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The Analysis of Rigid Body Motion From Measured Data

In this paper, a new method for analyzing rigid body motion from measured data is presented. The approach is numerically stable, explicitly accounts for the errors inherent in measured data and those introduced by floating point arithmetic, automatically accommodates any number of rigid body particles, and is computationally efficient. The sole restriction on the data is that it represent 3 noncollinear particles of a rigid body.

1 Introduction

Chasles (1830) showed that the most general motion (both finite and infinitesimal) of a rigid body could be described as a rotation about an axis in space and a translation along the same axis. Since that time, numerous authors have provided geometrical and analytical proofs of that theorem (for example c.f. Goldstein, 1981; Routh, 1960; Whittaker, 1937). Additional authors (e.g., Ball, 1900; Beatty, 1966; Beatty, 1977b; Beatty, 1977a; Paul, 1963; Rodrigues, 1840; Schwartz, 1963; Thurnauer, 1967; Wittenburg, 1977) have shown how to determine the next position from the previous position and the finite displacement parameters, the velocity from the position and the infinitesimal displacement parameters, and the orthogonal and skew-symmetric matrices which characterize finite and infinitesimal motion from the respective displacement parameters.

Calculating the position (or velocity) of any particle on a rigid body given an initial position of the particle and the parameters of the displacement is an example of a "forward" kinematic problem. The related "backward" problem (calculating the parameters of the displacement given successive positions or the position and velocity of several particles on the body) is receiving renewed interest as an adjunct to realtime control of multi-degree of freedom mechanical systems (e.g., robots which incorporate endpoint sensing, space structures, etc.).

Many methods exist for determining the parameters of a rigid body displacement when exact position and/or velocity data are available for three particles of the body (c.f. Angeles, 1982; Angeles, 1986a; Angeles, 1986b; Angeles, 1987; Beggs, 1966; Beggs, 1983; Bottema and Roth, 1979; Hamidi, 1981; Laub and Shiflett, 1982; Laub and Shiflett, 1983; Milne, 1948; Ravani and Q. J., 1993; Spoor and Veldpals, 1980; Strom and Svensson, 1976; Suh and Radcliffe, 1978). However, even for exact data, many of these methods contain numerical singularities which cause them to fail when presented with particular sets of data. The numerical problem is exacerbated by the fact that exact data are only available in theoretical situations. In addition to the problems introduced by floating point representation of exact data, Potthoff (1968) and Roth (1969) have pointed out that inexact measurements further complicate the calculations since the resulting imprecise data can cause many algorithms to fail even when the data do not directly lead to singularities. In actual practice, algorithms for solving the backward problem must compensate for the possibility of measured data with concomitant measurement errors, errors introduced by the very nature of floating point arithmetic, and potential errors due to particular rigid body poses and motion. Angeles (1990) is one of the few researchers to address the problems introduced by imprecise data.

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This paper presents a new method for analyzing rigid body motion which is explicitly designed to correctly handle "perturbed" data. The method minimizes the error of the calculated motion parameters in a least squares sense, automatically incorporates information from redundant data (i.e., from more than three particles), and is both numerically stable and computationally efficient. The sole restriction is that the data represent a minimum of three noncollinear particles of a rigid body.

2 Rigid Body Motion Relations for Imprecise Data

2.1 Position-Position Problems (Finite Displacements). The linear algebraic form of the relation between the positions of a set of rigid body particles before and after a finite displacement is given by (Laub and Shiflett, 1982)

$$\mathbf{P}_2 = \mathbf{R}\mathbf{P}_1 + \mathbf{a}\mathbf{h}^T \tag{1}$$

where the columns of $\mathbf{P}_1$ and $\mathbf{P}_2$ contain the x, y, and z position components of n rigid body particles. However, this equation is only valid in a theoretical sense. If, for instance, there are measurement errors in the data, the equality will no longer hold.

For practical considerations, we can assume that $\mathbf{P}_1$ and $\mathbf{P}_2$ have been perturbed by position error arrays, $\mathbf{E}_1$ and $\mathbf{E}_2$, which contain random errors with zero mean and a variance proportional to the resolution of the measuring device. Letting $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ represent such perturbed data, we have

$$\tilde{\mathbf{P}}_2 \approx \mathbf{R}\tilde{\mathbf{P}}_1 + \mathbf{a}\mathbf{h}^T \tag{2}$$

while an exact relation for this imprecise data is given by

$$\tilde{\mathbf{P}}_2 = \tilde{\mathbf{R}}\tilde{\mathbf{P}}_1 + \tilde{\mathbf{a}}\mathbf{h}^T \tag{3}$$

Here, $\tilde{\mathbf{R}} = \mathbf{R} + \mathbf{E}_R$ and is no longer a proper orthogonal matrix (i.e., orthogonal with a determinant $= +1$) while $\tilde{\mathbf{a}} = \mathbf{a} + \mathbf{e}_a$.

2.2 Position-Velocity Problems (Infinitesimal Displacements). The theoretical relation between position and velocity given by (Laub and Shiflett, 1983)

$$\dot{\mathbf{P}} = \mathbf{W}\mathbf{P} + \mathbf{b}\mathbf{h}^T \tag{4}$$

is also valid only for exact data.

Again, we can perturb the measured positions and velocities with random error arrays, $\mathbf{E}$ and $\dot{\mathbf{E}}$, which have zero mean and a variance proportional to the resolution of the measuring devices. For such perturbed data the relation becomes

$$\dot{\tilde{\mathbf{P}}} \approx \mathbf{W}\tilde{\mathbf{P}} + \mathbf{b}\mathbf{h}^T \tag{5}$$

while the exact relationship is

$$\dot{\tilde{\mathbf{P}}} = \tilde{\mathbf{W}}\tilde{\mathbf{P}} + \tilde{\mathbf{b}}\mathbf{h}^T \tag{6}$$

where $\tilde{\mathbf{W}} = \mathbf{W} + \dot{\mathbf{E}}$ is no longer skew-symmetric and $\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{e}_b$.

2.3 Position-Acceleration Problems. Finally, the theoretical relation between position and acceleration for exact data is given by

$$\ddot{\mathbf{P}} = (\dot{\mathbf{W}} + \mathbf{W}^2)\mathbf{P} + \mathbf{c}\mathbf{h}^T \tag{7}$$

Once again, the measured positions and accelerations may be perturbed by random error arrays, $\mathbf{E}$ and $\ddot{\mathbf{E}}$, with zero mean and a variance proportional to the resolution of the measuring devices. For imprecise data the approximate relation is

$$\tilde{\ddot{\mathbf{P}}} \approx (\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}} + \mathbf{c}\mathbf{h}^T \tag{8}$$

while the exact relationship becomes

$$\tilde{\ddot{\mathbf{P}}} = (\tilde{\dot{\mathbf{W}}} + \tilde{\mathbf{W}}^2)\tilde{\mathbf{P}} + \tilde{\mathbf{c}}\mathbf{h}^T \tag{9}$$

In this case, neither $\tilde{\dot{\mathbf{W}}} = \dot{\mathbf{W}} + \ddot{\mathbf{E}}$ nor $\tilde{\mathbf{W}} = \mathbf{W} + \dot{\mathbf{E}}$ are skew-symmetric and $\tilde{\mathbf{c}} = \mathbf{c} + \mathbf{e}_c$.

In the next section, we will derive methods for determining $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{R}$, $\mathbf{W}$, and $\dot{\mathbf{W}}$ from a knowledge of the measured position, velocity, and acceleration of three or more particles of a rigid body. The only restriction on the data is that at least three of the particles must be non-collinear.

3 Method of Solution

3.1 The Position-Position (Finite Displacement) Solution. The finite displacement of a rigid body may be described by Eq. (1) and Eq. (2) for exact and measured data respectively. When measured data are used, a scalar measure of the error may be defined as

$$\epsilon(\mathbf{R}, \mathbf{a}) = \operatorname{tr}((\mathbf{R}\tilde{\mathbf{P}}_1 + \mathbf{a}\mathbf{h}^T - \tilde{\mathbf{P}}_2)^T(\mathbf{R}\tilde{\mathbf{P}}_1 + \mathbf{a}\mathbf{h}^T - \tilde{\mathbf{P}}_2)) \geq 0 \tag{10}$$

with the equality holding when $\tilde{\mathbf{P}}_2 = \mathbf{P}_2$ and $\tilde{\mathbf{P}}_1 = \mathbf{P}_1$. Since each element of $\mathbf{R}\tilde{\mathbf{P}}_1 + \mathbf{a}\mathbf{h}^T - \tilde{\mathbf{P}}_2$ represents the difference or error between $\mathbf{R}\tilde{\mathbf{P}}_1 + \mathbf{a}\mathbf{h}^T$ and $\tilde{\mathbf{P}}_2$, $\epsilon(\mathbf{R}, \mathbf{a})$ as defined by Eq. (10) is simply the sum of the squares of the individual errors.

An average value of $\mathbf{a}$ can be calculated by postmultiplying Eq. (2) by $\mathbf{h}$ and rearranging to get

$$\mathbf{a} \approx \frac{1}{\mathbf{h}^T\mathbf{h}}(\tilde{\mathbf{P}}_2 - \mathbf{R}\tilde{\mathbf{P}}_1)\mathbf{h} = \frac{1}{n}(\tilde{\mathbf{P}}_2 - \mathbf{R}\tilde{\mathbf{P}}_1)\mathbf{h} \tag{11}$$

Substituting Eq. (11) into Eq. (10) and rearranging yields

$$\epsilon(\mathbf{R}, \mathbf{a}) = \operatorname{tr}\left(\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)(\mathbf{R}\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_2)^T (\mathbf{R}\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_2)\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\right) \tag{12}$$

We would like to minimize the error function subject to the constraint that $\mathbf{R}$ is orthogonal (i.e., $\mathbf{R}\mathbf{R}^T = \mathbf{I}$) so we define an augmented scalar error function, $\epsilon^*(\mathbf{R}, \mathbf{a}, \mathbf{M})$, such that

$$\epsilon^*(\mathbf{R}, \mathbf{a}, \mathbf{M}) = \epsilon(\mathbf{R}, \mathbf{a}) + \operatorname{tr}(\mathbf{M}^T(\mathbf{R}^T\mathbf{R} - \mathbf{I})) \tag{13}$$

For compatibly dimensioned matrices, $\operatorname{tr}\mathbf{A}\mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B})^T = \operatorname{tr}(\mathbf{B}\mathbf{A})^T = \operatorname{tr}\mathbf{B}\mathbf{A}$ and $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$. Therefore Eq. (13) becomes

$$\begin{aligned}\epsilon^*(\mathbf{R}, \mathbf{a}, \mathbf{M}) = {} & \operatorname{tr}\left(\mathbf{R}\left(\tilde{\mathbf{P}}_1\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T + \mathbf{M}^T\right)\mathbf{R}^T\right) \\ & - 2\operatorname{tr}\left(\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_2^T\mathbf{R}\tilde{\mathbf{P}}_1\right) \\ & + \operatorname{tr}\left(\tilde{\mathbf{P}}_2\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_2^T - \mathbf{M}^T\right)\end{aligned} \tag{14}$$

To find an $\mathbf{R}$ such that $\epsilon^*(\mathbf{R}, \mathbf{a}, \mathbf{M})$ is minimized set

$$\begin{aligned}\frac{\partial}{\partial\mathbf{R}}(\epsilon^*(\mathbf{R}, \mathbf{a}, \mathbf{M})) = {} & \frac{\partial}{\partial\mathbf{R}}\operatorname{tr}\left(\mathbf{R}\left(\tilde{\mathbf{P}}_1\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T + \mathbf{M}^T\right)\mathbf{R}^T\right) \\ & - \frac{\partial}{\partial\mathbf{R}}2\operatorname{tr}\left(\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_2^T\mathbf{R}\tilde{\mathbf{P}}_1\right) \\ & + \frac{\partial}{\partial\mathbf{R}}\operatorname{tr}\left(\tilde{\mathbf{P}}_2\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_2^T - \mathbf{M}^T\right) = 0\end{aligned} \tag{15}$$

Taking the appropriate matrix derivatives (Graham, 1981) yields

Nomenclature

$\mathbf{E}_1$ = a $3 \times n$ array of initial position errors
$\mathbf{E}_2$ = a $3 \times n$ array of final position errors
$\mathbf{E}_P$ = a $3 \times n$ array of position errors
$\mathbf{E}_R$ = a 3×3 error array added to $\mathbf{R}$ to create a nonorthogonal matrix close to $\mathbf{R}$
$\dot{\mathbf{E}}$ = a 3×3 error array added to $\mathbf{W}$ to create a nonskew-symmetric matrix close to $\mathbf{W}$
$\ddot{\mathbf{E}}$ = a 3×3 error array added to $\dot{\mathbf{W}}$ to create a nonskew-symmetric matrix close to $\dot{\mathbf{W}}$
$\mathbf{I}$ = an identity matrix (either 3×3 or $n \times n$ as appropriate)
$\mathbf{M}$ = a 3×3 array of Lagrangian multipliers
$\mathbf{P}$ = a $3 \times n$ array containing the x, y, and z position components of n particles
$\tilde{\mathbf{P}}$ = $\mathbf{P} + \mathbf{E}_P$
$\mathbf{P}_1$ = a $3 \times n$ array of the initial positions of the n particles
$\mathbf{P}_2$ = a $3 \times n$ array of the final positions of the n particles
$\tilde{\mathbf{P}}_1$ = $\mathbf{P}_1 + \mathbf{E}_1$
$\tilde{\mathbf{P}}_2$ = $\mathbf{P}_2 + \mathbf{E}_2$
$\dot{\mathbf{P}}$ = a $3 \times n$ array containing the x, y, and z velocity components of n particles
$\tilde{\dot{\mathbf{P}}}$ = $\dot{\mathbf{P}} + \dot{\mathbf{E}}$
$\ddot{\mathbf{P}}$ = a $3 \times n$ array containing the x, y, and z acceleration components of n particles
$\tilde{\ddot{\mathbf{P}}}$ = $\ddot{\mathbf{P}} + \ddot{\mathbf{E}}$
$\mathbf{R}$ = the 3×3 orthogonal (with a determinant = +1) rotation matrix characterizing a finite displacement
$\tilde{\mathbf{R}}$ = $\mathbf{R} + \mathbf{E}_R$
$\mathbf{W}$ = the 3×3 skew-symmetric angular velocity matrix characterizing an infinitesimal displacement
$\tilde{\mathbf{W}}$ = $\mathbf{W} + \mathbf{E}_W$
$\dot{\mathbf{W}}$ = the 3×3 skew-symmetric angular acceleration matrix characterizing an infinitesimal displacement
$\tilde{\dot{\mathbf{W}}}$ = $\dot{\mathbf{W}} + \mathbf{E}_W$
$\mathbf{a}$ = a 3×1 vector that includes the rigid body translation for finite displacements
$\tilde{\mathbf{a}}$ = $\mathbf{a} + \mathbf{e}_a$
$\mathbf{b}$ = a 3×1 vector that includes the rigid body linear velocity for infinitesimal displacements
$\tilde{\mathbf{b}}$ = $\mathbf{b} + \mathbf{e}_b$
$\mathbf{c}$ = a 3×1 vector that includes the rigid body linear acceleration for infinitesimal displacements
$\tilde{\mathbf{c}}$ = $\mathbf{c} + \mathbf{e}_c$
$\mathbf{e}_a$ = an $n \times 1$ perturbation vector for $\mathbf{a}$
$\mathbf{e}_b$ = an $n \times 1$ perturbation vector for $\mathbf{b}$
$\mathbf{e}_c$ = an $n \times 1$ perturbation vector for $\mathbf{c}$
$\mathbf{h}$ = an $n \times 1$ vector of ones
n = the number of rigid body particles being tracked
$\boldsymbol{\omega}$ = the angular velocity vector and
$\dot{\boldsymbol{\omega}}$ = the derivative with respect to time of $\boldsymbol{\omega}$

$$\mathbf{R}\left(2\tilde{\mathbf{P}}_1\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T+\mathbf{M}^T+\mathbf{M}\right) - 2\tilde{\mathbf{P}}_2\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T = 0 \quad (16)$$

or, after a slight rearrangement,

$$\tilde{\mathbf{P}}_2\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T = \mathbf{R}\left(\tilde{\mathbf{P}}_1\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T+\frac{\mathbf{M}^T+\mathbf{M}}{2}\right) \quad (17)$$

Suppose we define a symmetric matrix $\mathbf{A}$ and a nonsymmetric matrix $\mathbf{B}$ by

$$\mathbf{A} = \tilde{\mathbf{P}}_1\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T+\frac{\mathbf{M}^T+\mathbf{M}}{2} \quad (18)$$

$$\mathbf{B} = \tilde{\mathbf{P}}_2\left(\mathbf{I}-\frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}_1^T \quad (19)$$

respectively. Equation (17) becomes

$$\mathbf{RA} = \mathbf{B} \quad (20)$$

from which we can solve for $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{B}\mathbf{A}^{-1} \quad (21)$$

Premultiplying each side of Eq. (20) by the respective transposes and using the facts that $\mathbf{R}$ is orthogonal ($\mathbf{R}^T = \mathbf{R}^{-1}$) and $\mathbf{A}$ is symmetric ($\mathbf{A}^T = \mathbf{A}$) we find

$$\mathbf{A}^2 = \mathbf{B}^T\mathbf{B} \quad (22)$$

Since $\mathbf{B}^T\mathbf{B}$ is symmetric and positive definite, it has a symmetric positive definite square root $(\mathbf{B}^T\mathbf{B})^{1/2} = \mathbf{A}$. Thus, from Eq. (21) we have

$$\mathbf{R} = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1/2} \quad (23)$$

An alternative solution for $\mathbf{R}$ begins with a singular value decomposition of $\mathbf{B}$; i.e.,

$$\mathbf{B} = \mathbf{U}\Sigma\mathbf{V}^T \quad (24)$$

in which $\mathbf{U}$ and $\mathbf{V}$ are 3×3 orthogonal matrices and Σ is a 3×3 diagonal matrix containing the singular values. Substituting Eq. (24) into Eq. (20) gives

$$\mathbf{RA} = \mathbf{U}\Sigma\mathbf{V}^T \quad (25)$$

Premultiplying both sides of Eq. (25) by the respective transposes leads to

$$\mathbf{A}^2 = \mathbf{V}\Sigma^2\mathbf{V}^T \quad (26)$$

from which $\mathbf{A}$ is found to be

$$\mathbf{A} = \mathbf{V}\Sigma\mathbf{V}^T \quad (27)$$

Substituting Eq. (27) into Eq. (25) and rearranging gives

$$(\mathbf{RV} - \mathbf{U})\Sigma\mathbf{V}^T = 0 \quad (28)$$

In order for Eq. (28) to be always satisfied, the solution for $\mathbf{R}$ is given by

$$\mathbf{R} = \mathbf{U}\mathbf{V}^T \quad (29)$$

Obviously, letting $\mathbf{B} = \mathbf{U}\Sigma\mathbf{V}$ in Eq. (23) gives Eq. (29). Numerically, however, the SVD approach is to be preferred as it avoids the numerically unsatisfactory "squaring up" associated with the formation of $\mathbf{B}^T\mathbf{B}$.

Although $\mathbf{R}$ as calculated by Eq. (29) is orthogonal, the singular value decomposition does not guarantee that the $\mathbf{R}$ so calculated is a rotation rather than a reflection. This problem results from the fact that the "handedness" of $\mathbf{U}$ and $\mathbf{V}$ is irrelevant to the SVD. To guarantee that $\mathbf{R}$ as calculated by Eq. (29) is a rotation, $\mathbf{U}$ and/or $\mathbf{V}$ may need to be modified. Let $\mathscr{R}_r^{m\times n}$ denote the set of all $m \times n$ matrices of rank r with real coefficients and $\mathbf{A} \in \mathscr{R}_r^{m\times n}$ have the singular value decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$ where $\mathbf{U} \in \mathscr{R}_r^{m\times m}$ and $\mathbf{V} \in \mathscr{R}_r^{n\times n}$ are orthogonal and $\Sigma \in \mathscr{R}_r^{m\times n}$ has zero elements everywhere except for its first r diagonal elements which are the singular values ordered as $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$. Let $\mathbf{U}$ have columns $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ and $\mathbf{V}$ have columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then the SVD can be written in the form

$$\mathbf{A} = \mathbf{U}_R\Sigma_R\mathbf{V}_R^T = \sum_{i=1}^{r}\sigma_i\mathbf{u}_i\mathbf{v}_i^T \quad (30)$$

Now consider the special case of $\mathbf{A} \in \mathscr{R}_r^{3\times 3}$. The SVD of $\mathbf{A}$ becomes

$$\mathbf{A} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \sigma_3\mathbf{u}_3\mathbf{v}_3^T \quad (31)$$

Thus, when $\mathbf{A} \in \mathscr{R}_2^{3\times 3}$, $\sigma_3 = 0$ so $\mathbf{u}_3$ and $\mathbf{v}_3$ may be changed independently (and still not affect the SVD of $\mathbf{A}$) as long as orthogonality is maintained; i.e., the signs of $\mathbf{u}_3$ and $\mathbf{v}_3$ may be manipulated independently.

The following theorem will be useful:

Theorem 3.1.0.1. If $\mathbf{A} \in \mathscr{R}_r^{m\times n}$ with an SVD $\mathbf{U}\Sigma\mathbf{V}^T$, then $\mathbf{U}\mathbf{V}^T$ is the unique orthogonal matrix closest to $\mathbf{A}$ in the sense of any unitarily invariant matrix norm; e.g., the matrix spectral norm $\|\cdot\|_2$ or the matrix Euclidean/Frobenius norm $\|\cdot\|_F$ (see Fan and Hoffman (1955) or Mirsky (1960) for proof).

Now, if $\mathbf{A}$ is not full rank, it is still true that $\mathbf{U}\mathbf{V}^T$ is the nearest orthogonal matrix to $\mathbf{A}$ but it may no longer be unique. In particular, for $\mathbf{A} \in \mathscr{R}_2^{3\times 3}$, it is easily seen that both $\mathbf{U}_-\mathbf{V}^T$ and $\mathbf{U}_+\mathbf{V}^T$ are nearest orthogonal matrices to $\mathbf{A}$ where

$$\mathbf{U}_+ = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad +\mathbf{u}_3] \quad (32)$$

$$\mathbf{U}_- = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad -\mathbf{u}_3] \quad (33)$$

Clearly, we have

$$\mathbf{U}\mathbf{V}^T = [\mathbf{u}_1\mathbf{v}_1^T + \mathbf{u}_2\mathbf{v}_2^T \pm \mathbf{u}_3\mathbf{v}_3^T] \quad (34)$$

Although the signs of $\mathbf{u}_3$ and $\mathbf{v}_3$ make no difference to the SVD when $\sigma_3 = 0$, for our purposes we still need a means to distinguish the proper orthogonal solution from the improper orthogonal solution. The following theorem suggests how such a task may be accomplished:

Theorem 3.1.0.2. Let $\mathbf{N} \in \mathscr{R}^{3\times 3}$ be orthogonal with elements ν_{ij}. Further, let $\mathbf{N} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ where $\mathbf{n}_j^T = (\nu_{1j}, \nu_{2j}, \nu_{3j})^T$. Then $\det \mathbf{N} = \pm 1$ as $\mathbf{n}_1 \times \mathbf{n}_2 = \pm\mathbf{n}_3$.

The proof follows from standard properties of the dot and cross products; i.e., $\det \mathbf{N} = \det \mathbf{N}^T = \mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{n}_3) = (\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_3 = \pm\mathbf{n}_3 \cdot \mathbf{n}_3 = \pm 1$.

This theorem suggests a very efficient means for determining if a 3×3 orthogonal matrix, $\mathbf{N}$, is proper or improper. Simply examine the first nonzero element (there must, of course, be at least one) of the third column of $\mathbf{N}$, $\mathbf{n}_3$. Suppose ν_{13} is the first nonzero element. Then $\mathbf{N}$ is proper if

$$\nu_{21}\nu_{32} - \nu_{22}\nu_{31} = +\nu_{13} \quad (35)$$

and improper (i.e., a reflection) if

$$\nu_{21}\nu_{32} - \nu_{22}\nu_{31} = -\nu_{13} \quad (36)$$

In practice, it is only necessary to check the signs of the first non-zero element of $\mathbf{n}_3$ and the corresponding element of $\mathbf{n}_1 \times \mathbf{n}_2$. If they are the same, $\mathbf{N}$ is proper. Otherwise $\mathbf{N}$ is improper.

The above results may now be combined to solve the problem of determining the nearest proper orthogonal matrix $\mathbf{R}$. First, compute $\mathbf{B}$ according to Eq. (19). Second, perform a singular value decomposition of $\mathbf{B}$. Third, form $\mathbf{R} = \mathbf{U}_+\mathbf{V}^T$ as in Eq. (34). Fourth, check whether the result is proper (this check may be efficiently done by a test such as Eq. (35) and Eq.

(36)). Fifth, if $\mathbf{U}_+\mathbf{V}^T$ is proper, then $\mathbf{R}$ is correct. If $\mathbf{U}_+\mathbf{V}^T$ is improper, then the nearest proper orthogonal matrix is formed by $\mathbf{R} = \mathbf{U}_-\mathbf{V}^T$. Finally, with $\mathbf{R}$ known, the average value for $\mathbf{a}$ is determined from Eq. (11).

The method just described has the important advantages of numerical stability and the ability to handle any number of particles. Although the accuracy of the results is affected by the accuracy of the data, it is not necessary to modify the calculated values for $\mathbf{R}$ and $\mathbf{a}$ to compensate for errors in the data. The following section describes a technique with similar advantages for analyzing position-velocity problems for infinitesimal displacements.

3.2 The Position-Velocity (Infinitesimal Displacement) Solution. Postmultiplying the rigid body transformation of Eq. (5) by $\mathbf{h}$ and rearranging gives the average value of $\mathbf{b}$ for all n particles as

$$\mathbf{b} \approx \frac{1}{n}(\dot{\tilde{\mathbf{P}}} - \mathbf{W}\tilde{\mathbf{P}})\mathbf{h} \tag{37}$$

We may again define a scalar error function as

$$\epsilon(\mathbf{W}, \mathbf{b}) = \operatorname{tr}((\mathbf{W}\tilde{\mathbf{P}} + \mathbf{b}\mathbf{h}^T - \dot{\tilde{\mathbf{P}}})^T(\mathbf{W}\tilde{\mathbf{P}} + \mathbf{b}\mathbf{h}^T - \dot{\tilde{\mathbf{P}}})) \geq 0 \tag{38}$$

into which Eq. (37) may be substituted to give

$$\epsilon(\mathbf{W}, \mathbf{b}) = \operatorname{tr}\left(\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)(\mathbf{W}\tilde{\mathbf{P}} - \dot{\tilde{\mathbf{P}}})^T (\mathbf{W}\tilde{\mathbf{P}} - \dot{\tilde{\mathbf{P}}})\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\right) \tag{39}$$

Since $\mathbf{W}^T = -\mathbf{W}$, we once more use a 3×3 array of Lagrangian multipliers to define an augmented scalar error function as

$$\epsilon^*(\mathbf{W}, \mathbf{b}, \mathbf{M}) = \epsilon(\mathbf{W}, \mathbf{b}) + \operatorname{tr}(\mathbf{M}^T(\mathbf{W} + \mathbf{W}^T)) \tag{40}$$

With $\epsilon(\mathbf{W}, \mathbf{b})$ defined by Eq. (39), the augmented error function becomes

$$\epsilon^*(\mathbf{W}, \mathbf{b}, \mathbf{M}) = \operatorname{tr}\left(\mathbf{W}\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T\mathbf{W}^T\right) - 2\operatorname{tr}\left(\dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T\mathbf{W}^T\right) + \operatorname{tr}\left(\dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\dot{\tilde{\mathbf{P}}}^T\right) + \operatorname{tr}((\mathbf{M} + \mathbf{M}^T)\mathbf{W}) \tag{41}$$

To find the skew-symmetric $\mathbf{W}$ such that the augmented scalar error function is minimized, we set the partial derivative of Eq. (41) with respect to $\mathbf{W}$ equal to zero and take the appropriate matrix derivatives to get

$$\frac{\partial}{\partial \mathbf{W}}(\epsilon^*(\mathbf{W}, \mathbf{b}, \mathbf{M})) = 2\mathbf{W}\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T + \mathbf{M} + \mathbf{M}^T - 2\dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T = 0 \tag{42}$$

The Lagrangian multipliers may be eliminated by rearranging Eq. (42) as

$$\mathbf{W}\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T = \dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \frac{\mathbf{M} + \mathbf{M}^T}{2} \tag{43}$$

and subtracting the transpose of Eq. (43) from Eq. (43) itself to get

$$\mathbf{W}\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T\mathbf{W}^T = \dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\dot{\tilde{\mathbf{P}}}^T \tag{44}$$

Defining a symmetric matrix $\mathbf{A}$ and a skew-symmetric matrix $\mathbf{C}$ as

$$\mathbf{A} = \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T \tag{45}$$

$$\mathbf{B} = \dot{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\dot{\tilde{\mathbf{P}}}^T \tag{46}$$

we may write Eq. (44) as

$$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A} = \mathbf{B} \tag{47}$$

Although the solution of Eq. (47) can be written in the form

$$\mathbf{W} = -\int_0^{+\infty} e^{-t\mathbf{A}}\mathbf{B}e^{-t\mathbf{A}}dt \tag{48}$$

this form is not particularly useful. A better solution can be found by exploiting the fact that the matrices involved are all 3×3. First, because $\mathbf{A}$ is a symmetric, positive semi-definite matrix, Eq. (47) may be written as the Kronecker sum (Graham, 1981)

$$(\mathbf{A} \oplus \mathbf{A}) \operatorname{vec} \mathbf{W} = \operatorname{vec} \mathbf{B} \tag{49}$$

where the vec operator forms a column vector from the operand by sequentially placing the columns of the operand under one another.

Second, since $\mathbf{B}$ and $\mathbf{W}$ are both 3×3 skew-symmetric matrices, Eq. (49) may be reduced to 3 independent equations. Representing $\mathbf{A}$, $\mathbf{C}$, $\mathbf{y}$, $\mathbf{W}$, and $\boldsymbol{\omega}$ as

$$\mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} 0 & -\psi_z & \psi_y \\ \psi_z & 0 & -\psi_x \\ -\psi_y & \psi_x & 0 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix};$$

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}; \quad \text{and} \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{50}$$

the expansion of the Kronecker sum in Eq. (49) gives the following 3 independent equations for the off-diagonal components of $\mathbf{W}$:

$$\begin{bmatrix} \alpha_{22} + \alpha_{33} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{12} & \alpha_{11} + \alpha_{33} & -\alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & \alpha_{11} + \alpha_{22} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix} \tag{51}$$

or

$$((\operatorname{tr} \mathbf{A})\mathbf{I} - \mathbf{A})\boldsymbol{\omega} = \mathbf{y} \tag{52}$$

The angular velocity vector $\boldsymbol{\omega}$ is then found from

$$\boldsymbol{\omega} = ((\operatorname{tr} \mathbf{A})\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \tag{53}$$

with $\mathbf{W}$ itself determined by putting the components of $\boldsymbol{\omega}$ into $\mathbf{W}$ as shown in Eq. (50).

Although it is essentially obvious that $(\operatorname{tr} \mathbf{A})\mathbf{I} - \mathbf{A}$ is nonsingular, a simple direct proof begins by applying an eigendecomposition to $\mathbf{A}$ to give

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T \tag{54}$$

where $\mathbf{Q}$ is orthogonal, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Since $\det(\mathbf{Q}^T((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A})\mathbf{Q}) = \det((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A})$ and $\text{tr }\mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3$, we have

$$\begin{aligned}\det((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A}) &= \det(\mathbf{Q}^T((\text{tr }\mathbf{A})\mathbf{Q}\mathbf{Q}^T - \mathbf{Q}\Lambda\mathbf{Q}^T)\mathbf{Q}) \\ &= \det(\mathbf{Q}\mathbf{Q}^T((\text{tr }\mathbf{A})\mathbf{I} - \Lambda)\mathbf{Q}\mathbf{Q}^T) \\ &= \det((\text{tr }\mathbf{A})\mathbf{I} - \Lambda) \\ &= (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) \quad (55)\end{aligned}$$

Since the determinant is obviously nonzero, $((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A})$ is not singular.

The general procedure for solving problems of this type reduces to: 1) Calculate $\mathbf{A}$ and $\mathbf{B}$ according to Eqs. (45) and (46); 2) form $\mathbf{y}$ according to Eq. (50); 3) calculate $\boldsymbol{\omega}$ with Eq. (53); and 4) with $\mathbf{W}$ determined from $\boldsymbol{\omega}$, calculate $\mathbf{b}$ with Eq. (37). As was the case in Section 3.1, this method is also numerically stable and automatically includes redundant data from any number of particles.

3.3 The Position-Acceleration Solution. The solution in this is similar to that of the preceding section. Postmultiplying the rigid body transformation (Eq. (8)) by $\mathbf{h}$ and rearranging gives the average value of $\mathbf{c}$ for all n particles as

$$\mathbf{c} \approx \frac{1}{n}(\tilde{\tilde{\mathbf{P}}} - (\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}})\mathbf{h} \quad (56)$$

We may again define a scalar error function as

$$\epsilon(\dot{\mathbf{W}}, \mathbf{c}) = \text{tr}\,(((\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}} + \mathbf{c}\mathbf{h}^T - \tilde{\tilde{\mathbf{P}}})^T ((\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}} + \mathbf{c}\mathbf{h}^T - \tilde{\tilde{\mathbf{P}}})) \geq 0 \quad (57)$$

into which Eq. (56) may be substituted to give

$$\epsilon(\dot{\mathbf{W}}, \mathbf{c}) = \text{tr}\left(\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)((\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}} - \tilde{\tilde{\mathbf{P}}})^T ((\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}} - \tilde{\tilde{\mathbf{P}}})\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\right) \quad (58)$$

Since $\dot{\mathbf{W}}^T = -\dot{\mathbf{W}}$, we again use a 3×3 array of Lagrangian multipliers to define an augmented scalar error function as

$$\begin{aligned}\epsilon^*(\dot{\mathbf{W}}, \mathbf{c}, \mathbf{M}) &= \epsilon(\dot{\mathbf{W}}, \mathbf{c}) + \text{tr}\,(\mathbf{M}^T(\dot{\mathbf{W}} + \dot{\mathbf{W}}^T)) \\ &= \text{tr}\left((\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T(\dot{\mathbf{W}} + \mathbf{W}^2)^T\right) \\ &\quad - 2\,\text{tr}\left(\tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T(\dot{\mathbf{W}} + \mathbf{W}^2)^T\right) \\ &\quad + \text{tr}\left(\tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\tilde{\mathbf{P}}}^T\right) \\ &\quad + \text{tr}\,((\mathbf{M} + \mathbf{M}^T)\dot{\mathbf{W}}) \quad (59)\end{aligned}$$

To minimize the augmented scalar error function with respect to a skew-symmetric $\dot{\mathbf{W}}$ we take the partial derivative of Eq. (59) with respect to $\dot{\mathbf{W}}$ and set it equal to zero; i.e.,

$$\frac{\partial}{\partial \dot{\mathbf{W}}}(\epsilon^*(\dot{\mathbf{W}}, \mathbf{c}, \mathbf{M})) = 2(\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T + \mathbf{M} + \mathbf{M}^T - 2\tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T = 0 \quad (60)$$

The Lagrangian multipliers may be eliminated by rearranging Eq. (60) as

$$(\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T = \tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \frac{\mathbf{M} + \mathbf{M}^T}{2} \quad (61)$$

and subtracting the transpose of Eq. (61) from Eq. (61) itself to get

$$(\dot{\mathbf{W}} + \mathbf{W}^2)\tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T(\dot{\mathbf{W}} + \mathbf{W}^2)^T = \tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\tilde{\mathbf{P}}}^T \quad (62)$$

Defining a symmetric matrix $\mathbf{A}$ and a skew-symmetric matrix $\mathbf{C}$ as

$$\mathbf{A} = \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T \quad (63)$$

$$\mathbf{B} = \tilde{\tilde{\mathbf{P}}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\mathbf{P}}^T - \tilde{\mathbf{P}}\left(\mathbf{I} - \frac{\mathbf{h}\mathbf{h}^T}{n}\right)\tilde{\tilde{\mathbf{P}}}^T + \mathbf{A}\mathbf{W}^2 - \mathbf{W}^2\mathbf{A} \quad (64)$$

Eq. (62) becomes

$$\mathbf{A}\dot{\mathbf{W}} + \dot{\mathbf{W}}\mathbf{A} = \mathbf{B} \quad (65)$$

As in Section 3.2, Eq. (65) may be written as the Kronecker sum

$$(\mathbf{A} \oplus \mathbf{A})\ \text{vec}\ \dot{\mathbf{W}} = \text{vec}\ \mathbf{B} \quad (66)$$

which leads to 3 independent equations for the components of $\dot{\mathbf{W}}$. Again representing $\mathbf{A}$, $\mathbf{B}$, $\mathbf{y}$, $\dot{\mathbf{W}}$, and $\dot{\boldsymbol{\omega}}$ as

$$\mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} 0 & -\psi_z & \psi_y \\ \psi_z & 0 & -\psi_x \\ -\psi_y & \psi_x & 0 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix};$$

$$\dot{\mathbf{W}} = \begin{bmatrix} 0 & -\dot{\omega}_z & \dot{\omega}_y \\ \dot{\omega}_z & 0 & -\dot{\omega}_x \\ -\dot{\omega}_y & \dot{\omega}_x & 0 \end{bmatrix}; \quad \text{and} \quad \dot{\boldsymbol{\omega}} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} \quad (67)$$

the expansion of the Kronecker sum in Eq. (66) gives the following 3 independent equations for the components of $\dot{\mathbf{W}}$:

$$\begin{bmatrix} \alpha_{22} + \alpha_{33} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{12} & \alpha_{11} + \alpha_{33} & -\alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & \alpha_{11} + \alpha_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{bmatrix} \quad (68)$$

or

$$((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A})\dot{\boldsymbol{\omega}} = \mathbf{y} \quad (69)$$

The angular acceleration vector $\dot{\boldsymbol{\omega}}$ is then found from

$$\dot{\boldsymbol{\omega}} = ((\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \quad (70)$$

As shown in Section 3.2 $(\text{tr }\mathbf{A})\mathbf{I} - \mathbf{A}$ is nonsingular so the required inverse always exists.

The procedure for solving problems of this type reduces to: 1) Calculate $\mathbf{A}$ and $\mathbf{B}$ according to Eqs. (63) and (64); 2) form y according to Eq. (67); 3) calculate $\dot{\boldsymbol{\omega}}$ with Eq. (70); and 4) with $\dot{\boldsymbol{\omega}}$ (and therefore $\dot{\mathbf{W}}$) and $\mathbf{W}$ determined by Eqs. (70) and (53) respectively, $\mathbf{c}$ may be calculated from Eq. (56). As was the case in Section 3.1, this method is also numerically stable and automatically includes redundant data from any number of particles.

3.4 Numerical Example. In what follows, we present the results of applying the above methods to actual perturbed data. We started with an set of exact positions and, using known motion parameters, calculated a corresponding set of exact positions, velocities, and/or accelerations. The exact data were then perturbed and the perturbed data used to calculate the corresponding motion parameters.

Perturbations to the exact data were created using a random number generator that produced a normal distribution with 0 mean and unit variance. Each perturbation was then multiplied 0.01 (thereby creating a normal distribution with zero mean and a variance of 0.01) before being added to an individual element of the exact data.

The results presented were generated with the commercial software program MatLab (Moler et al., 1987) version 3.5 running on a Macintosh II computer. We took advantage of MatLab's ability to count the actual number of floating point operations (flops) in order to report on the efficiency of the algorithms. The number of flops we report are "honest"; i.e., no attempt was made to optimize the code to reduce the number of flops. The sole gesture to efficiency was that $\mathbf{hh}^T$ (which is nothing more than an $n \times n$ array of ones) was not determined by actually postmultiplying $\mathbf{h}$ by $\mathbf{h}^T$ but by assigning all n^2 elements of $\mathbf{hh}^T$ the value 1.

3.4.1 Position-Position Example. The motion parameters relating exact position data

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_2 = \frac{1}{3}\begin{bmatrix} 5 & 8 & 8 \\ 2 & -1 & 2 \\ 14 & 14 & 11 \end{bmatrix}$$

are given by

$$\mathbf{R} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

The data were perturbed so that $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ became

$$\tilde{\mathbf{P}}_1 = \begin{bmatrix} 0.9994 & 0.0065 & -0.0067 \\ -0.0047 & 1.0081 & -0.0099 \\ 0.0026 & -0.0071 & 1.0104 \end{bmatrix} \quad \text{and}$$

$$\tilde{\mathbf{P}}_2 = \begin{bmatrix} 1.6661 & 2.6654 & 2.6614 \\ 0.6584 & -0.3332 & 0.6738 \\ 4.6862 & 4.6599 & 3.6540 \end{bmatrix}$$

The resulting calculated motion parameters are:

$$\mathbf{R} = \begin{bmatrix} -0.3228 & 0.6778 & 0.6606 \\ 0.6570 & -0.3420 & 0.6719 \\ 0.6813 & 0.6509 & -0.3349 \end{bmatrix} \quad \text{and}$$

$$\mathbf{a} = \begin{bmatrix} 1.9925 \\ 0.0022 \\ 4.0032 \end{bmatrix}$$

The scalar measure of the error as determined by Eq. (10) is only 2.8180e-4—approximately the square of the assumed perturbation variance. The calculations required a total of 847 flops of which most (543 to be exact) were used by MatLab's SVD algorithm.

Time measurements in MatLab are skewed by MatLab's overhead and the limited resolution of the timer (1/60th of a second on the Macintosh II used for the examples). A more realistic estimate of the time required for these calculations may be obtained using performance figures for various computer systems. Dongarra (1994) has published such figures for a wide range of hardware/software configurations. Dongarra reports three values of flops per second for each hardware/software combination. The first (and most conservative) is for a relatively small problem solved with standard linear algebra software coded in Fortran. Three widely used hardware/software configurations were selected from the list compiled by Dongarra—a SUN SPARCstation 2 (f77), an Apple Macintosh Quadra 950 (Absoft Fortran), and a Compaq Deskpro 486/331-120 w/ 487 (Lahey F77). The conservative speeds reported for the SPARCstation, Quadra, and Deskpro are 4.0 Mflop/s, 2.0 Mflop/s, and 1.1 Mflop/s, respectively. Dividing the flops by the reported speeds leads 0.212 ms, 0.424 ms, and 0.77 ms for the time needed by the SPARCstation, Quadra, and Deskpro to calculate $\mathbf{R}$ and $\mathbf{b}$.

3.4.2 Position-Velocity Example. The motion parameters relating exact position and velocity data

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \dot{\mathbf{P}} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 2 & 3 \\ 4 & 2 & 3 \end{bmatrix}$$

are

$$\mathbf{W} = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Perturbing the position and velocity data so that $\tilde{\mathbf{P}}$ and $\tilde{\dot{\mathbf{P}}}$ are given by

$$\tilde{\mathbf{P}} = \begin{bmatrix} 1.0130 & -0.0011 & 0.0064 \\ 0.0053 & 0.9923 & -0.0066 \\ -0.0072 & 0.0104 & 0.9983 \end{bmatrix} \quad \text{and}$$

$$\tilde{\dot{\mathbf{P}}} = \begin{bmatrix} 1.0110 & -1.0026 & 0.0174 \\ 3.9961 & 1.9874 & 2.9841 \\ 3.9978 & 1.9967 & 3.0101 \end{bmatrix}$$

leads to the calculated motion parameters

$$\mathbf{W} = \begin{bmatrix} 0 & -2.0090 & -1.0007 \\ 2.0090 & 0 & 1.0013 \\ 1.0007 & 1.0013 & 0 \end{bmatrix} \quad \text{and}$$

$$\mathbf{b} = \begin{bmatrix} 1.0063 \\ 1.9730 \\ 2.9926 \end{bmatrix}$$

The scalar measure of the error as determined by Eq. (37) is only 1.9639e-4—again, approximately the square of the assumed position/velocity perturbation variance. These calculations required only 488 flops with the matrix inversion requiring 112 flops. Again using the data from Dongarra, the SPARCstation, Quadra, and Deskpro would perform these calculations in 0.122 msec, 0.244 msec, and 0.444 msec respectively.

3.4.3 Position-Acceleration Example. Using the same position/velocity data as in the preceding section, the motion parameters relating exact position and acceleration data

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \dot{\mathbf{P}} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 2 & 3 \\ 4 & 2 & 3 \end{bmatrix}; \quad \text{and}$$

$$\ddot{\mathbf{P}} = \begin{bmatrix} -8.5 & -6.5 & -7.5 \\ 16.5 & 6.5 & 11.5 \\ 7 & 3 & 5 \end{bmatrix}$$

are

$$\dot{\mathbf{W}} = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -3.5 \\ 11.5 \\ 7 \end{bmatrix}$$

With $\tilde{\mathbf{P}}$ and $\tilde{\dot{\mathbf{P}}}$ perturbed as in the previous section and a perturbed $\tilde{\ddot{\mathbf{P}}}$ given by

$$\tilde{\ddot{\mathbf{P}}} = \begin{bmatrix} -8.4879 & -6.5078 & -7.4930 \\ 16.4844 & 6.5084 & 11.5062 \\ 7.0085 & 2.9936 & 5.0063 \end{bmatrix}$$

the calculated motion parameters are

$$\dot{\mathbf{W}} = \begin{bmatrix} 0 & -4.0162 & -2.0425 \\ 4.0162 & 0 & 1.9848 \\ 2.0425 & 1.9848 & 0 \end{bmatrix} \quad \text{and}$$

$$\mathbf{c} = \begin{bmatrix} -3.4373 \\ 11.4693 \\ 6.9811 \end{bmatrix}$$

with $\mathbf{W}$ and $\mathbf{b}$ as previously calculated.

The scalar measure of the error as determined by Eq. (56) is 7.1439e-3. The relatively larger error is explained by the fact that the acceleration calculations required use of a calculated $\mathbf{W}$ based on the same perturbed data. The acceleration calculations required an additional 668 flops with the matrix inversion again requiring 112 flops. Once more using the data from Dongarra, the SPARCstation, Quadra, and Deskpro would perform these calculations in 0.167 ms, 0.334 ms, and 0.607 ms respectively.

4 Summary

We have presented a new method for analyzing rigid body motion which is explicitly designed to correctly handle "perturbed" data. The sole restriction is that the data represent a minimum of three noncollinear particles of a rigid body. The method is based on minimization of the rigid body motion error subject to certain characteristics of the motion parameters themselves. It automatically incorporates information from redundant data (i.e., from more than three particles) and, as more data are included, the calculated motion parameters become more representative of the motion. Finally, the method is both numerically stable and computationally efficient.

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