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[GTN LXIII:2] TWO CLASSIFICATION PROBLEMS IN MATROID THEORY

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Abstract

Matroids are a modern type of synthetic geometry in which the behavior of points, lines, planes, and higher-dimensional spaces are governed by combinatorial axioms. In this paper we describe our work on two well-known classification problems in matroid theory: determine all binary matroids M such that for every element e, either deleting the element ($M\backslash e$) or contracting the element (M/e) is regular; and determine all binary matroids M having an element e such that, both $M\backslash e$ and M/e are regular.

1. Introduction

Matroid theory can trace its origins to the 1935 paper by Hassler Whitney *On the abstract properties of linear dependence* **[1]**. Whitney gave an axiomatic treatment of matroids and approached it as a generalization of graphs and matrices. He defined a *matroid* M as a set E of n elements and a family of subsets of E called *independent sets* such that; the empty set is independent; any subset of an independent set is independent; and if I_1 and I_2 are independent sets such that $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup e$ is independent. A set that is not independent is called *dependent*. A *simple matroid* is one for which all the 1-element and 2-element sets are independent.

The first example of a matroid usually given is that of an r by n matrix A over a field F with its columns labeled $\{1, 2, \ldots, r, \ldots, n\}$. Let E be the set of column labels and let I be subsets of column labels that correspond to linearly independent sets of columns in the *vector space* $V(r, F)$. Then I satisfies the three matroid axioms and the resulting matroid, denoted by $M[A]$, is called a *vector matroid*.

A second example is a graph G with edges labeled $1, 2, \ldots, n$. Let E be the set of edges and let I be subsets of edges that correspond to tree subgraphs of G. Then I satisfies the three matroid axioms and the resulting matroid, denoted by $M(G)$, is called a *graphic matroid*. Thus, matroids encompass matrices and graphs, both well-established subjects in their own right.

A *maximal independent set* is called a *basis* and a *minimal dependent set* is called a *circuit*. In a graph, a basis corresponds to a *spanning tree* and a circuit corresponds to a *cycle*. The *rank* of a set X, denoted by $r(X)$, is the cardinality of a largest independent set contained in X. The *closure* of X, denoted by $\mathrm{cl}(X)$, is defined by $\mathrm{cl}(X) = \{e \in E : r(X \cup e) = r(X)\}$. Almost immediately after Whitney's paper, Birkhoff showed that a simple matroid can be interpreted as a *geometric lattice* (that is, a finite semi-modular lattice in which each element is a join of atoms) **[2]**. MacLane showed that a simple matroid can be interpreted as a schematic geometric figure composed of points, lines, planes, and so forth with certain combinatorially defined incidences **[3]**. Thus, for example, a rank 3 matroid is a collection of points and subsets of points (called lines) that satisfy three conditions: any two distinct points belong to precisely one line; any line contains at least two distinct points; and there are at least three non-colinear points. The many different perspectives on matroids makes them interesting objects that underlie other well-studied mathematical objects. For more information on matroids see the survey article by Kingan **[4]**. The notation and all unexplained concepts are available in the books on matroid theory by J.G. Oxley **[5]** and by G. Gordon and J. McNulty **[6]**.

Our story of the *two problems* begins in 1980 when Paul Seymour developed a decomposition theory for the class of regular matroids **[7]**. A *regular matroid* is one that can be represented by a *totally unimodular matrix* over the reals. That is, a matrix each subdeterminant of which is $\{0, 1, -1\}$. In 1958 Tutte proved that a matroid is representable over all fields if and only if it can be represented by a totally unimodular matrix over the reals **[8]**. This result suggested that such matroids would be important in matroid structure theory, thus he called them *regular*. Totally unimodular matrices are also important in *linear programming* because if the constraint matrix is totally unimodular, then we are guaranteed an integer optimal solution.

A *minor* of a graph is a special type of substructure obtained by deleting edges (and any resulting isolated vertices) or by contracting edges. *Contraction* of edges amounts to taking the two vertices of an edge, identifying them, and throwing away the resulting loop. A *minor in a matroid* is defined in a similar manner. We denote the *deletion* of an element by $M\backslash e$ and contraction of an element by M/e. Consider a class of matroids $\mathcal{M}$ that is closed under minors and isomorphism; such as graphic matroids, regular matroids, or F-representable matroids. A matroid M is a *minimal excluded minor* for $\mathcal{M}$, if it is not in $\mathcal{M}$, but for every element e, both $M\backslash e$ and M/e are in $\mathcal{M}$. In 1930 Kuratowski proved that a graph is planar if and only if it has no subgraph that is a subdivision of K_5 or $K_{3,3}$ **[9]**. A few years later, in 1937, Wagner proved that a graph is planar if and only if it had no minor isomorphic with K_5 or $K_{3,3}$ **[10]**. This result is considered the first of many excluded minor characterization for classes of graphs and matroids. In 1959 Tutte proved that a matroid is binary if and only if it has no $U_{2,4}$-minor and a binary matroid is regular if and only if it has no F_7, or F_7^*-minor **[11]**. The matroid F_7 is the well-known *Fano matroid*. The matroid F_7^* is the dual of F_7.

Duality is a central feature of a matroid. Every matroid M is accompanied by a *dual matroid*, denoted by M^*. This is because if $\mathcal{B}$ is a basis for M, the set $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$ satisfies the basis axioms, and therefore, determines a matroid (called the dual matroid). The *dual of a planar graph* is just its geometric dual. The dual of a non-planar graph does not exist as a graph, but it does exist as a matroid. It is called a *cographic matroid*. For example $M^*(K_5)$ and $M^*(K_{3,3})$ are cographic matroids. For a rank-r, n-element F-representable matroid represented by matrix $A = [I_r|D]$ over F, the dual is represented by the orthogonal matrix $A^* = [-D^T|I_{n-r}]$.

As with most mathematical objects, it is useful to know how to construct new objects from old ones. The *clique-sum* of two graphs is obtained by joining the graphs along a common subgraph and then deleting all identified edges. If G_1 and G_2 are graphs, each having a K_n-subgraph for $n \geq 1$, the *n-sum* of G_1 and G_2 is formed by pairing the vertices of the chosen K_n-subgraph of G_1 with the vertices of the chosen K_n-subgraph of G_2. The vertices are then identified, as are the corresponding edges, and then all identified edges are deleted (see **[5]** page 354). Figure 1 shows examples of a 1-sum, 2-sum, and 3-sum of two graphs, denoted by $\oplus$, $\oplus_2$, and $\oplus_3$, respectively. Note that 1-sum is also called *direct sum*.

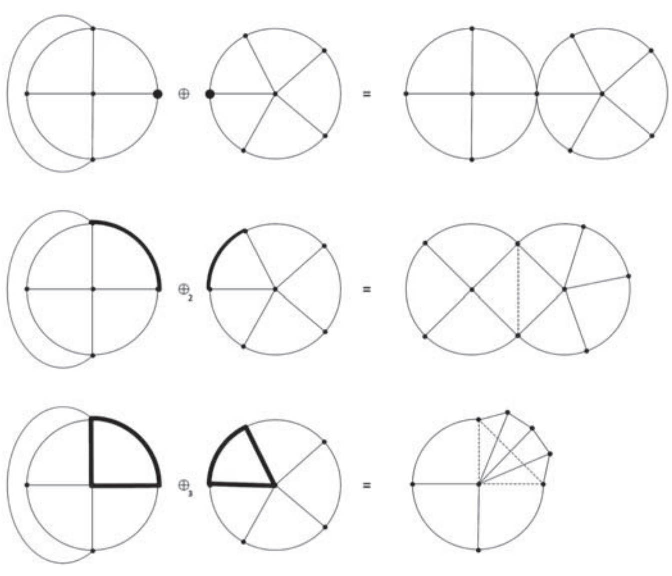


Figure 1: 1-sum, 2-sum, and 3-sum of $K_5\backslash e$ and W_5

Seymour extended this concept to matroids. A *cycle of a binary matroid* is a disjoint union of circuits. Let M_1 and M_2 be binary matroids with non-empty ground sets E_1 and E_2, respectively. Seymour defined a new binary matroid $M_1 \Delta M_2$ as the matroid with ground set $E_1 \Delta E_2$ and with cycles having the form $C_1 \Delta C_2$, where C_1 and C_2 are cycles of M_1 and M_2, respectively. When $E_1 \cup E_2 = \varnothing$, then $M_1 \Delta M_2$ is called a *1-sum* of M_1 and M_2. When $|E_1|, |E_2| \geq 3$, $E_1 \cap E_2 = \{z\}$, and z is not a loop or coloop of M_1 or M_2, then $M_1 \Delta M_2$ is called a *2-sum* of M_1 and M_2. When $|E_1|, |E_2| \geq 7$, $E_1 \cap E_2 = T$, and T is triangle in M_1 and M_2, then $M_1 \Delta M_2$ is called a *3-sum* of M_1 and M_2.

In classification problems we seek to completely identify the members of a particular infinite class of objects. It is sufficient to determine only the 3-connected members of a class because, in 1972, Bixby proved that every matroid that is not 3-connected can be constructed from its 3-connected proper minors using direct-sums and 2-sums **[12]**. If the class lends itself to a clean classification, then typically there are infinite families of objects identified based on some pattern. If such a precise determination is not possible, then it may be possible to decompose members of the class into smaller objects in a well-defined manner. For example, in 1963 Dirac proved that a simple 3-connected graph has no $(K_5 \backslash e)^*$-minor if and only if it is isomorphic with W_n for some $n \geq 3$; $K_5, K_5 \backslash e, K_{3,p}, K'_{3,p}, K''_{3,p}$, or $K'''_{3,p}$ for some $p \geq 3$ **[13]**. Thus, the class of graphs with no minor isomorphic with $(K_5 \backslash e)^*$ is completely classified. The graphs are shown in Figure 2. The infinite family of *wheel graphs*, W_n, play an important role in this paper.

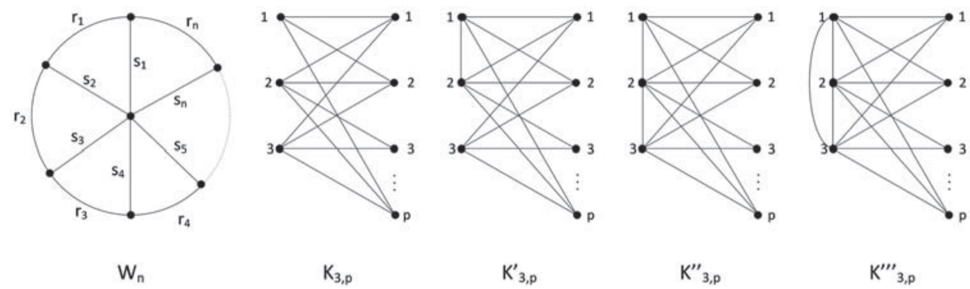


Figure 2: Dirac's infinite family of graphs with no $(K_5 \backslash e)^*$ minor

It is worth noting that Dirac's original result talked about 3-connected graphs without two vertex-disjoint cycles. Excluding two vertex-disjoint cycles in a 3-connected graph is equivalent to excluding $(K_5 \backslash e)^*$ as a minor. An earlier 1935 result due to Wagner characterizes the class of matroids with no K_5-minor. Wagner proved that a connected graph has no minor isomorphic to K_5 if and only if it can be obtained from planar graphs and subgraphs of V_8 by means of 1-sums, 2-sums, and 3-sums. The graph V_8 consists of a cycle with eight vertices with diametrically opposite vertices joined by edges **[14]**.

Seymour proved that a matroid is regular if and only if it can be obtained from graphic matroids, cographic matroids, or R_{10} by means of 1-sums, 2-sums, and 3-sums **[7]**. His 1980 paper is significant, not only for this result, but because it contained techniques for proving such decomposition results. This was the paper that introduced the *splitter theorem* (a generalization of *wheels and whirls theorem* due to Tutte) and the *decomposition theorem*. The splitter theorem describes how 3-connected matroids can be systematically built-up and the decomposition theorem describes the conditions under which a specific type of separation in a matroid gets carried forward to all matroids containing it as a minor.

The splitter theorem asserts that if N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or whirl then M has no larger wheel or whirl, respectively; then there exists $e \in E(M) - E(N)$ such that either $M \backslash e$ or M/e is 3-connected and has N as a minor. Therefore, we can build up from N to M by performing a sequence of deletions and contractions and maintaining 3-connectivity throughout. Oxley noted that Tan also proved this result in his Ph.D. dissertation in 1981 (see **[5]**, 12.1).

Connectivity plays an important role in decomposition results. Let M be a matroid and X be a subset of the ground set E. The *connectivity function* λ is defined by $\lambda(X) = r(X) + r(E - X) - r(M)$. Observe that $\lambda(X) = \lambda(E - X)$. For $k \geq 1$, a partition (A, B) of E is called a *k-separation* if $|A| \geq k$, $|B| \geq k$, and $\lambda(A) \leq k - 1$. When $\lambda(A) = k - 1$, we call (A, B) an *exact k-separation*. When $\lambda(A) = k - 1$ and $|A| = k$ or $|B| = k$ we call (A, B) a *minimal exact k-separation*. For example, a simple matroid is 3-connected if $\lambda(A) \geq 2$ for all partitions (A, B) with $|A| \geq 3$ and $|B| \geq 3$.

The *decomposition theorem* (see **[7]**, 9.1) is rather long to state, but the idea is easy to understand. If a 3-connected matroid has a non-minimal exact 3-separation, (A, B) for which $\lambda(A) = 2$ and $|A| \geq 4$, $|B| \geq 4$, then under certain conditions this separation carries forward to all matroids (in the class under consideration) containing it as a minor.

The proof of the decomposition of regular matroids consists of three main parts. The first part establishes that a 3-connected regular matroid is graphic or cographic or has a minor isomorphic with R_{10} or R_{12}. Matrix representations for R_{10} and R_{12} are shown below. These two matroids and other special matroids like them play a significant role in matroid structure theory.

$$R_{10} = \left[I_5 \middle| \begin{array}{ccccc} 1&1&0&0&1 \\ 1&1&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&1&1 \\ 1&0&0&1&1 \end{array} \right] \qquad R_{12} = \left[I_6 \middle| \begin{array}{cccccc} 1&1&1&0&0&0 \\ 1&1&0&1&0&0 \\ 1&0&0&0&1&0 \\ 0&1&0&0&0&1 \\ 0&0&1&0&1&1 \\ 0&0&0&1&1&1 \end{array} \right]$$

The matroid R_{10} is a 4-connected, rank-5, 10-element self-dual matroid. It is a *splitter* for regular matroids. This means that no 3-connected regular matroid contains R_{10} as a proper minor. So the building-up process stops at R_{10}. The matroid R_{12} is a 3-connected, rank-6, 12-element self-dual matroid that has a non-minimal exact 3-separation. The second part of the decomposition result establishes that this 3-separation in R_{12} carries forward in all regular matroids containing R_{12} as a minor. The third part establishes that 3-connected regular matroids can be pieced together from graphic and cographic matroids using the operation of 3-sums.

The two problems in the title of the current paper are related to the notion of a minimal excluded minor. Both were posed by Seymour and appear in **[5]**.

(1) Determine all binary matroids M such that for every element e, either $M\backslash e$ or M/e is regular.

(2) Determine all binary matroids M having an element e such that, both $M\backslash e$ and M/e are regular.

Matroids that satisfy the first problem are called *almost-regular matroids*. An element e for which both $M\backslash e$ and M/e are regular is called a *regular element*.

2. The First Problem

In this section we present our work on the first problem **[15]**. This problem is an instance of a more general problem: For a class $\mathcal{M}$ that is closed under minors and isomorphism, characterize those matroids that are not in $\mathcal{M}$, but for every element e, either $M\backslash e$ or M/e is in $\mathcal{M}$. Matroids of this type are called *almost-*$\mathcal{M}$ to reflect their connection to the original class $\mathcal{M}$. When $\mathcal{M}$ is the class of graphic matroids or regular matroids, almost-$\mathcal{M}$ is the class of almost-graphic or almost-regular matroids (respectively). There is no reason to expect that these classes will lend themselves to tidy characterizations. That they do is quite surprising.

For any matroid M not in $\mathcal{M}$, define

$$C(M) = \{e \in E(M) : M/e \notin \mathcal{M}\}$$
$$D(M) = \{e \in E(M) : M\backslash e \notin \mathcal{M}\}.$$

Thus, M is almost-$\mathcal{M}$ if $C(M) \cap D(M) = \varnothing$. In this case, define

$$R(M) = E(M) - (C(M) \cup D(M)).$$

If M is almost-$\mathcal{M}$, then its dual M^* is almost-$\mathcal{M}$ with $C(M^*) = D(M)$, $D(M^*) = C(M)$, and $R(M^*) = R(M)$. Let H be a minor of M such that $H \notin \mathcal{M}$. Observe that, $C(H) \subseteq C(M)$, $D(H) \subseteq D(M)$, and $R(M) \subseteq R(H)$.

Now we place additional hypotheses on the class $\mathcal{M}$. Suppose that $\mathcal{M}$ is also closed under the operations of 1-sums and 2-sums. Our first result shows that we may focus on just the 3-connected members of the class almost-$\mathcal{M}$ (see **[15]**, 3.1).

Theorem 2.1: Suppose M is an almost-$\mathcal{M}$ matroid. Then, either M is a 3-connected matroid or M is a series extension or a parallel extension of a 3-connected matroid. ■

If, in addition to 1-sum and 2-sum, we require $\mathcal{M}$ to be closed under *generalized parallel connection* (another operation that the classes of graphic and regular matroids satisfy) then we can give a method for constructing new almost-$\mathcal{M}$ matroids from old ones. Let M be a matroid with a triangle. Label the triangle as follows: $\Delta = \{s_1, s_n, r_n\}$. Consider the rank-$n$ wheel W_n with spokes labeled $s_1, \ldots, s_n$ and rim elements labeled $r_1, \ldots, r_n$. The matroid formed by attaching a type-1 fan across the triangle Δ is obtained by identifying the edges s_1 and s_n of the triangle $\{s_1, s_n, r_n\}$ in W_n with the edges s_1 and s_n of the triangle $\{s_1, s_n, r_n\}$ in M and deleting the additional r_n. It is very similar to the 3-sum operation except we do not delete the spokes in the triangle. We refer to the spokes as the basis of the fan. The remaining triangles of W_n may be viewed as a fan attached to M in the place of the triangle. We prove that a matroid obtained by attaching a type-1 fan along a triangle of an almost-$\mathcal{M}$ matroid is also almost-$\mathcal{M}$ (see **[15]**, 3.3).

Theorem 2.2: If M is an almost-$\mathcal{M}$ matroid and N is obtained from M by attaching a type-1 fan across a triangle, then N is almost-$\mathcal{M}$. ■

Theorem 2.2 provides a way to construct infinite families of almost-regular matroids from a specified almost-regular matroid. Consider the *Fano matroid*, F_7. It is almost graphic. View F_7 as a single-element extension of W_3 obtained by adding a point that forms a circuit with the three spokes of W_3. Consider the infinite family of binary non-graphic matroids F_{2n+1}, for $n \geq 3$, represented by the matrix $[I_n|D_n|x]$ where $[I_n|D_n]$ is the matrix representing W_n, and the additional element x forms a circuit with all the spokes (darkened in Figure 3). A matrix representation and a visual aid (not a graph) for F_{2n+1} is shown in Figure 3.

$$\begin{array}{c} s_1 \quad s_1 \ldots s_n \qquad r_1 \quad r_2 \quad \cdots \quad r_{n-1} \quad r_n \quad x \\ \left(\begin{array}{c|cccccc} & 1 & 0 & \cdots & 0 & 1 & 1 \\ & 1 & 1 & \cdots & 0 & 0 & 1 \\ & 0 & 1 & \cdots & 0 & 0 & 1 \\ I_{n+1} & 0 & 0 & \cdots & 0 & 0 & 1 \\ & \vdots & \vdots & & \vdots & \vdots & \\ & 0 & 0 & \cdots & 0 & 0 & 1 \\ & 0 & 0 & \cdots & 1 & 0 & 1 \\ & 0 & 0 & \cdots & 1 & 1 & 1 \end{array}\right). \end{array}$$

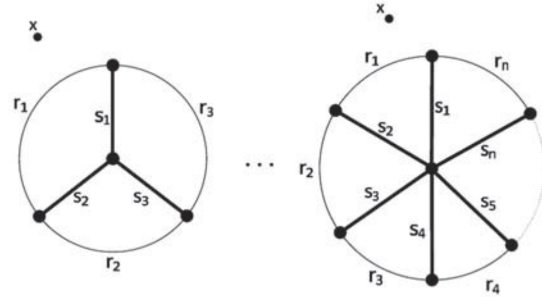


Figure 3: The infinite family F_{2n+1}

The next result establishes that F_{2n+1} is non-regular when n is odd and cographic when n is even. Moreover, when $n \geq 5$ is odd F_{2n+1} is almost-graphic with one regular element (see **[15]**, 4.4 and 4.5). When $n = 3$, we have the Fano matroid F_7 and $R(F_7) = 7$.

Theorem 2.3: For $n \geq 3$, F_{2n+1} is non-regular if and only if n is odd. Moreover, when $n \geq 5$, $C(F_{2n+1}) = \{r_1, r_2, \ldots, r_n\}$, $D(F_{2n+1}) = \{s_1, s_2, \ldots, s_n\}$, and $R(F_{2n+1}) = \{x\}$. ■

The infinite family F_{2n+1} encapsulates all the single-element extensions of the wheels and allows for their complete characterization. Denote a binary 3-connected single element extension of W_n by $M(S, n)$, where S is the subset of spokes and the new element x forms a circuit with S; that is, the set $S \cup \{x\}$ is a circuit. The matrix representing $M(S, n)$ is $[I_n|D_n|x]$, where the non-zero elements of column x correspond to the spokes in S. Observe that, for $n \geq 3$, $M(S, n)$ is non-regular if and only if $|S|$ is odd. Moreover, when $|S|$ is even, $M(S, n)$ is graphic if $|S| = 2$ and cographic otherwise (see **[15]**, 4.4). Most importantly, observe that $M(S, n)$ is obtained from F_{2n+1} by attaching fans along triangles.

For $n \geq 5$ the triangles in F_{2n+1} have the form $\{s_i, s_{i+1}, s_{i+2}\}$, where $i \in \{1, \ldots, n\}$, $s_{n+1} = s_1$, and $s_{n+2} = s_2$. Therefore, for $|S| \geq 5$, $M(S, n)$ is almost-graphic with $C(M) = \{r_1, r_2, \ldots, r_n\}$, $D(M) = \{s_1, s_2, \ldots, s_n\}$, and $R(M) = \{x\}$. For $|S| = 3$ there are three cases:

(1) If the three spokes in S are consecutive, say $\{s_1, s_2, s_3\}$, then $M(S, n)$ is almost-graphic with $R(M) = \{r_1, r_2, s_2, x\}$;

(2) If only two of the three spokes in S are consecutive, say $\{s_1, s_2\}$, then $M(S, n)$ is almost-graphic with $R(M) = \{r_1, x\}$; and

(3) If S has no consecutive spokes, then $M(S, n)$ is almost-graphic with $R(M) = \{x\}$.

As noted previously, for $n = 3$ we obtain F_7, for which deletion and contraction of every element is regular. Moreover, F_7 has triangles that are different from the triangles mentioned above. Construction of infinite families is more complicated in this case. Consider the three triangles with no common elements in F_7: $\{s_1, r_2, x\}$, $\{s_2, r_3, x\}$, and $\{s_3, r_1, x\}$. We can attach fans along these triangles in two different ways to obtain two different infinite families. For $m, n, r \geq 1$:

(1) The infinite family $\mathcal{F}_1(m, n, r)$ obtained by attaching fans along triangles with basis points s_1x, s_2x, and r_1x; and

(2) the infinite family $\mathcal{F}_2(m, n, r)$ obtained by attaching fans along triangles with basis points s_1x, s_2x, and s_3x.

Both infinite families are almost-graphic with zero regular elements. (For details see **[8]**, 4.6, and the preceding discussion.)

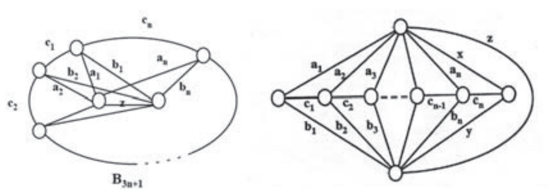
As mentioned already, some matroids like R_{10} and R_{12} are more important for structural results than others. We flag three 10-element binary non-regular matroids, S_{10}, E_4, and E_5 and two 12-element matroids T_{12} and X_{12} that play important roles in the classification of almost-regular matroids. The matroid S_{10} is a 10-element, rank-4, almost-graphic matroid with one regular element. The matroid E_5 is a self-dual, 10-element, rank-5, almost-regular (but not almost-graphic) matroid with zero regular elements. The matroid E_4 is a self-dual, 10-element, rank-5, almost-graphic matroid with one regular element. The matroid X_{12} is a self-dual, rank-6, 12-element matroid that is almost-regular (but not almost graphic) with zero regular elements. It is a splitter for the class of almost-regular matroids with no E_5-minor. The matroid T_{12} is a 4-connected, self-dual, rank-6, 12-element matroid. It is not almost-regular, but its single-element deletion $T_{12}\backslash e$ and single-element contraction T_{12}/e are almost-graphic with one regular element. Matrix representations are shown below.

$$S_{10} = \left[I_4 \middle| \begin{array}{cccccc} 1&0&0&1&1&0 \\ 1&1&0&0&1&1 \\ 0&1&1&0&1&1 \\ 0&0&1&1&0&1 \end{array} \right], \quad E_4 = \left[I_5 \middle| \begin{array}{ccccc} 0&1&1&1&1 \\ 1&0&1&1&0 \\ 1&1&0&1&0 \\ 1&1&1&1&0 \\ 1&1&0&0&1 \end{array} \right], \quad E_5 = \left[I_5 \middle| \begin{array}{ccccc} 0&1&1&1&1 \\ 1&0&1&1&0 \\ 1&1&0&1&1 \\ 1&1&1&1&0 \\ 1&1&0&0&0 \end{array} \right],$$

$$X_{12} = \left[I_6 \middle| \begin{array}{cccccc} 0&1&1&1&1&1 \\ 1&0&1&1&1&0 \\ 1&1&0&1&0&0 \\ 1&1&1&1&0&1 \\ 1&1&0&0&0&0 \\ 1&0&0&1&0&0 \end{array} \right], \quad T_{12} = \left[I_6 \middle| \begin{array}{cccccc} 1&1&0&0&0&1 \\ 1&0&0&0&1&1 \\ 0&0&0&1&1&1 \\ 0&0&1&1&1&0 \\ 0&1&1&1&0&0 \\ 1&1&1&0&0&0 \end{array} \right].$$

The matroid S_{10} gives rise to an infinite family, S_{3n+1}, that forms the centerpiece of the classification. First consider the well-known bicycle wheel graph with rank $n+1$ and $3n+1$ elements, B_{3n+1}, for $n \geq 3$. This graph is similar to the wheel graph, except that it has two middle vertices joined by an edge. Label the n edges of the rim by $c_1, \ldots, c_n$. Label the spokes originating from one of the middle vertices by $a_1, \ldots, a_n$ and from the other by $b_1, \ldots, b_n$. Label the edge joining the two middle vertices z.

For $n \geq 3$, S_{3n+1} is obtained from B_{3n+1} by replacing edge c_n with an element that forms a circuit with each of a_1, a_n, z, and b_1, b_n, z. A binary matrix representation for S_{3n+1} and a representation (S_{3n+1} is not a graph) are shown in Figure 4. Observe that, identifying x with a_1 and y with b_1 gives the graph B_{3n+1}. On the other hand, identifying x with b_1 and y with a_1 gives an illustration of the triangles in the binary non-regular matroid S_{3n+1}. It is almost-graphic with $C(S_{3n+1}) = \{c_1, \ldots, c_n\}$, $D(S_{3n+1}) = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, and $R(S_{3n+1}) = \{z\}$.



$$
\begin{array}{cc}
\begin{array}{cccccccccccc} a_1 & c_1 \ldots c_n & a_2 & a_3 & \cdots & a_n & b_1 & b_2 & \cdots & b_n & z \end{array} \\
\left(\begin{array}{c|cccccccccc}
 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 \\
 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 1 \\
 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 \\
I_{n+1} & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 \\
 & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \\
 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 \\
 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 \\
 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1
\end{array}\right)
\end{array}
$$

Figure 4: The infinite family S_{2n+1}

Let A and B be the classes of almost-regular matroids with no S_{10}-minor and no S_{10}^* -minor, respectively. We give a detailed structural characterization of these excluded minor classes. The proof of the next result spans many sections and consists of several theorems each of which is useful in its own right (see **[15]**, 6.2, 5.3, 7.4, 7.5, 7.2, and 8.1).

Theorem 2.4: Let M be a 3-connected, binary, almost-regular matroid. Then

(1) M has no S_{10}-minor and no S_{10}^* -minor if and only if M or M^* is isomorphic with $M(E_5)$, F_{11}, $\mathcal{F}_1(m, n, r)$, $\mathcal{F}_2(m, n, r)$, or $M(S, n)$ for $n \geq 3$ and $|S| = 3$ with at least two consecutive spokes, or their 3-connected single-element deletions;

(2) M has no S_{10}-minor if and only if M is isomorphic with F_7^* or a coextension of $\mathcal{F}_1(m, n, r)$, $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$, or $M(S, n)$ for $n \geq 3$ and $|S|$ odd;

(3) M has no S_{10}^* -minor and $|R(M)| \geq 1$ if and only if M is isomorphic with a 3-connected restriction of S_{3n+1} , where $n = r(M) - 1$. Moreover, for $n \geq 3$, $M(S, n)^*$ is a 3-connected restriction of S_{3n+1} ;

(4) If M has both S_{10}- and S_{10}^* -minors, then $|R(M)| = 0$;

(5) If M has an S_{10}-minor or an S_{10}^* -minor, and $|R(M)| = 0$, then $M \cong X_{12}$ or M has an E_5-minor. ■

In the next two results we identify the almost-regular matroids with at least one regular element (see **[15]**, 7.6). The only almost-regular matroids left to classify must have zero regular elements. We also proved an excluded minor result that says these matroids must have an E_5-minor (see **[15]**, 8.2).

Theorem 2.5: M is a binary, 3-connected, almost-regular matroid with $|R(M)| \geq 1$ if and only if M or M^* is isomorphic with a 3-connected restriction of S_{3n+1} for $n \geq 3$. ■

Theorem 2.6: Suppose M is a 3-connected, binary, almost-regular matroid with no E_5-minor. Then $M \cong X_{12}$ or M or M^* is isomorphic with a 3-connected restriction of S_{3n+1} for $n \geq 3$; or $\mathcal{F}_1(m, n, r)$ or $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$. ■

The next result is derived from all the previous results and gives a complete characterization of binary, non-regular, almost-graphic matroids.

Theorem 2.7: A binary, non-regular, 3-connected matroid M is almost-graphic if and only if M or M^* is isomorphic with a 3-connected restriction of S_{3n+1} for $n \geq 3$; or $\mathcal{F}_1(m, n, r)$ or $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$. ■

We now turn our attention to regular almost-graphic matroids and non-binary, almost-graphic matroids. A *cographic matroid* M is almost-graphic if for every element e, $M\backslash e$ or M/e is graphic. Because the class of cographic matroids is closed under minors, this amounts to finding graphic non-planar matroids such that for every element e, $M\backslash e$ or M/e is planar. Their duals will be cographic matroids that are almost-graphic.

In 1996 Bradley Gubser found the almost-planar graphs **[16]**. For $n \geq 3$, let μ_{2n} denote the *Mobius ladder* with $2n$ vertices; that is, the graph formed from the cycle with $2n$ vertices and edges labeled $1, 2, \ldots, 2n$, so that every pair of diametrically opposite vertices is joined by an edge. Observe that $\mu_6 \cong K_{3,3}$ and that $\mu_8 \cong V_8$. Let $M(K'''_{3,3})$ be the graph labeled as shown in Figure 5.

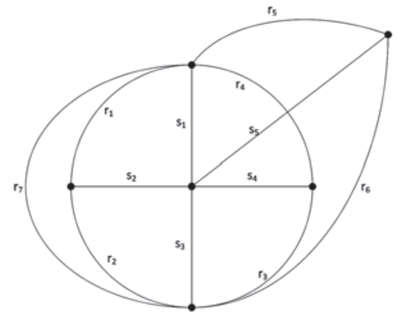


Figure 5: $M(K'''_{3,3})$ with appropriately labeled triangles

Consider the triangles $\{s_1, s_5, r_5\}$, $\{s_3, s_4, r_3\}$, and $\{s_1, s_5, r_1\}$. For $m, n, r \geq 1$, define $\mathcal{H}_1(m, n, r)$ to be the infinite family obtained by attaching type-1 fans of length m, n, and r along the above triangles with basis points s_1s_5, s_3s_4, and s_1r_1. Define $\mathcal{H}_2(m, n, r)$ to be the infinite family obtained by attaching type-1 fans of length m, n, and r along the above triangles with basis points s_1s_5, s_3s_4, and s_1s_2. It is an easy consequence of Gubser's result that if G is a simple, 3-connected, almost-planar graph, then G is isomorphic with a 3-connected, non-planar minor of B_{3n+1} for $n \geq 3$, μ_{2n} for $n \geq 3$, or $\mathcal{H}_1(m, n, r)$ or $\mathcal{H}_2(m, n, r)$ for $m, n, r \geq 1$.

Combining Theorem 2.7 with Seymour's result (a 3-connected regular matroid is either graphic or cographic or has a minor isomorphic to one of R_{10} or R_{12}) we obtain the regular, almost-graphic matroids. Observe that, R_{10} is almost-graphic since every single-element deletion is isomorphic with $M(K_{3,3})$ and every single-element contraction is isomorphic with $M^*(K_{3,3})$. Since it is a splitter for regular matroids, no further matroid with an R_{10} minor is relevant. The matroid R_{12} is not almost-graphic since for every element $e \in \{3, 4, 7, 8, 11, 12\}$, both $R_{12} \backslash e$ and R_{12}/e are cographic (see **[15]**, 2.1).

Theorem 2.8: Suppose M is a 3-connected, regular, almost-graphic matroid. Then $M \cong R_{10}$ or $M = M^*(G)$, where G is isomorphic with a 3-connected, non-planar restriction of B_{3n+1} for $n \geq 3$, μ_{2n} for $n \geq 3$, $\mathcal{H}_1(m, n, r)$ or $\mathcal{H}_2(m, n, r)$ for $m, n, r \geq 1$. ■

Finally, we consider non-binary, almost-graphic matroids. Given a matroid M with a set X that is both a circuit and a hyperplane, we define a new matroid M' whose bases are the original bases of M and the set X. (It is easy to see that this new family of sets is indeed a set of bases, see **[5]**, page 39). The matroid M' is called a *relaxation* of M and this operation is called *relaxing a circuit-hyperplane*. In 1990 Oxley gave a constructive classification of almost-binary matroids **[17]**. He proved that M is a 3-connected, almost-binary matroid if and only if M is isomorphic with $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$ or both the rank and corank of M exceed two and M can be obtained from a 3-connected, binary matroid by relaxing a circuit-hyperplane. It follows that if M is a binary, almost-$\mathcal{M}$ matroid with a circuit-hyperplane and M' is obtained from M by relaxing a circuit-hyperplane, then M' is almost-$\mathcal{M}$. Thus, we must determine which of the binary, almost-regular matroids have circuit-hyperplanes (see **[15]**, 9.3, 9.1, and 9.2).

Theorem 2.9:

(1) $M(W_n)$ for $n \geq 3$ are the only 3-connected, regular matroids with a circuit hyperplane:

(2) S_{3n+1} for $n \geq 3$ and $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$ and their 3-connected restrictions have circuit-hyperplanes;

(3) $\mathcal{F}_1(m, n, r)$ for $m, n, r \geq 1$ does not have a circuit-hyperplane;

(4) If M is a 3-connected, binary matroid with an E_5-minor, then $C(M)$ is a circuit-hyperplane. ■

Now suppose M is a 3-connected, almost-graphic matroid. If M is regular, then Theorem 2.8 completely identifies the regular, almost-graphic matroids. If M is binary and non-regular, then Theorem 2.7 completely identifies the non-regular, almost-graphic matroids. If M is non-binary, then Theorem 2.9 implies that the binary, almost-regular matroids with circuit-hyperplanes are the 3-connected restrictions of S_{3n+1}, $\mathcal{F}_2(m, n, r)$, and

W_n. The matroid obtained by relaxing the circuit-hyperplane in the wheel graph is the *whirl graph* W^n. Thus, we put everything together to obtain a complete characterization of the almost-graphic matroids (a problem posed by Oxley and published in the first edition of his book in 1992, see **[15]**, 2.2).

Theorem 2.10: A 3-connected matroid M is almost-graphic if and only if M or M^* is non-graphic and is isomorphic with
(1) A 3-connected restriction of R_{10}, S_{3n+1} for $n \geq 3$, $\mathcal{F}_1(m, n, r)$ or $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$;
(2) The dual of a 3-connected restriction of $M(B_{3n+1})$ for $n \geq 3$, $M(\mathcal{H}_1(m, n, r))$ or $M(\mathcal{H}_2(m, n, r))$ for $m, n, r \geq 1$;
(3) W^n for $n \geq 3$, $U_{2,n}$ or $U_{n-2,n}$ for $n \geq 4$; or
(4) A relaxation of a 3-connected, non-graphic restriction of S_{3n+1} for $n \geq 3$ or $\mathcal{F}_2(m, n, r)$ for $m, n, r \geq 1$. ■

Finally, although we did not use any of Truemper's work on almost-regular matroids, we would be remiss if we did not mention his constructive characterization for a subclass of the almost-regular matroids **[18]**. Truemper defined:

$$\textit{con elements} = \{e \in E(M) : M/e \text{ is regular}\}$$
$$\textit{del elements} = \{e \in E(M) : M\backslash e \text{ is regular}\}$$

and he required that the con elements form a cocircuit-cohyperplane and the del elements form a circuit-hyperplane. Observe that, *del elements* $\subseteq C(M) \cup R(M)$ and *con elements* $\subseteq D(M) \cup R(M)$ with equality if and only if $|R(M)| = 0$. Therefore, Truemper's condition requires the existence of a circuit-hyperplane C such that $C(M) \subseteq C \subseteq C(M) \cup R(M)$ and a cocircuit-cohyperplane D such that $D(M) \subseteq D \subseteq D(M) \cup R(M)$. As we saw already, the infinite family $\mathcal{F}_1(m, n, r)$ for $m, n, r \geq 1$ does not have a circuit-hyperplane.

3. The Second Problem

In the second problem, we want to classify all matroids with at least one element e such that $M\backslash e$ and M/e are both regular. In other words, we want to determine all matroids with at least one regular element. As mentioned in the introduction, given a class of matroids $\mathcal{M}$ that is closed under minors, a minimal excluded minor for $\mathcal{M}$ is a matroid M that is not in $\mathcal{M}$, but for every element e of M, both $M\backslash e$ and M/e are in $\mathcal{M}$. The condition in the second problem in a sense weakens the excluded-minor condition with "for every element e of M" replaced by "for some element e of M." It would be reasonable to call such matroids *weak excluded minors* for $\mathcal{M}$. One would expect these weak excluded minors to be somehow related to the minimal excluded minors for $\mathcal{M}$. The question is how are they related and whether they lend themselves to some sort of classification.

Recall that Tutte proved a matroid is binary if and only if it has no $U_{2,4}$-minor. In 1987 Oxley proved that $U_{2,4}$ is the only 3-connected non-binary matroid having an element e such that both $M\backslash e$ and M/e are binary **[19]**. In this case, the minimal excluded minor and the weak excluded minor coincide. Tutte also proved a binary matroid is regular if and only if it has no minor isomorphic with F_7 or F_7^*. Thus, one would expect the weak excluded minors for regular matroids to be somehow related to the Fano matroid. As it turns out we can show a surprising connection by giving a Seymour-type decomposition result for the class of matroids with at least two regular elements.

Suppose M is a binary, 3-connected, non-regular matroid. Then Seymour's *splitter theorem* implies that M can be obtained from F_7 or F_7^* by a sequence of extensions and coextensions. Observe that $|R(F_7)| = 7$. The matroid F_7 has no binary 3-connected single-element extensions and two binary 3-connected single-element coextensions, namely, S_8 and $AG(3, 2)$, both of which are self-dual. Observe that $AG(3, 2)$ is not almost-regular since every single-element deletion and contraction of $AG(3, 2)$ is isomorphic with F_7 and F_7^*, respectively. The matroid S_8 is almost-graphic with $|R(S_8)| = 6$. A matrix representation for S_8 is shown below.

$$S_8 = \left[I_4 \middle| \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{matrix} \right]$$

The next result gives us a bound on the number of regular elements (see **[15]**, 5.1 and **[20]**, 3.1).

> **Theorem 3.1:** Let M be a 3-connected, binary, non-regular matroid. Then $|R(M)| \leq 4$ for all matroids other than F_7, F_7^*, and S_8. Moreover, if $|R(M)| = 4$, then $R(M)$ is both a circuit and a cocircuit. ■

Let F_7^p and S_8^p be the matroids obtained from F_7 and S_8, respectively, by adding an element in parallel with an element belonging to at least two triangles. The next result is the main result of this section and completely classifies non-regular matroids with at least two regular elements. The proof, although not as long as our results on the first problem, spans several pages and requires a thorough description of 3-separations in the context of regular elements (see **[20]**, 1.1).

> **Theorem 3.2:** Let M be a 3-connected, non-regular matroid. Then M has at least two regular elements if and only if
> (1) M is $U_{2,4}$, F_7, F_7^*, or S_8; or
> (2) M is the 3-sum of F_7 or S_8 with a 3-connected regular matroid (with the possible exception of elements in parallel with the 3-sum triangle); or
> (3) M is the 3-sum of F_7^p or S_8^p with two 3-connected regular matroids (with the possible exception of elements in parallel with the 3-sum triangle). These two 3-sums are made along two disjoint triangles of F_7^p or S_8^p. ■

The proof of Theorem 3.2 uses a 2004 result by Xiangqian Zhou, in which he determined the internally 4-connected matroids of a subclass of binary matroids **[21]**. If a class of matroids has too many 3-connected members with no hope of classifying precisely infinite families, then one strategy is to raise the connectivity. However, we immediately run into difficulty. The concept of 3-connectivity in matroids corresponds to 3-connected and simple in graphs. This is not the case for 4-connected matroids. By definition, a 4-connected matroid has no 3-separations and therefore, has no triangle and triad, whereas triangles and triads are permitted in 4-connected graphs. This means the complete graphs and the projective geometries, which for all practical purposes are highly connected objects, do not meet the definition of 4-connectivity.

An intermediate concept called internal 4-connectivity was developed, in which some 3-separations are allowed. A 3-connected matroid is *internally 4-connected* if $\lambda(A) \geq 3$ for all partitions (A, B) with $|A| \geq 4$ and $|B| \geq 4$. In this case $\lambda(A) = 2$ is allowed only when either $|A|$ or $|B|$ has size at most 3. In other words, the matroid is 4-connected except for the presence of triangles and triads. Seymour's decomposition for regular matroids implies that R_{10} is the only internally 4-connected non-graphic and non-cographic regular matroid. Zhou characterized the internally 4-connected matroids in a subclass of binary matroids.

> **Theorem 3.3:** A non-regular, internally 4-connected, binary matroid other than F_7 and F_7^* contains one of the following matroids as a minor: E_5, S_{10}, S_{10}^*, $T_{12}\backslash e$, or T_{12}/e. ■

Here we see again those same matroids flagged as important. As we have already seen S_{10}, S_{10}^*, $T_{12}\backslash e$, and T_{12}/e each have one regular element and E_5 has zero regular elements. Combining Theorem 3.3 with Theorem 3.2 gives us the following result (see **[9]**, 1.4).

> **Theorem 3.4:** If M is an internally 4-connected, binary, non-regular matroid having at least two regular elements, then M is isomorphic with F_7 or F_7^*. ■

Thus, if we raise the connectivity, the excluded minors and the weak-excluded minors are once again the same. Such a neat characterization does not exist for non-regular matroids with one regular element because there are infinite families of internally 4-connected matroids with one regular element; for example, S_{3n+1}. However, the classification, when it is eventually completed, is bound to be interesting.

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