. A *factor* of length $|f|\geq 0$ of w is word such that $f=a_ia_{i+1}\dots a_{i+|f|-1}$.

We say that the index i is the position of the letter a_i in the word w. The rate r_i of the letter a_i in finite word w is defined as the fraction $r_i = \frac{|w|_i}{|w|}$ where $|w|_i$ denotes the number of occurrences of the index i in the word w. An infinite sequence $w = w_1 w_2 \dots$ for which $||u|_i - |v|_i| \le \delta$ for all i with with $|u| = |v|$ is called δ-balanced. We denote the infinite repetition of a finite word w by $w^* = ww \dots$. An infinite word s is called periodic if $s = w^*$ for some finite word w. A finite word w is called symmetric if $w = w^R$ where w^R is a mirror reflection of w. An infinite balanced word s is called symmetric and periodic if $s = w^*$ for some finite symmetric word w.

One way of building an infinite word on finite letter alphabet $\mathcal{A}$ using the numbers $\frac{2j-1}{2r_i} = \frac{(j-1)D}{d_i} + \frac{D}{2d_i}$ is described in Kubiak [19]. It builds an infinite word as follows. Label the points $\{\frac{(j-1)D}{d_i} + \frac{D}{2d_i}\ \ , j \in \mathcal{N}\}$ by the letter i, consider $\cup_i^n \{\frac{(j-1)D}{d_i} + \frac{D}{2d_i}\ \ , j \in \mathcal{N}\}$ and the corresponding sequence of labels. Break the tie by choosing i over i' when $i < i'$ giving higher priority to a lower index whenever a conflict needs to be resolved. Thus a word with angle vector $\alpha = (\frac{D}{d_1}, \frac{D}{d_2}, \dots, \frac{D}{d_n})$ and the starting point $\beta = (\frac{D}{2d_1}, \frac{D}{2d_2}, \dots, \frac{D}{2d_n})$, referred to as an *hyperbolic billard* word in Vuillon [42], is obtained. Let w be an infinite word associated with *n-dimensional* hypercubic billards of angle α and starting point β. Then w is *d=n-1-balanced* on each letter. Moreover, the bound for the balance is always reached Vuillon [42].

Jost [13] proves that for any finite sequence of total demand $D = \sum_i^n d_i$ with maximum deviation B for n product rates d_i, any infinite periodic word w of period s is 1-balanced, 2-balanced or 3-balanced on each product i, if $B < \frac{1}{2}$, $B < \frac{3}{4}$ and $B < 1$, respectively. Any sequence with $d_i = 1$ for all $i = 1, \dots, n$, is a 1-balanced word though its maximum deviation $B = 1 - \frac{1}{n} > \frac{1}{2}$ for $n \ge 3$, Kubiak [19]. However, the maximum deviation B is greater than $\frac{3}{4}$ for the 2-balanced word $a_1 a_1 a_2 a_2 \dots a_n a_n$ with $d_i = 2$ for each $i = 1, \dots, n$. Likewise, the maximum deviation B is greater than 1 for the 3-balanced word $a_1 a_1 a_1 a_2 a_2 a_2 \dots a_n a_n a_n$ with $d_i = 3$ for each $i = 1, \dots, n$ with $n \ge 3$. Thus we have

Theorem 6 *Let s be a finite sequence of length $D = \sum_i^n d_i$ with maximum deviation B for n rates d_i, and let $S^{\frac{1}{2}}$, $S^{\frac{3}{4}}$ and S^1 be the sets of sequences with $B < \frac{1}{2}$, $B < \frac{3}{4}$ or $B < 1$, respectively. Then $S^{\frac{1}{2}}$, $S^{\frac{3}{4}}$ and S^1 are properly contained in the sets of 1-balance, 2-balance and 3-balance words, respectively.*

The result of Vuillon [42] shows that the priority based conflict resolution applied whenever there is a competition for an ideal position yields d being almost the same size of the alphabet, that is n. Theorem 6 shows that the conflict resolution provided by any algorithm minimizing maximum deviation leads to d being constant. Thus, it is clear that the conflict resolution provided by any algorithm minimizing maximum deviation yields a better balance than the priority based conflict resolution applied whenever there is a competition for an ideal position with model $n \ge 3$.

Theorem 6 combined with Theorem 4 guarantees the existence of an optimal solution in the set of all 3-balanced words. However, it is an open question whether there always exists a 2-balanced word that optimizes AMDJIT. For $n \ge 3$, the standard instance satisfies the property of 1-balanced words Kubiak [22], Brauner, Jost and Kubiak [7]. Kubiak [22] proves that there exists a periodic, symmetric and 1-balanced word on $n \ge 3$ letters with densities $r_1 \le r_2 \le \dots \le r_n$, if and only if the densities satisfy $r_i = \frac{2^{i-1}}{2^n - 1}$ (see Theorem 5). It is easy to construct symmetric, periodic, 1-balanced word with densities $\frac{2^{i-1}}{2^n-1}$: given such a sequence with $n \geq 3$ letters, one fixes a new letter and inserts it between every consecutive letters of s as well as at the beginning and end of s to obtain a sequence for $n+1$, letters with required properties. The number of instances with 1-balanced property is infinite in case of $n = 2$ as Brauner and Crama [6], Kubiak [22] prove that the optimal value of the AMDJIT problem is less than $\frac{1}{2}$ if and only if one of the demands is even and the other is odd.

6 The Cyclic MDJIT and SDJIT Sequences

In this section, we discuss the existence of cyclic sequence that are optimal. As all existing algorithms have time complexities depending on the magnitude of the demands $d_1, \ldots, d_n$ and hence on D, the existence of cyclic schedule reduces computational time. Therefore, the question whether the concatenation s^m of $m \geq 1$ copies of an optimal sequence s for $d_1, d_2, \ldots, d_n$ is optimal for $md_1, md_2, \ldots, md_n$ is important for JIT sequencing.

Miltenburg [30], Miltenburg and Sinnamon [31] observe the existence of cyclic schedules for sum of squared deviations in single-level. The min-sum problem have such a cyclic optimal solution if $f_i = f$ for all i, where f is convex and symmetric function with $f(0) = 0$, Bautista, Companys and Corominas [4]. Kubiak and Kovalyov [23] prove that if $f_i(x) = f(x)$ for all i with $x \in (0, 1)$ for symmetric and convex function f, then the cyclic schedule for min-sum problem is optimal. Moreover, they give a counterexample to show that the answer is negative if at least one f_i is asymmetric.

All the affirmative answers have been based on the following two observations. The first observation is that if $w = uv$ where u and v are sequences for the instances $\beta d_1, \ldots, \beta d_n$ and $\gamma d_1, \ldots, \gamma d_n$, respectively, where β, γ are positive integers, then $F_{SD}(w) = F_{SD}(u) + F_{SD}(v)$, Miltenburg [30]. The second observation is that even if one relaxes the constraints $x(w)_{iD} = d_i$, $i = 1, 2, \ldots, n$, then there still exists an optimal sequence w^* such that $x(w^*)_{iD} = d_i$, $i = 1, 2, \ldots, n$, Bautista, Companys and Corominas [4]. The latter conclusion does not hold if f_i are different though convex and symmetric f_i having the values zeros at 0, Kubiak and Kovalyov [23].

Kubiak [21] proves that the set of all optimal sequences for min-sum single-level problem includes *cyclic* sequences for symmetric, convex and nonnegative functions. In his proof a different exchange method is used.

Theorem 7 *Given $d_1, \ldots, d_n$, let s be an optimal sequence for the single-level min-sum problem SDJIT with convex, symmetric and nonnegative f_i, $i = 1, \ldots, n$, all assuming minimum 0 at 0. Then s^m, $m \geq 1$, is optimal sequence to SDJIT for $md_1, md_2, \ldots, md_n$.*

A similar result for single-level min-max problem MDJIT could be proved for l_∞-norm.

Steiner and Yeomans [35] show that the set of optimal sequences for both weighted as well as un-weighted single-level min-max problems for absolute deviations include *cyclic* sequences.

We conjecture that cyclic JIT sequences in multi-level problem are optimal.

7 Relations Between Different Objectives

Corominas and Moreno [8] prove the following.

Theorem 8 *Let s be any sequence for single-level JIT sequencing problem. Then $F_{ASD}(s) - F_{QSD}(s) = H_0 - H(s)$ where the constant $H_0 \geq 0$ depends only on the problem instance, and $H(s) = 0$ if s is an 1-bounded solution and positive otherwise.*

Furthermore, the min-sum problems for absolute and squared deviations have the same set of optimal solutions on $\mathcal{X}_1$, where $\mathcal{X}_1$ is the set of all 1-bounded solutions, [8]. Moreover, any 1-bounded solution optimal for min-sum absolute deviation problem (if exists) is also optimal for min-sum squared deviation problem, and hence, all optimal solutions for the latter problem are 1-bounded [8]. If none of the min-sum optimal solution for squared deviation is 1-bounded, then the problem for absolute deviation also does not have 1-bounded solution.

An optimal solution to the min-sum problem with absolute deviation which is not 1-bounded may not be optimal for the min-sum problem with squared deviation [8]. There may exist a 1-bounded optimal solution to the latter problem even though none of the optimal solution to the former problem is 1-bounded. Moreover, either of these problems may have 1-bounded optimal solutions [8].

Unlike the absolute deviation and squared deviation objectives for min-sum problems, the sets of 1-bounded optimal solutions with other convex, symmetric and nonnegative functions are not the same, Corominas and Moreno [8].

The empirical results of Kovalyov, Kubiak and Yeomans [18] refute number of conjectures about the relationships between optimal solutions for different objective functions.

8 Complexity and Dynamic Programming

The question of the exact complexity of single-level JIT sequencing problem remains open Kubiak [26]. As the input size of any instance $(d_1, \ldots, d_n)$ is $O(\sum_{i=1}^{n} \log d_i) = O(n \log D)$, an algorithm which is polynomial in n and D is only pseudo-polynomial but not polynomial in the input size. The problem MDJIT is in Co-NP but it is still open if the problem is Co-NP-Complete or polynomially solvable Brauner and Crama [6].

Kubiak [26] proves that a version of multi-level min-sum problem, referred to as *Output Rate Variation Problem*, is NP-hard. The multi-level min-max problem with absolute deviation objective is strongly NP-hard, Kubiak, Steiner and Yeomans [24].

However, Kubiak, Steiner and Yeomans [24] present following dynamic programming approach for multi-level, min-max and min-sum problems.

Let $\mathbf{d} = (d_{11}, \ldots, d_{n_1 1}) = (d_1, \ldots, d_{n_1})$ be the demand vector at level 1 and let e_i be a unit vector of dimension n_1 with unity in the ith row. Redefine states in a schedule by $\mathbf{X} = (x_1, \ldots, x_{n_1})$, where x_i denotes the cumulative production of the product i with $x_i \leq d_i$ and the *cardinality* of a state $\mathbf{X}$ as $|\mathbf{X}| = \sum_{i=1}^{n_1} x_i$. The minimum value of the maximum deviation for all products and parts over all partial schedules which lead to state $\mathbf{X}$ is defined by $\psi(\mathbf{X})$. The maximum norm $||\Gamma\mathbf{X}||_1$ represents the maximum deviation of actual production from desired one over all products and parts in state $\mathbf{X}$ at stage $k = |\mathbf{X}|$ (see Section 2.1 for the definition of Γ). Following *dynamic programming* DP_{AMD} recursion holds for $\psi(\mathbf{X})$ [43]:

$\psi(\emptyset) = \psi(\mathbf{X} \,:\, x_i = 0,\ i = 1, 2, \ldots, n_1) = 0,$
$\psi(\mathbf{X}) = \min_i \{\max\{\psi(\mathbf{X} - e_i),\ ||\Gamma\mathbf{X}||_1\} \,:\, x_i \geq 1,\ i = 1, 2, \ldots, n_1\}.$

The space and time complexities of DP_{AMD} are $O(\prod_{i=1}^{n_1}(d_i+1))$ and $O(nn_1\prod_{i=1}^{n_1}(d_i+1))$, respectively.

Kubiak, Steiner and Yeomans [24] give extensive experiments to problems of practical size. They consider Toyota's scheduling application described in Monden [32] which requires the production of $D_1 = 500$ products for one 8-hour production shift. Two filtering heuristics were introduced to reduce potentially vast state space to be examined in dynamic programming. They tested four level randomly generated problems with total product demands $D_1 = 500$ and unit weights. For a problem with $n_1 = 16$, $L = 4$ and $D_1 = 400$, time required to implement the algorithm is 342.38 minutes, for instance. The ratio of heuristic solution 79.857 to the optimal solution 76.136 is 1.03. Moreover, they conclude that the solution time of a problem strongly depends on the number of different products n but only slightly not on the range of part requirements.

The dynamic programming for multi-level min-max problem is modified for multi-level min-sum problem [24, 43].

The minimum total squared deviation for all products and parts over all partial schedules of $\mathbf{X}$ is defined by $\phi(\mathbf{X})$. For the amount of product produced $\mathbf{X}$, let $(||\Omega\mathbf{X}||_2)^2$ denoted by $\theta(\mathbf{X})$ be the squared sum of the deviations of actual production from the desired one for all products and parts (see Section 2.1 for the definition of Ω). Then the following *dynamic programming* DP_{QSD} recursion holds for $\phi(\mathbf{X})$ [24, 43]:

$$\phi(\emptyset) = \psi(\mathbf{X} \; : \; x_i = 0, \; i = 1, 2, \ldots, n_1) = 0,$$
$$\phi(\mathbf{X}) = \min_i\{\phi(\mathbf{X} - e_i) + \theta(\mathbf{X}) \; : \; x_i \geq 1, \; i = 1, 2, \ldots, n_1\}.$$

9 Concluding Remarks

In this paper we reviewed some research in JIT sequencing that has been carried out till now. A number of outstanding and interesting questions have been explored which are still open and challenging.

The single-level min-sum problems with any convex, symmetric, nonnegative functions which take the value zero only at zero deviation are solvable by reduction to the assignment problem. This approach applies to min-max problems as well.

A pseudo-polynomial binary search for a feasible B-bounded sequence obtained through perfect matching in bipartite graph solves the single-level min-max absolute-deviation problem. This approach can be applied to other convex, symmetric, nonnegative functions.

Regardless of the methods, obtaining common solutions to different objective functions would significantly save the complexity cost. However, the 1-bounded solutions obtained via incomplete bipartite graphs does not guarantee an optimal solution for min-sum problems. The question, what is minimum B such that optimal solution for min-sum problem is B-bounded?, remains open.

Although most of the single-level JIT problems had been efficiently solved by pseudo-polynomial algorithms depending on the input size of the demands, their complexity status is not yet clear. Even the basic min-man absolute-deviation problem is $Co-NP$ but it is still open whether the problem is Co-NP-Complete or polynomially solvable.

The multi-level problems for two or more levels are strongly NP-hard. However, they are efficiently solvable if either the products require approximately the same number and mix of parts or the pegging assumptions are imposed. Therefore, searching for special properties in this class of problems for which efficient algorithms exist or looking for good approximation algorithms would be an interesting direction of research in this area.

The existence of optimal schedules that are cyclic considerably reduces the computational requirements for any type of JIT optimization problem. This problem has been resolved for single-level problems. We conjecture that cyclic Just-in-Time sequences in multi-level are optimal as well.

One way to deal with JIT problems is the elegant concept of balanced words. However, 1-balanced words cannot be obtained for some rates. The set of all 3-balanced words always contains an optimal sequence for AMDJIT. It is an open question whether there always exists a 2-balanced word that is optimal for any given instance of AMDJIT. Characterizations of balance words to min-max squared-deviation and min-sum problems would be an interesting problem for further research.

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