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On an Inverse Problem for a Class of Dirac Operator with Discontinuous Coefficient and a Spectral Parameter in the Boundary Condition

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Abstract

In this paper, it is examined on the half line the inverse problem of scattering theory for a class Dirac operator with discontinuous coefficient and a spectral parameter in the boundary condition . The scattering function is defined as scattering data and its properties are investigated. It is obtained Gelfand-Levitan-Marchenko type main equation which plays an important role in the solution of inverse problem and it is shown the uniqueness of the solution of the inverse problem by using Fredholm alternative.

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1 Introduction

The inverse problem of scattering theory for Dirac operator on the half line was examined in [1, 3, 5] (see [11]). The similar problem for Dirac operator of order two with discontinuous coefficients was considered in [6, 7]. In this study as different from other studies it is used new integral presentation, not operator transformation. The uniqueness of the solution of the inverse scattering problem for Dirac operator with discontinuous coefficients for not containing parameter was shown in [13]. In this paper in the case of discontinuous coefficients, it is investigated an inverse problem with a spectral parameter in the boundary condition.

We remind that the inverse problem of scattering theory for Sturm-Liouville operator on the half line was solved completely in [2, 14], for discontinuous case in [4, 8, 12].

On the half line $[0,\infty)$, we consider the boundary-value problem generated by the differential equation

$$BY' + \Omega(x) Y = \lambda\rho(x) Y \tag{1}$$

and the boundary condition

$$Y_1(0) + \lambda Y_2(0) = 0, \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \ Y = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real measurable functions, λ is a spectral parameter and

$$\rho(x) = \begin{cases} \alpha, & 0 \le x \le a, \\ 1, & a < x < \infty, \end{cases}$$

and $1 \neq \alpha > 0$.

Assume that the condition

$$\int_0^\infty \|\Omega(x)\| \, dx < \infty \tag{3}$$

is satisfied for Euclidean norm of the matrix function $\Omega(x)$. Let

$$\mu(x) = \begin{cases} a + \alpha(x-a), & 0 \le x \le a, \\ x, & x > a. \end{cases}$$

It is easily shown that the vector function

$$f^0(x,\lambda) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda\mu(x)}$$

is a solution of the equation (1) when $\Omega(x) \equiv 0$.

As known from [6], when the condition (3) is satisfied, for $\operatorname{Im}\lambda \ge 0$ the equation (1) has an solution $f(x,\lambda)$ which satisfies the condition

$$\lim_{x\to\infty} f(x,\lambda)\, e^{-i\lambda x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and can be expressed uniquely as

$$f(x,\lambda) = f^0(x,\lambda) + \int\limits_{\mu(x)}^{\infty} K(x,t) \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda t} dt. \tag{4}$$

Moreover, the elements of the matrix kernel $K(x,t)$ are summable on the positive half line and for the Euclidean norm of $K(x,t)$, the inequality

$$\int\limits_{\mu(x)}^{\infty} \|K(x,t)\|\, dt \le e^{\sigma(x)} - 1 \tag{5}$$

is satisfied, where $\sigma(x) = \int\limits_{x}^{\infty} \|\Omega((t))\|\, dt.$

Also,

$$\rho(x)\{BK(x,\mu(x)) - K(x,\mu(x))B\} = \Omega(x) \tag{6}$$

and if the matrix function $\Omega(x)$ is absolutely continuous, then the kernel $K(x,t)$ satisfies:

$$BK_x(x,t) + \Omega(x)K(x,t) = -\rho(x)K_t(x,t)B.$$

Let $Y(x,\lambda)$ and $Z(x,\lambda)$ be vector solutions of equations system (1). The expression

$$\begin{aligned} W[Y(x,\lambda), Z(x,\lambda)] &= Y^T(x,\lambda)BZ(x,\lambda) = (Y_1,Y_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ &= Y_1Z_2 - Y_2Z_1 \end{aligned}$$

is called Wronskian of the vector functions $Y(x,\lambda)$ and $Z(x,\lambda)$.

Since $p(x)$ and $q(x)$ are real valued functions, the vector functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ are solutions of the equation (1) for all real λ. The Wronskian of the vector functions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ doesn't depend on x and is equal to $2i$.

We denote by $\varphi(x,\lambda)$ the solution of the equation (1) satisfying the conditions

$$\varphi_1(0,\lambda) = -\lambda, \qquad \varphi_2(0,\lambda) = 1.$$

Let us define the function

$$E(\lambda) \equiv f_1(0,\lambda) + \lambda f_2(0,\lambda).$$

The paper is organized as follows: In Section 1 it is used the new integral representation (not operator transformation) for the solution of the equation (1), also, we find the scattering function corresponding to the boundary value problem (1)-(2) and its properties are investigated in Section 2. The Gelfand-Leviatan-Marchenko type integral equation or main equation of the inverse problem of scattering theory for the boundary value problem (1)-(2) which is related the scattering function is derived and it is obtained Levinson formula in Section 3. Finally, the solvability of main equation is proved and the unique recovery of the potential from the solution of the main equation is shown in Section 4.

2 The Properties of Scattering Function

The following lemma is valid.

Lemma 1 *For real λ, the identity*

$$\frac{2i\varphi(x,\lambda)}{f_1(0,\lambda) + \lambda f_2(0,\lambda)} = \overline{f(x,\lambda)} - S(\lambda) f(x,\lambda) \tag{7}$$

holds, where

$$S(\lambda) = \frac{\overline{f_1(0,\lambda)} + \lambda \overline{f_2(0,\lambda)}}{f_1(0,\lambda) + \lambda f_2(0,\lambda)} \tag{8}$$

and

$$|S(\lambda)| = 1.$$

Proof. Since $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ constitute the fundamental system for real numbers λ,

$$\varphi(x,\lambda) = c_1 f(x,\lambda) + c_2 \overline{f(x,\lambda)}$$

is written. Taking into account the following relations

$$W[f(x,\lambda), \varphi(x,\lambda)] = f_1(0,\lambda)\varphi_2(0,\lambda) - f_2(0,\lambda)\varphi_1(0,\lambda) = f_1(0,\lambda) + \lambda f_2(0,\lambda)$$

and

$$W\left[\overline{f(x,\lambda)}, \varphi(x,\lambda)\right] = \overline{f_1(0,\lambda)}\varphi_2(0,\lambda) - \overline{f_2(0,\lambda)}\varphi_1(0,\lambda) = \overline{f_1(0,\lambda)} + \lambda \overline{f_2(0,\lambda)}$$

it is obtained

$$c_1 = -\frac{\overline{f_1(0,\lambda)} + \lambda\overline{f_2(0,\lambda)}}{2i}, \qquad c_2 = \frac{f_1(0,\lambda) + \lambda f_2(0,\lambda)}{2i}. \tag{9}$$

Hence

$$\varphi(x,\lambda) = -\frac{\overline{f_1(0,\lambda)} + \lambda\overline{f_2(0,\lambda)}}{2i} f(x,\lambda) + \frac{f_1(0,\lambda) + \lambda f_2(0,\lambda)}{2i}\overline{f(x,\lambda)}.$$

Since $p(x)$ and $q(x)$ are real valued functions, $f_1(0,\lambda) + \lambda f_2(0,\lambda) \neq 0$ for real λ. In fact, on the contrary there exits a real number λ_0 such that $f_1(0,\lambda_0) = -\lambda f_2(0,\lambda_0)$. Then

$$f_1(0,\lambda_0)\overline{f_2(0,\lambda_0)} - f_2(0,\lambda_0)\overline{f_1(0,\lambda_0)} = 2i$$

is valid according to expression of Wronskian. It is obtained a contradiction from here and the assumption is not true. Dividing both sides of last expression by $E(\lambda) \equiv f_1(0,\lambda) + \lambda f_2(0,\lambda)$, the identity (7) is obtained. The equality $|S(\lambda)| = 1$ is found directly from (8). The lemma is proved. ■

Lemma 2 *The function $E(\lambda)$ has no zeros in the upper plane* $(\operatorname{Im}\lambda \geqslant 0)$.

Proof. We showed above that the function $E(\lambda)$ had no zeros in the real line. It is clear from the expression (4) of solution that the functions $f_1(0,\lambda)$ and $f_2(0,\lambda)$ can be continued as analytical and are continuous on the whole line. It is seen from the expression of $E(\lambda)$ that these properties are also satisfied for $E(\lambda)$.

Since

$$f(0,\lambda) \to \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad |\lambda| \to \infty,$$

the zeros of $E(\lambda)$ in the upper plane are not more than countable and constitute a bounded set.

Let us show that the function $E(\lambda)$ has no zeros in the half plane $(\operatorname{Im}\lambda > 0)$. Assume the contrary. Let λ $(\operatorname{Im}\lambda > 0)$ be a zero of the function $E(\lambda)$

$$\begin{aligned} Bf'(x,\lambda) + \Omega(x) f(x,\lambda) &= \lambda\rho(x) f(x,\lambda), \\ -\dot{f}^*(x,\lambda) B + f^*(x,\lambda)\Omega(x) &= \overline{\lambda}\rho(x) f^*(x,\lambda). \end{aligned}$$

Taking this into account, multiplying the first equation by $f^*(x,\lambda)$ and the second equation by $f(x,\lambda)$. Substracting the first equality from the second one, and finally integrating this relation from 0 to ∞, we get

$$W\left\{\overline{f(x,\lambda)}, f(x,\lambda)\right\}|_{x=0} + \left(\lambda - \overline{\lambda}\right)\int_0^\infty f^*(x,\lambda) f(x,\lambda)\rho(x)\,dx = 0.$$

On the other hand

$$E(\lambda) = f_1(0,\lambda) + \lambda f_2(0,\lambda) = 0$$

or

$$f_1(0,\lambda) = -\lambda f_2(0,\lambda).$$

Hence,

$$\begin{aligned} W\left[\overline{f(x,\lambda)}, f(x,\lambda)\right] &= \overline{f_1(0,\lambda)} f_2(0,\lambda) - \overline{f_2(0,\lambda)} f_1(0,\lambda) \\ &= -\overline{\lambda}\left|f_2(0,\lambda)\right|^2 + \lambda\left|f_2(0,\lambda)\right|^2 = \left(\lambda-\overline{\lambda}\right)\left|f_2(0,\lambda)\right|^2 \end{aligned}$$

and taking it into account

$$\left(\lambda-\overline{\lambda}\right)\left\{\left|f_2(0,\lambda)\right|^2 + \int_0^\infty f^*(x,\lambda) f(x,\lambda)\rho(x)\,dx\right\} = 0$$

it is found $\lambda = \overline{\lambda}$ from here. It is contrary to assumption. This contradiction shows that $E(\lambda)$ has no zeros in the half plane $(\operatorname{Im}\lambda > 0)$. The lemma is proved. ■

The function $S(\lambda)$ defined by (8) is called *scattering function* of the boundary value problem (1),(2).

From the definition of $S(\lambda)$ we have the following lemma.

Lemma 3 *For large λ, as $|\lambda| \to \infty$ the following asymptotic form holds*

$$S(\lambda) = S_0(\lambda) + O\left(\frac{1}{\lambda}\right), \tag{10}$$

where

$$S_0(\lambda) = -e^{-2i\lambda a(1-\alpha)}.$$

Proof. Substituting (4) into the expression $E(\lambda)$, we obtain

$$\begin{aligned} E(\lambda) &= e^{i\lambda a(1-\alpha)} + \int_{a(1-\alpha)}^\infty \left(K_{11}(0,t) - iK_{12}(0,t)\right)e^{i\lambda t}dt \\ &\quad +\lambda\left\{-ie^{i\lambda a(1-\alpha)} + \int_{a(1-\alpha)}^\infty \left(K_{21}(0,t) - iK_{22}(0,t)\right)e^{i\lambda t}dt\right\}. \end{aligned}$$

Here, by using partial integration we get

$$E(\lambda) = \lambda\left[-ie^{i\lambda a(1-\alpha)} + O\left(\frac{1}{\lambda}\right)\right].$$

Taking into account (8) and using properties of $K_{ij}(x,t)$, $i,j = 1,2$, we obtain

$$\begin{aligned} -ie^{-2i\lambda a(1-\alpha)} - S(\lambda) &= -ie^{-2i\lambda a(1-\alpha)} - \frac{\lambda\left[ie^{-i\lambda a(1-\alpha)} + O\left(\frac{1}{\lambda}\right)\right]}{\lambda\left[-ie^{i\lambda a(1-\alpha)} + O\left(\frac{1}{\lambda}\right)\right]} \\ &= \frac{O\left(\frac{1}{\lambda}\right)}{-ie^{i\lambda a(1-\alpha)} + O\left(\frac{1}{\lambda}\right)} \\ &= O\left(\frac{1}{\lambda}\right), \qquad |\lambda| \to \infty \end{aligned}$$

Therefore, for $|\lambda| \to \infty$ we obtained that the asymptotic form (10) holds and this completes the proof of lemma. ■

The function $S(\lambda) - S_0(\lambda)$ is continuous on the real axis $-\infty < \lambda < +\infty$ and by using the Lemma2 we have $S(\lambda) - S_0(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $|\lambda| \to \infty$. Then the function $S(\lambda) - S_0(\lambda)$ is integrable square in the neighborhood of $+\infty$ $(-\infty)$, and is the Fourier transform of some function in $L_2(-\infty, +\infty)$.

3 The Main Equation

Now, we shall obtain the main equation that contributes to construct the potential $\Omega(x)$ in the equation (1) to scattering function. For this, we rewrite the identity (7) as the following form

$$\begin{aligned} &\frac{2i\varphi(x,\lambda)}{E(\lambda)} + S_0(\lambda)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)} - \begin{pmatrix}1\\i\end{pmatrix}e^{-i\lambda\mu(x)} \\ = &\int_{\mu(x)}^{\infty} K(x,t)\begin{pmatrix}1\\i\end{pmatrix}e^{-i\lambda t}dt - S_0(\lambda)\int_{\mu(x)}^{\infty} K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt \\ &+ \left[S_0(\lambda) - S(\lambda)\right]\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)} + \left[S_0(\lambda) - S(\lambda)\right]\int_{\mu(x)}^{\infty} K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda t}dt. \end{aligned}$$

Multiplying this equality by $\frac{1}{2\pi}(1,-i)\,e^{i\lambda y}$ and integrating it in $(-\infty,\infty)$ to λ, we get

$$\begin{aligned}
&\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2i\varphi(x,\lambda)}{E(\lambda)}+S_0(\lambda)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda\mu(x)}-\begin{pmatrix}1\\i\end{pmatrix}e^{-i\lambda\mu(x)}\right](1,-i)\,e^{i\lambda y}d\lambda\\
=\ &\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\i\end{pmatrix}(1,-i)\,e^{-i\lambda(t-y)}dtd\lambda \qquad (11)\\
&-\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)\int_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)\,e^{i\lambda(t+y)}dtd\lambda\\
&+\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda)-S(\lambda)]\int_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)\,e^{i\lambda(t+y)}dtd\lambda\\
&+\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda)-S(\lambda)]\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)\,e^{i\lambda(\mu(x)+y)}d\lambda.
\end{aligned}$$

It is easily shown

$$\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}1\\i\end{pmatrix}(1,-i)\,e^{-i\lambda(t-y)}d\lambda=\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda(t-y)}\begin{pmatrix}1&-i\\i&1\end{pmatrix}d\lambda=\delta(t-y)\,E_2$$

where $\delta(t)$ is a delta function and $E_2=\operatorname{Re}\begin{pmatrix}1&-i\\i&1\end{pmatrix}$. Thus

$$\begin{aligned}
\int_{\mu(x)}^{\infty}K(x,t)\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{pmatrix}1\\i\end{pmatrix}(1,-i)\,e^{-i\lambda(t-y)}d\lambda dt &= \int_{\mu(x)}^{\infty}K(x,t)\,\delta(t-y)\,E_2dt\\
&= K(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)\int_{\mu(x)}^{\infty}K(x,t)\begin{pmatrix}1\\-i\end{pmatrix}(1,-i)\,e^{i\lambda(t+y)}dtd\lambda\\
=\ &\int_{\mu(x)}^{\infty}K(x,t)\left(\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)\begin{pmatrix}1&-i\\-i&-1\end{pmatrix}e^{i\lambda(t+y)}d\lambda\right)dt.
\end{aligned}$$

Substituting $S_0(\lambda)$, it is calculated the following integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\lambda)\, e^{i\lambda(t+y)} d\lambda = -\delta\left(t-y+2a(1-\alpha)\right).$$

Substituting this on the right hand of (11), we get

$$K(x,y) + \operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty} [S_0(\lambda) - S(\lambda)] \int_{\mu(x)}^{\infty} K(x,t)\begin{pmatrix} 1 \\ -i \end{pmatrix}(1,-i)\, e^{i\lambda(t+y)} dt d\lambda$$

$$+\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty} [S_0(\lambda) - S(\lambda)]\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} e^{i\lambda(\mu(x)+y)} d\lambda$$

$$-\operatorname{Re}\int_{\mu(x)}^{\infty} K(x,t)\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\delta\left(t-y+2a(1-\alpha)\right) dt$$

$$= \quad K(x,y) + F(\mu(x)+y) + \int_{\mu(x)}^{\infty} K(x,t)\, F(t+y)\, dt + \operatorname{Re} K\left(x, -y+2a(1-\alpha)\right)$$

where

$$F(x) = \operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty} [S_0(\lambda) - S(\lambda)]\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} e^{i\lambda x} d\lambda, \tag{12}$$

and

$$K\left(x, 2a(1-\alpha) - y\right) = 0$$

for $y > \mu(x)$. Hence, the right hand of (11) has the form

$$K(x,y) + F(\mu(x)+y) + \int_{\mu(x)}^{\infty} K(x,t)\, F(t+y)\, dt$$

for $y > \mu(x)$. The left hand of (11) is also

$$\operatorname{Re}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2i\varphi(x,\lambda)}{E(\lambda)}(1,-i)\, e^{i\lambda y} d\lambda + \operatorname{Re}\frac{1}{2\pi}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\int_{-\infty}^{\infty} S_0(\lambda)\, e^{i\lambda(\mu(x)+y)} d\lambda$$

$$-\operatorname{Re}\frac{1}{2\pi}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\int\limits_{-\infty}^{\infty}e^{-i\lambda(\mu(x)-y)}d\lambda=0$$

As a result for $y>\mu(x)$ we get

$$K(x,y)+F(\mu(x)+y)+\int\limits_{\mu(x)}^{\infty}K(x,t)F(t+y)\,dt=0 \qquad (13)$$

where $F(x)$ is defined by (12). This equation is called the *main equation* of the inverse problem of scattering theory for the boundary value problem (1),(2). Thus the following theorem is proved.

Theorem 4 *For each fixed $x\geq 0$, the kernel of the special solution (4) satisfies the main equation.*

The continuity of the function $S(\lambda)$ at real points is a straightforward consequence of Lemma1. The increment of logarithm of $S(\lambda)$ is equal to zero. In fact, we apply the argument principle to the function $E(\lambda)$. This function is regular in the upper half plane, continuous in the closed upper half plane $\operatorname{Im}\lambda\geq 0$. The increment of argument of $E(\lambda)$ as λ runs over the real axis from $-\infty$ to ∞ is equal to zero:

$$\arg E(+\infty)-\arg E(-\infty)=2\pi(N-P)$$

where N is number of zeros of $E(\lambda)$ and P is number of poles Since $N=P=0$ and

$$\ln S(\lambda)=-2i\arg E(\lambda)$$

we get

$$\ln S(+\infty)-\ln S(-\infty)=-2i\{\arg E(+\infty)-\arg E(-\infty)\}=0$$

The last relation is called *Levinson formula*.

4 Solvability of the Main Equation

By given the scattering function $S(\lambda)$, it is found the function $F(x)$ by the formula (12) and then write the main equation (13). In the main equation (13) we can take kernel $K(x,t)$ of equation as an unknown and regard it as a matrix equation of Fredholm type in the space of matrix functions with elements in $L_2(\mu(x),\infty)$ for every fixed x.

Theorem 5 *For every fixed $x \geq 0$, the main equation has an unique vector solution with elements in $L_2(\mu(x),\infty)$.*

Proof. It is obtained from Lemma3.3.1 in [14] that the main equation is generated by compact operator. We rewrite the homogeneous equation

$$\begin{aligned}
&\begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix} + \int_{\mu(x)}^{\infty} \begin{pmatrix} K_{11}(x,t) & K_{12}(x,t) \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix} \qquad (14)\\
&\times \frac{1}{4\pi} \int_{-\infty}^{\infty} (S_0(\lambda) - S(\lambda)) \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} e^{i\lambda(t+y)} d\lambda dt \\
&+ \int_{\mu(x)}^{\infty} \begin{pmatrix} K_{11}(x,t) & K_{12}(x,t) \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(\overline{S_0(\lambda) - S(\lambda)}\right) \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} e^{-i\lambda(t+y)} d\lambda dt \\
= \; & 0.
\end{aligned}$$

From here it follows that

$$\begin{aligned}
&K_{11}(x,y) + \frac{1}{4\pi} \int_{\mu(x)}^{\infty} \int_{-\infty}^{\infty} \{K_{11}(x,t) - iK_{12}(x,t)\} (S_0(\lambda) - S(\lambda)) e^{i\lambda(t+y)} d\lambda dt \\
&+ \frac{1}{4\pi} \int_{\mu(x)}^{\infty} \int_{-\infty}^{\infty} \left(\overline{S_0(\lambda) - S(\lambda)}\right) \{K_{11}(x,t) + iK_{12}(x,t)\} e^{-i\lambda(t+y)} d\lambda dt = 0
\end{aligned}$$

and

$$\begin{aligned}
&K_{12}(x,y) + \\
&+ \frac{1}{4\pi} \int_{\mu(x)}^{\infty} \int_{-\infty}^{\infty} \{-iK_{11}(x,t) - K_{12}(x,t)\} (S_0(\lambda) - S(\lambda)) e^{i\lambda(t+y)} d\lambda dt \\
&+ \frac{1}{4\pi} \int_{\mu(x)}^{\infty} \int_{-\infty}^{\infty} \left(\overline{S_0(\lambda) - S(\lambda)}\right) \{iK_{11}(x,t) - K_{12}(x,t)\} e^{-i\lambda(t+y)} d\lambda dt = 0.
\end{aligned}$$

Multiplying the second equation $-i$ and adding the first equation we get

$$K_{11}(x,y)-iK_{12}(x,y) \tag{15}$$
$$+\frac{1}{2\pi}\int_{\mu(x)}^{\infty}\int_{-\infty}^{\infty}\left(\overline{S_0(\lambda)}-\overline{S(\lambda)}\right)\{K_{11}(x,t)+iK_{12}(x,t)\}e^{-i\lambda(t+y)}d\lambda dt=0.$$

We put

$$\begin{aligned} g(y) &:= K_{11}(x,y)+iK_{12}(x,y) \\ F_s(z) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}(S_0(\lambda)-S(\lambda))e^{i\lambda z}d\lambda. \end{aligned}$$

Multiplying the equation (15) by $\overline{g(y)}$, integrating it from $-\infty$ to ∞ according to y, we obtain

$$\left(\overline{g(y)},\overline{g(y)}\right)+\left(\int_{\mu(x)}^{\infty}g(t)F_s(t+y)dt,\overline{g(y)}\right)=0$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{\overline{g(\lambda)}}\right|^2d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{g(-\lambda)}(S_0(\lambda)-S(\lambda))\widetilde{g(\lambda)}d\lambda=0.$$

Since $\widetilde{g(\lambda)}$ is the Fourier transform of a function $g(y)$ which vanishes for $y<\mu(x)$, $\widetilde{g(-\lambda)}e^{-2i\lambda\mu(0)}$ is the Fourier transform of a function $g(-y+2\mu(0))$, which vanishes for $y>\mu(x)$ .Hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-2i\lambda\mu(0)}\widetilde{g(-\lambda)}\widetilde{g(\lambda)}d\lambda=\int_{\mu(x)}^{\infty}g(-y+2\mu(0))g(y)dy=0$$

and

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left|\overline{\widetilde{g(\lambda)}}\right|^{2}d\lambda+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{g(-\lambda)}S_{0}(\lambda)\,\widetilde{g(\lambda)}d\lambda-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{g(-\lambda)}S(\lambda)\,\widetilde{g(\lambda)}d\lambda$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left|\overline{\widetilde{g(\lambda)}}\right|^{2}d\lambda-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{g(-\lambda)}S(\lambda)\,\widetilde{g(\lambda)}d\lambda$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\overline{\widetilde{g(-\lambda)}}\widetilde{g(-\lambda)}d\lambda-\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{g(-\lambda)}S(\lambda)\,\widetilde{g(\lambda)}d\lambda$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left\{\overline{\widetilde{g(-\lambda)}}-S(\lambda)\,\widetilde{g(\lambda)}\right\}\widetilde{g(-\lambda)}d\lambda=0$$

This shows that the function $z(\lambda)=\overline{\widetilde{g(-\lambda)}}-S(\lambda)\,\widetilde{g(\lambda)}$ is orthogonal to $\overline{\widetilde{g(-\lambda)}}$ in $L_2(-\infty,\infty)$. But then

$$\left\|\widetilde{g(-\lambda)}\right\|^{2}=\left\|S(\lambda)\,\widetilde{g(\lambda)}\right\|^{2}=\left\|\widetilde{g(-\lambda)}-z(\lambda)\right\|^{2}=\left\|\widetilde{g(-\lambda)}\right\|^{2}+\left\|z(\lambda)\right\|^{2}$$

which is possible if and only if $z(\lambda)=0$.We find that

$$\frac{\overline{\widetilde{g(-\lambda)}}}{\overline{f_1(0,\lambda)}+\lambda\overline{f_2(0,\lambda)}}=\frac{\widetilde{g(\lambda)}}{f_1(0,\lambda)+\lambda f_2(0,\lambda)}.$$

Taking

$$z_1(\lambda)=\begin{cases}\frac{\overline{\widetilde{g(-\lambda)}}}{\overline{f_1(0,\lambda)}+\lambda\overline{f_2(0,\lambda)}},\operatorname{Im}\lambda\geq 0,\\ \frac{\widetilde{g(\lambda)}}{f_1(0,\lambda)+\lambda f_2(0,\lambda)},\operatorname{Im}\lambda\leq 0.\end{cases}$$

This formula shows that the function $z_1(\lambda)$ is regular in the upper and lower half plane. Therofere, the function $z_1(\lambda)$ is an entire function and tends to zero as $\lambda\to\infty$. Then $\widetilde{g(\lambda)}=0$ and we conclude that $g(t)\equiv 0$. Thus, the homogeneous equations (14) and (15) have only zero solution and from here it follows that the unique solvability of integral equations (13), which proves the theorem. ■

Theorem 6 *The scattering function determines the boundary value problem (1), (2) uniquely.*

Proof. Evidently, to form the main equation it is sufficient to know the matrix function $F(x)$ and in its turn, to find $F(x)$ it is sufficient to know the scattering function $S(\lambda)$. It is seen from Theorem 5 that the main equation (13), constructed only on the basis of the scattering function, and has a unique solution $K(x,y)$. Then the matrix function $\Omega(x)$ in the equation (1) can be uniquely found from (4). It is constructed the equation (1) by given algorithm. The theorem is proved. ■

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