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# Optimal Design of Rack-and-Pinion Steering Linkages

*A mathematical model is developed for the design synthesis of rack-and-pinion steering linkages. The general objective is to minimize the difference between the steering centers over the full range of steering angle inputs while fitting into a reasonable space. Because there is a substantial amount of design art in these systems and the mathematical representation is not clear, the model, constraints, and objective actually "evolve" to the eventual desired form. The problem has multiple optima, and practical and heuristic considerations are used to choose suboptimal but more realistic solutions, once satisfactory optimal solutions are identified. These involve manipulation of the objective function, constraint set, and intitial guesses. Both leading and trailing link designs are considered, the former being slightly better. Limitations of the model are also presented.*

## Introduction

While there is considerable literature on computer analysis, design, and optimization of mechanisms (see e.g. [1]), there appear to be few published applications to the design synthesis of vehicle steering systems. Visa and Alexandru address this problem in [2] although they present limited explicit results and conclusions; in this and a companion article [3], they are more concerned with cross-coupling effects due to the suspension-steering linkage interaction. Interestingly, these articles contain no citations from Western literature. Otherwise, most of the steering linkage literature is in vehicle-oriented publications and addresses qualitative factors affecting tire and linkage forces, tracking, and radius of curvature, e.g. [4–8]. In this paper, we attempt to relate this design art to specific criteria, mathematically expressed, and then employ computer-aided design techniques to synthesize "optimal" linkages. In so doing, the design synthesis problem actually evolves, since the problem as initially formulated was "incomplete"—as are most real design problems.

The system considered is the rack-and-pinion steering linkage, currently one of the most widely used systems. The advantages of rack-and-pinion steering are its superior response to steering inputs, simplicity, and relative ruggedness. There are two practical implementations of the rack-and-pinion steering mechanism, both of which are considered here:

- a trailing link design shown schematically in Fig. 1(*a*)
- a leading link design per Fig. 1(*b*)

Both designs are symmetric.

## Model

In positioning the front wheels of a car to guide its direction, ideally the center of curvature should be the same for all four wheels [4]. The greater the discrepancies between the respective centers ($C_1$ and $C_2$ in Fig. 1(*c*)), the more each wheel works against the others, producing increased wear on the tires, increased effort in the steering wheel, and additional drag which increases fuel consumption and decreases traction. For practical linkages of reasonable complexity and cost, it is impossible to make the centers coincide over a full range of steering, but we can seek to determine the dimensions of a feasible rack-and-pinion system to reduce the absolute value of the steering radius error to the minimum possible and/or to allocate the error in a most favorable way over the full range of steering inputs.

In Figs. 1(*a, b*), the actual links are $X(3)$, $R_1$, and $R_2$. However, the design can be specified in several ways, and the geometric relations are conveniently defined by the variables $X(i)$, $i = 1,2,3,4$. Using these variables and the steering input $U$ as shown, the model equations for both designs are identical, except for two sign changes. (Complete model equations are given in the Appendix.) In Fig. 1, trunnion width $TR$ (slightly less than tread width) and wheelbase $WB$ are vehicle size parameters. The vectors $C_1A$ and $C_2B$ are perpendicular to the respective steering angles and define the radii fo curvature as a function of $U$ and $\mathbf{X}$. With no change in the actual model equations, the model and results can be generalized by dividing through by $TR$. This results in a single vehicle size parameter $WB/TR = WB'$ and nondimensional variables denoted by the lower case variables, viz.,

$$\mathbf{x}:\quad x(i) = X(i)/TR,\ i = 1,2,3,4 \qquad u = U/TR \tag{1}$$

Unless otherwise stated, these variables are used, and the corresponding results are "per unit $TR$."

For practical computational reasons, the steering inputs are specified at discrete points $u_j$; usually the range was 0.02 to 0.08 in increments of 0.01. These correspond to actual movement of 2 to 12 cm, depending on $TR$. By symmetry,

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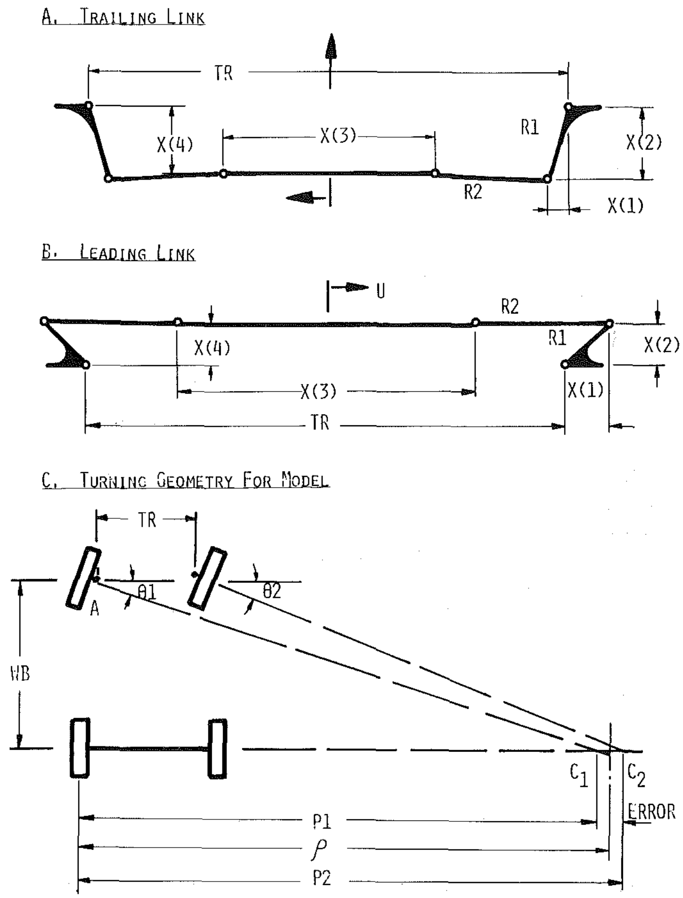


**Fig. 1 Steering linkages and turning geometry**

only the half-range need be considered explicitly. Errors for very small steering inputs are not particularly meaningful: for $u < 0.02$ the radius of curvature is usually so large that $\theta_1$ and $\theta_2$ in Fig. 1(*c*) are virtually identical; even though $|P_1 - P_2|$ may be fairly large, the actual amount of scrubbing or dragging of the tire as it is forced to move at an angle with respect to its normal rolling direction (i.e., side-slip) is very small. The maximum steering input $u$ would be extended when a minimum turning radius was desired that could not be satisfied in this range. (As $u$ increases, turning radius decreases.) While turning radius may be a more physically satisfying independent parameter than $u$, $u$ is more convenient at this stage of the modeling. Usually $u$ (max) = 0.08 was sufficiently large to satisfy turning circle considerations.

At any steering input $u_j$, the error (per unit *TR*) in the centers of curvature is (per Fig. 1(*c*))

$$e(u_j) = |P_1 - P_2| \text{ at that } u_j \tag{2}$$

For generality and for reasons to appear, the objective function is expressed here as

$$\text{minimize } E(x) = \sum_{j=1}^{J} q_j e(u_j) \tag{3}$$

by choice of $x(i)$, $i = 1,2,3,4$. The $q_j$, $j = 1,2, \ldots ,J$ are non-negative weightings on the error at each position $e(u_j)$. The $u_j$, $j = 1,2, \ldots ,J$ positions are specified. Accordingly, the objective is defined at the inputs for which the $q_i \neq 0$. In principle, the $q_i$ can be arbitrary, and several sets were considered, as discussed in a later section.

Equations (2, 3) comprise a kinematic minimum error objective (as suggested above and in [2, 4]) but do not consider certain dynamical considerations such as generation of lateral forces and the accompanying tire distortion, nor other geometric problems of fitting the mechanism into the vehicle. The latter can be partially considered in establishing system constraints, for example, by simple limits on the variables:

$$x_l(i) \leq x(i) \leq x_h(i), i = 1, \ldots ,4 \tag{4}$$

The baseline values are given in Table 1 and are suggested by practical considerations; we shall also consider the effects of these limits. Since the model can be solved explicitly (Appendix), there are no equality constraints.

Two other constraints were imposed, as needed: (*a*) the real

**Table 1 Variable limits and initial values (standard): $x^3$**

| | TRAILING LINK | | | LEADING LINK | | |
|---|---|---|---|---|---|---|
| | LOW | HIGH | $x^o$ | LOW | HIGH | $x^o$ |
| x(1) | 0.00 | 0.12 | .04 | .045 | .136 | .091 |
| x(2) | 0.10 | 0.20 | .15 | .045 | .136 | .091 |
| x(3) | 0.30 | 0.60 | .45 | .450 | .91 | .624 |
| x(4) | 0.08 | 0.20 | .14 | .045 | .136 | .091 |

## Nomenclature

- $C_1$ = center of curvature of back wheels and front wheel A, Fig. 1
- $C_2$ = center of curvature of back wheels and front wheel B, Fig. 1
- $E,E_1,E_2E_3,E_4$ = total error functions (see text)
- $e(u_j)$ = error $|P_1 - P_2|$ at $u_j$
- $i$ = index on design variables, $i$ = 1,2,3,4
- $j$ = index on normalized steering inputs
- $P_1$ = distance from left trunnion to $C_1$ along axis of back wheels, Fig. 1
- $P_2$ = distance from left trunnion to $C_2$ along axis of back wheels, Fig. 1
- $q_j$ = weighting factors at $u_j$
- $R_1,R_2$ = steering link lengths, Fig. 1
- $r_1$ = normalized steering link length, $R_1/WB$
- $r_2$ = normalized steering link length, $R_1/WB$
- $TR$ = steering trunnion width, Fig. 1
- $U$ = steering input, Fig. 1
- $u,u_j$ = normalized steering input, $U/TR$
- $WB$ = wheelbase, Fig. 1
- $WB'$ = normalized wheelbase, $WB/TR$
- $\mathbf{X},X(i)$ = actual design variables, Fig. 1
- $\mathbf{x},x(i)$ = normalized design variables, $x(i) = X(i)/TR$
- $\mathbf{x}^0$ = initial values of $\mathbf{x}$
- $\mathbf{x}^*$ = optimal values of $\mathbf{x}$
- $\mathbf{x}_l,x_l(i)$ = lower feasible values for $\mathbf{x}$
- $\mathbf{x}_h,x_h(i)$ = upper feasible values for $\mathbf{x}$
- $\rho$ = approximate overall center of curvature, $(P_1 + P_2)/2$
- $\theta_1,\theta_2$ = angle determined by steering linkage geometry when turning, Fig. 1

Table 2 Optimal designs for discrete values of $u$; trailing-link mechanism: $WB' = 1.85$; error curves in Fig. 3

| OPTIMIZED FOR u | $\underline{x}^*$: x(1)<br>x(2)<br>x(3)<br>x(4) | $R_1^*$<br>$R_2^*$ | TURNING RADIUS | AT u | THEORETICAL LIMIT (BINDING) u |
|---|---|---|---|---|---|
| .02(1) | .04445<br>.13412<br>.49732<br>.15481 | .14129<br>.20792 | 3.6475 | .06 | .06+ |
| .03 | .04446<br>.13548<br>.49228<br>.15325 | .14258<br>.21017 | 3.7918 | .06 | .063 |
| .04 | .04375<br>.13684<br>.48458<br>.15170 | .14367<br>.21447 | 3.9247 | .06 | .067 |
| .05 | .04309<br>.13892<br>.47971<br>.14942 | .14545<br>.21730 | 3.1596 | .07 | .07+ |
| .06 | .04243<br>.14102<br>.46302<br>.14637 | .14726<br>.22612 | 3.4276 | .06 | .077 |
| .07 | .04177<br>.14389<br>.45253<br>.14266 | .14983<br>.23197 | 2.8550 | .08 | .08+ |
| .08 | .04112<br>.14756<br>.44022<br>.13905 | .15318<br>.23892 | 3.1702 | .08 | .087 |

Note (1): $\underline{x}_0$ = .04, .15, .45, .14 for u = .02
$\underline{x}_0$ for u>.02 are $\underline{x}^*$ for proceeding u (see Text).

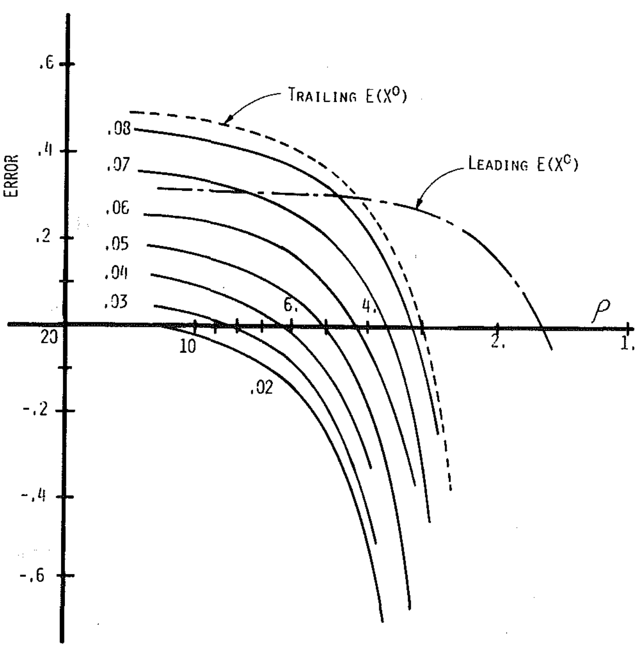


Fig. 2 Error curves when optimized for $E_1$, equation (7) for values of $u$ = 0.02, 0.03, . . . , 0.08; trailing link, $WB' = 1.85$. Also shown are $e(x^0)$ for trailing and leading link designs.

kinematic limit that the system not bind; and (*b*) ability to achieve a given minimum turning radius. Turning radius is approximated by

$$\rho = (P_1 + P_2)/2.$$

Binding becomes a factor if extremely large steering inputs are required to achieve the desired turning radius; usually this gives an impractical design. If within the reasonable range of $u$ the turning radius were inadequate, then (*b*) was imposed in

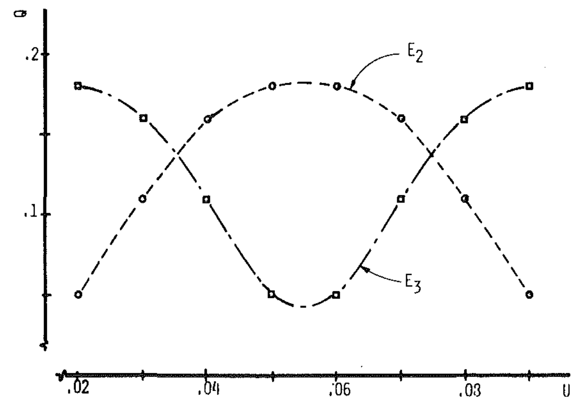


Fig. 3 Weightings for composite objective functions: $E_2$ and $E_3$

a subsequent run. Usually, satisfactory turning radii could be achieved; 2 to 2.5* $WB'$ is a typical minimum value.

Three values of the vehicle size parameter $WB'$ are considered: 1.70, 1.85, and 2.00. These span the current range of short subcompacts through large cars. For discussion purposes, the baseline size is $WB' = 1.85$, which corresponds to a large number of the current generation of popular designs—domestic, Japanese, and European. Nominal values would be as follows:

wheelbase (m) = 2.40 – 2.68
track (m) = 1.30 – 1.44

## Problem Solution

The preciseness of modeling and computer solutions notwithstanding, at the outset one often is not sure as to the most representative objective function, which constraints are critical, and whether additional constraints may be desired or required. That is, a criterion could be verbalized and even expressed generally as in equations (2, 3), but the actual weights clearly determine different designs; it thus becomes necessary to evaluate the different results (a finite number!) and establish an objective function and constraint set which we would expect to "minimize steering radius error over the steering range." Thus, we need some insight into the "character" of the problem, one which (as it turns out) has multiple optima that depend upon the starting point as well as the objective function and constraints. Thus, this study presents a picture of what is often involved in such studies: the evolution and determination of the "most suitable" combination of model, constraints, and objective function. Selection among the several optima may involve more than merely identifying the smallest value of an objective function and may be difficult to quantify, but it is nevertheless a real part of all design activities.

**Solution Method.** The algorithm used for optimization is the PATSH version of the Hooke-Jeeves pattern search [9, 10] with an exterior SLUMT type penalty function [11]. Accordingly, we seek to minimize

$$F(\mathbf{x}, r_k) = f(\mathbf{x}) + 1/r_k \text{ (violated constraints)}^2 \quad (6)$$

by choice of $x(i)$, $i = 1,2,3,4$

for a specified sequence of $r_k$, $r_1 > r_2 > \ldots > 0$. Typically, values of $r_k$ were 1.0, 0.01, 0.0001.

**Computational Experience: Evolution of Problem Statement.** The initial studies and most of our experience were on the trailing-link design of Fig. 1(*a*) and led to our eventual complete problem statement for leading as well as trailing link designs.

We began by generating optimal designs for specific steering inputs, viz.,

**Table 3 Dimensions of optimal solutions; objective function $E_4$ of equation (9)**

| WB′ | x*:x(1)<br>x(2)<br>x(3)<br>x(4) | "Best"<br>f(x)*<br>R1<br>R2 | Turning Radius<br>ρ @u | Binding<br>@ u | Alternate-Better Constraint Satisfaction<br>x*:x(1)<br>x(2)<br>x(3)<br>x(4) | f(x*) |
|---|---|---|---|---|---|---|
| 1.70 | .05865<br>.19782†<br>.29979†<br>.20078† | .03869<br><br>.20633<br>.35012 | 3.565 @ .10 | >.13 | .06221<br>.19863†<br>.18402†<br>.2000 | .03117 |
| 1.85 | .1064<br>.20145†<br>.60259†<br>.07489 | .03286<br><br>.22782<br>.15681 | 5.088 @ 1.25 | >.125 | essentially the same as 1.70 | .03455 |
| 2.00 | .10735<br>.2000 †<br>.59441†<br>.09 | .03490<br><br>.22697<br>.14564 | 4.91 @ .125 | >.125 | .0622<br>.1986†<br>.2000†<br>.2000† | .0378 |
| 1.70 | .08284<br>.18782†<br>.40983<br>.19723† | .05345<br><br>.20527<br>.29523 | 3.55 @ .10 | >.13 | .08216<br>.11692<br>.45 †<br>.1361 † | .08049 |
| 1.85 | .15215†<br>.15375<br>.41163<br>.2184 † | .05432<br><br>.21631<br>.30121 | 3.64 @ .11 | .13 | .05662<br>.13230†<br>.45 †<br>.13593† | .08699 |
| 2.00 | .16005†<br>.15619<br>.40988<br>.22791† | .05881<br><br>.22363<br>.30364 | 4.06 @ .11 | >.13 | .0583<br>.13585†<br>.44926†<br>.14016† | .09207 |

† At or exceed a limit.

$$E_1 = e(u_j)\text{, a specific } u_j \tag{7}$$

then evaluating the errors at the other inputs and the feasible steering range. As might be expected, the error could be made zero at any given $u_j$, but the errors at the other points could become quite large. This criterion was used in [2] for one position only. This can be seen in Fig. 2; the designs appear in Table 2. Values for the initial design vector $\mathbf{x}^0$ are also shown in Fig. 2. In generating the optima, starting with $\mathbf{x}^0$ for $u_1$ = 0.02, each optimum served as the initial guess for the next position. The optima are denoted by $\mathbf{x}^*$. This yielded a "family" of designs, in which rather small changes in the design variables essentially translate a characteristic curve laterally. In this case, there are about 10 percent decreases in $x(1)$, $x(3)$, and $x(4)$ and a comparable increase in $x(2)$ for the optima at $u = 0.02$ and $u = 0.08$.

This suggested that some composite objective function should give better overall results, hence the form of equation (3). The error form of equation (2) was retained throughout, since it is physically meaningful and should provide well-defined optima (e.g., as compared to a quadratic error function). Examples of the weighting $q_i$ are shown in Fig. 3. Our initial logic suggested the weightings for the curve denoted "criterion $E_2$"; it soon became apparent that most variants which penalized intermediate points any significant amount would result in error curves generally as shown in Fig. 2: zero at some intermediate point and increasing in absolute value toward either end of the steering range. Hence, we essentially inverted this curve for the weights of curve denoted "criterion $E_3$" in Fig. 3. $E_3$ penalizes errors at the end points and gave greatly improved composite error curves, as will be discussed. Taking this "philosophy" to an extreme suggested that an objective function of the form

$$E(\mathbf{x}) = q_l e(u_{\text{low}}) + q_h e(u_{\text{high}}) \tag{8}$$

would further "flatten" the error curve. However, equation (8) turned out to be bistable. "Poor" results were obtained for $u_{\text{high}} > 0.085$. Since $u_h = 0.08$ generally gave acceptable

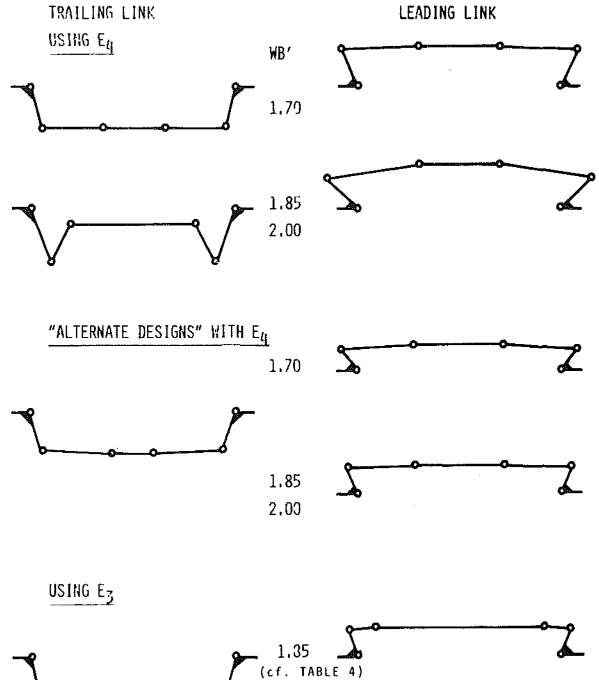


**Fig. 4 Optimal designs and selected alternates (to scale)**

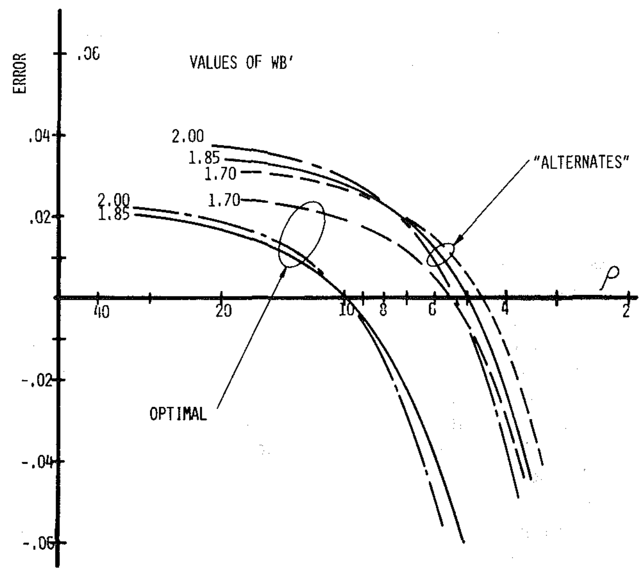


**Fig. 5 Error curves for trailing link optimal designs according to $E_4$**

minimum turning radii this value was used for $u_{\text{high}}$; $u_{\text{low}}$ = 0.02, per previous discussion. Using these, another "switch" between two different sets of solutions occurred around $q_l = 2q_h$. In all cases, the better solutions occurred when $q_h < q_l$.

A further "flattening" of the error characteristic was obtained by introducing an additional constraint on the errors at specified design points, typically the two end points, viz.

$$|e(0.02) - e(0.08)| \leq 0.01 \tag{9}$$

As a consequence of this experience, consistently "good" results were obtained with the constraints of equations (4) and (8), ad hoc imposition of the binding and turning radius constraints, and an objective function of

$$E_4 = e(0.02) + e(0.08) \tag{10}$$

This combination usually resulted in lower errors throughout the included steering range than did the composite objective

Table 4 Comparisons of objective function results $WB' = 1.85$

| Objective Function | Trailing Link $\underline{x}^*$:x(1)<br>x(2)<br>x(3)<br>x(4) | $f(\underline{x}^*)$ | Leading Link $\underline{x}^*$:x(1)<br>x(2)<br>x(3)<br>x(4) | $f(\underline{x}^*)$ |
|---|---|---|---|---|
| $E_1 = e(.05)$ | .04309<br>.13892<br>.47971<br>.14942 | 0.0 | | |
| $E_1 = e(.06)$ | .04243<br>.14102<br>.46302<br>.14637 | 0.0 | | |
| $E_2$ optimal | .1064<br>.20145*<br>.60257*<br>.07467 | .03286 | .1064<br>.20145<br>.60257<br>.07469 | .08409 |
| $E_2$ alternate | .06221<br>.19863*<br>.18402<br>.2000* | .03455 | .15215<br>.15375<br>.41163<br>.2184 | .08699 |
| $E_3$ | .0504<br>.1892<br>.6000*<br>.1871 | .1688<br>.1352 | .0454<br>.1290<br>.8324<br>.1352 | .1473 |
| $E_4$ | .0544<br>.1977*<br>.602*<br>.204* | .1743 | .0869<br>.0835<br>.8291<br>.1175 | .1653 |

* Variables at a limit, usually upper

functions. The range of $0.2 \leq u \leq 0.08$ generally spanned the required turning range.

## Results

For all objectives, the results showed some dependence on the starting point and to a lesser extent the variable limits, both of which are somewhat arbitrary. Often the initial guess was suggested by a previous solution, as above. The variable limits of equation (4) and Table 1 are largely practical, and therefore some violation is of little real consequence. Similarly, equation (9) was almost never satisfied, but it did tend to reduce the errors. Indeed, when a solution was infeasible, successive SLUMT stages might or might not give poorer answers (increased errors) as the solution was forced to meet the constraints. The limits were adjusted in a few studies to satisfy some extreme requirements—usually minimum steering radius without binding—or just to see what happened.

In presenting the results, generally the signed value of the error is plotted versus decreasing turning radius which corresponds to increasing input $u$, since the radius of turn is probably a more physically satisfying parameter. For convenience, the criteria are identified as follows:

| | | |
|---|---|---|
| 1. | $E_1$ | optimized for a specific input position, equation (7) |
| 2. | $E_2$ | equation (3), weighting of curve $E_2$, Fig. 3 |
| 3. | $E_3$ | equation (3), weighting of curve $E_3$, Fig. 3 |
| 4. | $E_4$ | equation (10) |

With criteria 2, 3, 4 the constraint set included equation (9). One selection of the "best" criterion and then the "best" design was to plot the range steering error for each computer-generated optimal design: criterion $E_4$ consistently produced the least magnitude error from $u = 0.02$ to $u = 0.08$; these $u$ typically gave $\rho > 15$ and $\rho < 4$, respectively. These designs are given in Table 3, and shown schematically in Fig. 4; the

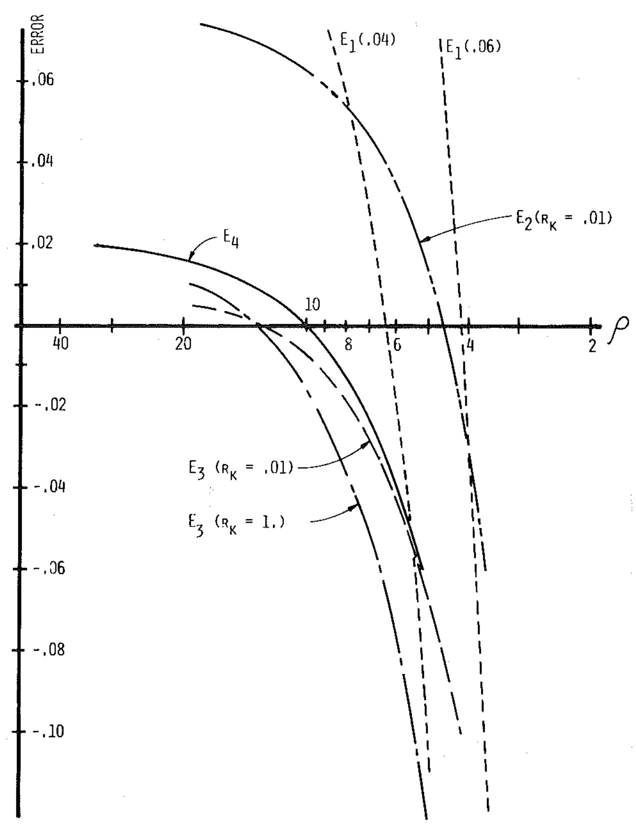


Fig. 6 Comparison of errors achieved for optima according to several objectives and at successive SLUMT stages; trailing link, $WB' = 1.85$

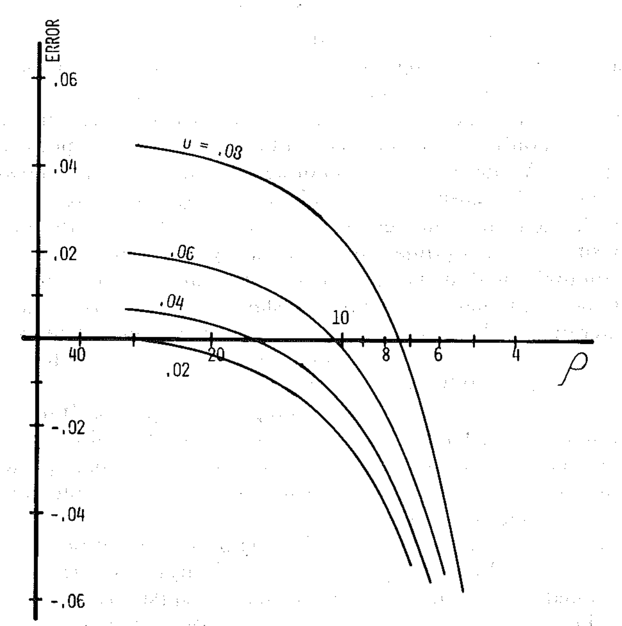


Fig. 7 Family of optima: trailing link, $WB' = 1.85$; $E_1$ minimized starting at $x^0 = x^*$ of $E_4$

error characteristics are shown in Figs. 5, 7, and 8. A subjective decision addressed the resulting linkage geometry and provides the reasons for also showing a slightly less optimal design, at least according to the mathematical criteria; these are discussed in the following sections. For comparison, designs and error curves for $WB' = 1.85$ (our typical car) for all criteria are given in Table 4 for both designs and in Fig. 6 for the trailing link design.

Note that the errors (abscissae) of curves 4–8 are one-tenth

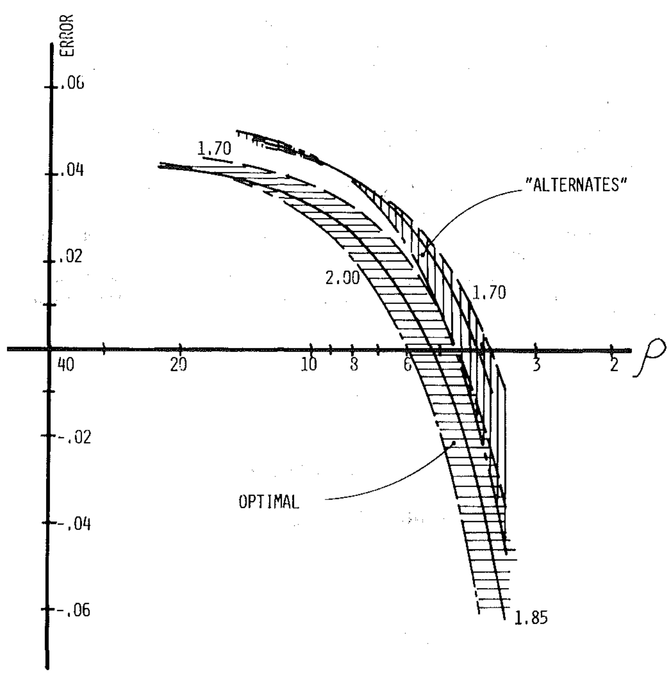


Fig. 8 Errors for optimal leading link designs using $E_4$

those of the initial guesses and the exploratory single-position optima (Fig. 2).

**Trailing Link Designs.** With $E_4$ and the same standard $\mathbf{x}^0$, the optimum design for $WB' = 1.70$ is substantially different than those for $WB' = 1.85$ and 2.00, which are essentially the same, per Fig. 4. This is true for the "best" leading link design, also. The latter designs will tend to induce large forces in the connecting arms, so we sought to find alternate optima using the optimal $\mathbf{x}$, i.e., $\mathbf{x}^*$, for $WB' = 1.70$ as the initial guess for $WB' = 1.85$ and 2.00. The result is the alternate design shown in Table 4; while quite good, the solutions show slightly larger errors than the previously found optima. For $WB' = 1.70$, the alternate actually appeared as a consequence of the successive SLUMT minimizations. The alternate designs for $WB' = 1.85$ and 2.00 never appeared at any SLUMT stage from the standard initial guess unless guided as above. All the alternate designs (for $\mathbf{x}^*$ for $WB' = 1.70$) are feasible; $\mathbf{x}^*$ for $WB' = 1.85$ and 2.00 give $x(2)^*$ and $x(3)^*$ at the upper limit.

Criteria $E_2$ and $E_3$ yield optima similar to the alternate, except that the connecting link $x(3)$ is longer (at the upper limit), as seen in Table 4 for $WB' = 1.85$. The results for $E_2$, $E_3$, and $E_4$ emphasize that weighting intermediate positions mainly penalizes points which will tend to have smaller errors than at the extremes. For example, designs for $E(0.05)$, $E(0.06)$, $E_2$, and $E_3$ are more similar to each other than to $E_4$. Even though these are all "optimal designs" clearly $E_4$ optima and the alternatives yield less error over the widest range. $E_3$ is the next best, per Fig. 6. However, despite the significantly different linkage, the actual error curves are not all that different: The alternate error curves are translated to the right. A further advantage of the alternate designs is the shorter turning radius for a given $u$. All of the alternate designs look very similar for the various $WB'$. Principally, the longer $WB'$ tend to increase the errors at the extremes.

What we apparently are seeing are multiple optima—"dips" in the design objective surface. As in the earlier study of the one-input-position optima (Table 2, Fig. 3 of which the results were distinctly suboptimal, relatively), we can generate families of optimal designs using the $\mathbf{x}^*$ of $E_4$ as starting points of a minimization for $E_1$ $(u)$. This was done for $WB' = 1.85$. Starting with $\mathbf{x}^0 = (0.110, 0.199, 0.590, 0.081) \approx \mathbf{x}^*$ for $E_4$, $x(1)$, $x(2)$, and $x(3)$ were essentially unchanged while the setback $x(4)$ increased from 0.0811 for $u = 0.02$ to 0.0949 for $u = 0.08$; errors were zero at the respective $u$. The error curves are shown in Fig. 7. Clearly this particular design is quite sensitive to $x(4)$.

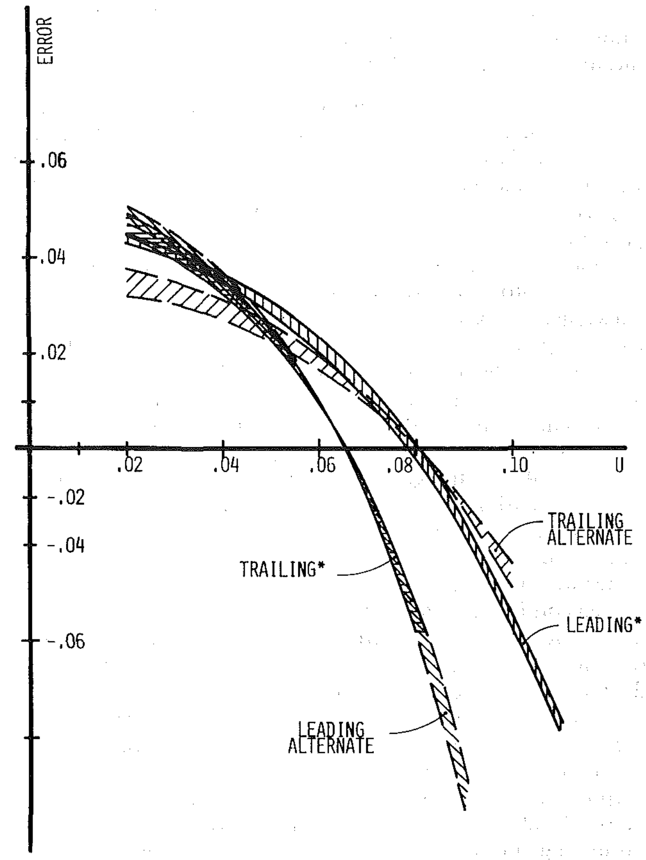


Fig. 9 Grouping error characteristics of optimal designs: leading and trailing link mechanism according to $E_4$

**Leading Link Design.** The properties and evolution for this arrangement are generally similar to the foregoing; one difference in the soluton process is that significantly different designs were found with successive SLUMT optimizations. Here, satisfaction of the constraints resulted in quite different designs, albeit with significantly larger values of $E_4$. In Fig. 4, the smallest error designs were substantially bowed and would probably interfere with the tires, especially for the longer $WB'$. The error curves of Fig. 8 are clearly grouped according to the specific design. The leading link designs are somewhat sharper-turning than the trailing-link designs.

## Discussion

**Comparison of Leading and Trailing Link Designs.** In all cases, the alternate designs (and the optima for $WB' = 1.70$) offer practical advantages, for example, less likelihood of interference, lower internal forces, and shorter turning radius. While we have presented the results versus turning radius, if presented versus $u$ it is seen that the various curves are essentially coincident, per Fig. 9. The minimum error point is usually at high $u$, due to the increasing slope of the error curve with $u$.

While the optimal and alternate trailing-link designs yield lower errors in the design range, sales-literature and parking-lot surveys indicated that the leading link designs predominate in recent vehicles, especially among the front-wheel drive vehicles. Incidentally, the leading link $\mathbf{x}^0$ for this study was based on a very popular subcompact import. This design shows virtually constant and large error when evaluated in this model, per Fig. 2, "Leading $E(\mathbf{x}^0)$" curve.

**Overall System Design Perspective.** In order to concentrate on the reduction of errors in the location of the turning center of the wheels of a vehicle, some important aspects of the steering problem have been disregarded or simplified, and it is important to recognize the limitations of such a design procedure.

First, the steering mechanism is part of a total design. It must fit into and around the dimensions of the vehicle-wheelbase, track width of the front tires, chassis or subframe, front suspension, and the engine space requirements. Clearly, these restrict the locations of steering linkage parts. Next, the linkage must not interfere with the suspension and its movements; note that the vertical movement of the suspension can induce a steering effect depending on the relative positions of the linkages. If the pivot points of the upper and lower suspension arms and the steering connecting rod are not aligned, the connecting rod would "push" or "pull" the wheel when the suspension moves, causing a steering effect. Other arrangements are possible with different amounts of suspension-induced steering. To some extent, the kingpin angle and the effective tire pivot point determine the true trunnion location, rather than the kinematic location. A dynamical consideration is the need for tire side forces to control turning; because of frictional effects these can alter the true center of curvature from the theoretical values calculated here. Some of these effects could be directly incorporated into the model or constraint set, although in the present study, we have imposed these "after the fact" i.e., by dismissing unacceptable designs. Inclusion might cause the results (designs) to change, but the approach and general tendencies should remain much the same.

## Closure

Our purposes here are to demonstrate the evolution of a suitable "complete" problem formulation including constraints and objective and then to illustrate the character of the design results. We have shown that multiple optima can be identified, but that these too can be preferentially evaluated.

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# APPENDIX

**Model (Ref. Figures 1)**

Centered Position

$$R_1 = \sqrt{X(1)^2 + X(2)^2}$$
$$ST = (TR - X(3))/2$$
$$R_2 = \sqrt{(ST \mp X(1))^2 + (X(2) - X(4))^2} \quad **$$
$$B_4 = \arctan(X(4)/ST)$$
$$D^2 = ST^2 + X(4)^2$$
$$A_4 = \arccos(R_1^{\,2} - R_2^{\,2} + D^2)/(2^* R_1{}^* D))$$
$$\text{ANGI} = A_4 + B_4$$

Right Turn: Input $U$ as Shown in Figs. 1($a$, $b$)

Right Side

$$DX_1 = ST + U$$
$$D_1 = \sqrt{DX_1^{\,2} + X(4)^2}$$
$$B_1 = \arctan(X(4)/DX_1)$$
$$A_1 = \arccos((R_1^{\,2} - R_2^{\,2} + D_1^{\,2})/(2^* R_1{}^* D_1))$$
$$\theta_1 = \text{ANGI} - (A_1 + B_1)$$

Left Side

$$DX_2 = ST - U$$
$$D_2 = \sqrt{DX_2^{\,2} + X(4)^2}$$
$$B_2 = \arctan(X(4)/DX2)$$
$$A_2 = \arccos((R_1^{\,2} - R_2^{\,2} + D_2^{\,2}/(2^* R_1{}^* D_2))$$
$$\theta_2 = (A_2 + B_2) - \text{ANGI}$$

ERROR

$$E(u) = |P_1 - P_2|, \text{ where}$$
$$P_1 = \pm TR + WB/\tan(\theta_1) \quad **$$
$$P_2 = WB/\tan(\theta_2)$$

* Upper sign for trailing-link, lower for leading

** Left turn is symmetric.

With $TR = 1$, all variables are normalized per text equation (1).