Mathematical Model of Cryospheric Response to Climate Changes

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Abstract: This paper focuses on the development of simplified mathematical models of the cryosphere which may be useful in further understanding possible global climate change impacts and in further assessing future impacts captured by global circulation models (GCMs). The mathematical models developed by leveraging the dominating effects of freezing and thawing within the cryosphere to simplify the relevant heat transport equations are tractable to direct solution or numerical modeling. In this paper, the heat forcing function is assumed to be a linear transformation of temperature (assumed to be represented by proxy realizations). The output from the governing mathematical model is total ice volume of the cryosphere. The basic mathematical model provides information as a systems modeling approach that includes sufficient detail to explain ice volume given the estimation of the heat forcing function. A comparison between modeling results in the estimation of ice volume versus ice volume estimates developed from use of proxy data are shown in the demonstration problems presented. **DOI:** 10.1061/(ASCE)CR.1943-5495.0000053.

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Introduction

The recent literature regarding assessment of the quality of information produced from global climate models (or global circulation models) (GCMs) has been mixed as far as the published commentaries and assessments of GCM modeling results. Some works point to successes achieved by the use of GCMs (for example, Randall et al. 2007), whereas other recently published works point to GCM modeling results that are classified as poor (for example, Anagnostopoulos et al. 2010). New research indicates that including additional processes in GCMs, such as down-regulation, caused by vegetative response to increased predicted carbon dioxide, may actually mitigate significant portions of the predicted increases in global temperature (Bounoua et al. 2010). The impact of such modeling enhancements integrated into coupled GCMs demonstrates the current incompleteness of the overall GCM structure and the possible importance of modeling omissions. Notwithstanding these mixed assessments, use of GCMs appears to be a main stream course in the analysis of climate change effects to develop public policy. Because the direction of GCM development is to include additional components of the heat budget and its several transport mechanisms, such advancement may need to address the well-known uncertainty problems that have been identified in the focused modeling efforts of those individual transport mechanisms.

For example, with the vast majority of greenhouse gases as water vapor, and with the feedback components to be modeled in coupled GCMs to include evapo-transpiration, rainfall, and other relevant hydrologic processes for moisture accounting, the established literature and experience with near-surface hydrologic and rainfall-runoff models in prediction of changing conditions or in simulating actual conditions may be noteworthy. That experience with hydrologic model development and application, primarily obtained during the 1970s and 1980s, demonstrates the significant magnitude of hydrologic and rainfall-runoff modeling uncertainty and modeling error in estimates and prediction. Further, the mistaken notion of modeling complexity necessarily providing more accurate results than the use of simpler models is also well understood, such as when applied to simulating rainfall-runoff processes (similar to other processes and feedbacks involved in the total global climate modeling setting). An indication of the problems inherent with rainfall-runoff models is provided in the U.S. Army Corps of Engineers (COE) Hydrologic Engineering Center (HEC) Research Note No. 6 (1979) where the Hydrocomp HSP continuous simulation computer model was applied to the West Branch DuPage River in Illinois. In the extensive study by Loague and Freeze (1985), several computer models of the near-earth surface processes were considered for rainfall-runoff modeling, including hydrologically simple models and hydrologically complex models. Loague and Freeze concluded in their paper that "... the fact that simpler models do as good as or better than more complex models is food for thought" (1985). Hence, from the previous studies and reports, among other references, model complexity may not be sufficient to mitigate overall model uncertainty, even when used as a component in a coupled GCM.

The current level of success in use of GCMs is reviewed in the recent paper by Anagnostopoulos et al. (2010), in which they examine whether GCMs provide credible quantitative estimates of future climate change, particularly at continental scales and above. In their paper, they examined the local performance of the models at 55 points and found that local projections do not correlate well with observed measurements. Furthermore, those authors found that the correlation at a large spatial scale, i.e., the contiguous United States, is worse than at the local scale. In their paper, they write that the most important question is not whether GCMs can produce credible estimates of future climate, but whether climate is at all predictable in deterministic terms.

The current authors suggest that some aspects of the climate may be predictable, depending on the time scale involved. For example, the evidence shows there has been approximately a 100,000-year cycle of glacials and interglacials occurring over the last two million years, and perhaps there are other aspects of the climate that are similarly predictable. However, even given the previous qualification regarding predictability of the climate, the ensemble of processes and feedback components may be strongly influenced by the uncertainty in their submodel components, such as the uncertainty in rainfall-runoff models as stated previously. Consequently, there is motivation to consider alternative methodologies in assessing climate change and causal factors, especially when the relative magnitude of the contemplated changes is small. In this paper, the authors propose a focus on the cryosphere as an alternative approach. The cryosphere is defined as the places on the planet where water is frozen. "The global cryosphere encompasses snow and ice in all its forms in the natural environment, including glaciers and ice sheets, sea ice, lake and river ice, permafrost, seasonal snow, and ice crystals in the atmosphere" (Bitz and Marshall 2012). Such a modeling approach may prove to be more accurate in its findings because of the dominating effects of the latent heat of phase change in freezing and thawing elements of the cryosphere.

In the work of The National Academy of Sciences, Research Articles, Geophysics, "Twentieth century climate change: Evidence from small glaciers" by Dyurgerov and Meier (1999), the authors write that evidence for rapid climate changes in the past has been derived from many sources, including glaciers and ice sheets. Further, these authors note that an examination of this relationship may be instructive for the study of paleoglacier evidence as indicative of past climate and for projecting the effects of future climate warming on cold regions of the world.

In Paper No. 58 of the Institute of Arctic and Alpine Research at the University of Colorado, Dyurgerov and Meier (1999) write that glacier variations are sensitive indicators of changes in climate and may have direct impacts on processes of global importance, such as sea-level rise, the hydrology of mountain-fed rivers, the freshwater balance of the oceans, natural disasters, and even the shape and rotation of the Earth.

Although GCMs in use today do not include all the relevant components of the climate budget (for example, Bounoua 2010), work continues toward including such relevant components. One of the key components in coupled GCMs is the cryosphere. Additionally, the thermal, hydrological, radiative, and in some cases, topographic characteristics of the cryosphere strongly impact the atmosphere (Bamber and Payne 2004). The dominating effects of freezing and thawing on the heat budget, as applied to the cryosphere, may allow interesting simplifications of the heat transport balance that result in a more tractable mathematical model. As such, for some simplifications, obtaining a closed-form solution of the governing mathematical relationships is possible. For other nonlinear formulations, numerical modeling can be accomplished.

In this paper, there are several objectives:

1. Develop a conceptual model of the heat budget applicable to a lumped parameter systems approach model of the cryosphere. Similar to the underpinnings of the Stefan model (e.g., Alexiades and Solomon 1993) for freezing and thawing soils or other medium, simplify the heat budget relationship caused by the dominating effects of freezing and thawing of water in the cryosphere. It is contemplated that by this analysis approach of focus on the cryosphere, it may be possible to gain insight not only in the heat transport processes involved with the cryosphere but also to develop an alternative model foundation that may be able to sidestep the many complications that currently impact the development of GCMs.
2. Assemble currently available data (such as isotopes of oxygen and hydrogen) typically presented in the literature that describes or relate to the volume of frozen material in the cryosphere, and relate to the temperature applicable to the cryosphere. These data include isotopes of oxygen and hydrogen, among others. Commonly used ratios of isotopes are used to provide proxy information of air temperature and ice volume, such as a ratio of hydrogen isotopes given by

$$dD = \left\{ \left[\left(\frac{H^2}{H^1} \right)_{\text{sample}} - \left(\frac{H^2}{H^1} \right)_{V-\text{SMOW}} \right] \left(\frac{H^2}{H^1} \right)_{V-\text{SMOW}} \right\} * 1000 \quad (1)$$

(Petit et al. 1999, 2001; Jouzel et al. 1987, 1993, 1996, 2007a, b) and a ratio of oxygen isotopes given by

$$d^{18}\text{O} = \left[\frac{\left(\frac{^{18}O}{^{16}O} \right)_{\text{sample}}}{\left(\frac{^{18}O}{^{16}O} \right)_{\text{standard}}} - 1 \right] * 1000‰ \quad (2)$$

(Petit et al. 1999, 2001; Lisiecki and Raymo 2005b). Table 1 defines the individual terms of the previously given equations. The previous two equations provide the mathematical description of the isotope ratios used in the current paper in development of the conceptual mathematical model.
3. Develop a conceptual mathematical model of the proposed simplified heat budget. This objective differs from the usual approach found in the literature of combining realizations of various relevant variables and contemplating hypotheses that may explain similarities and differences between the various plotted variables. The proposed conceptual model will directly relate estimates of temperature found in the literature (as derived from proxy data, such as the ratio dD; for example, Petit et al. 1999, 2001; Jouzel et al. 1987, 1993, 1996, 2007a, b) to estimates of ice volume also found in the literature (as derived from proxy data, such as the ratio d^{18}O; Petit et al. 1999, 2001; Lisiecki and Raymo 2005a). Given a mathematical analog of temperature and ice volume in partial differential equation (PDE) form, analytically solve the PDE. For PDE formulations not suitable for direct solution, develop a central-difference finite difference method analog by using computer program *EXCEL*.

Table 1. Definitions

Term	Definition	Relationship to climate change
dD	Ratio of the sampled hydrogen isotopic ratio compared to the V_{SMOW} standard isotopic ratio	Proxy for temperature
$\left(\frac{^{2}\mathrm{H}}{^{1}\mathrm{H}}\right)_{\mathrm{sample}}$	Isotopic ratio of ^{2}H versus ^{1}H in a sample	
$\left(\frac{^{2}\mathrm{H}}{^{1}\mathrm{H}}\right)_{V-SMOW}$	Hydrogen isotopic composition of the Vienna Standard Mean Ocean Water (V_{SMOW})	Standard as defined in 1968 by the International Atomic Energy Agency
$d^{18}\mathrm{O}$	Ratio of the sampled oxygen isotopic ratio compared to the V_{SMOW} standard isotopic ratio	Proxy for ice volume
$\left(\frac{^{18}\mathrm{O}}{^{16}\mathrm{O}}\right)_{\mathrm{sample}}$	Isotopic ratio of ^{18}O versus ^{17}O in a sample	
$\left(\frac{^{18}\mathrm{O}}{^{16}\mathrm{O}}\right)_{\mathrm{standard}}$	Oxygen isotopic composition of the Vienna Standard Mean Ocean Water (V_{SMOW})	Standard as defined in 1968 by the International Atomic Energy Agency

4. Calibrate the conceptual model to normalized data selected to represent the measure of frozen material, $I(t)$, as presented in the literature for the Pleistocene time period. To simplify the calibration process, the governing PDE is normalized so as to reduce the number of parameters. Some of the modeling parameters are directly calibrated to the selected proxy data representing the cumulative heat magnitude, $H(t)$, function. The other modeling parameters are calibrated to fit the modeling outcomes to the selected measure of frozen material $I(t)$. The calibrated model becomes the baseline model. At this stage, there exists a mathematical model linkage between the two selected proxy data types, one representing (in normalized $N(0,1)$ form) the model input realization, $H(t)$, and the other (also in normalized $N(0,1)$ form) representing the model output realization, $I(t)$.
5. Provide a simplified systems model of the proposed formulation (also used in the paper for a more simplified model development) appropriate for consideration in undergraduate mathematics modeling courses. Considerations of such model development and solution by using computer programs, such as Mathematica, are part of the material presented at the United States Military Academy at West Point, New York.

Conceptual Mathematical Model Development

In the consideration of the heat budget, as applied throughout the problem domain (i.e., the cryosphere), the freezing/thawing process typically dominates the heat transfer process such that several simplifications may be assumed applicable in the full-heat transport equation. For example, in the well-known Stefan solution (Alexiades and Solomon 1993) for estimating the freezing front depth in a column, steady state conditions are assumed above and below the freezing front, with heat exchange dominated by the phase change process at the freezing front.

In Guymon et al. (1980a, b, 1981a, b, 1984, 1993) and also in Hromadka et al. (1982, 1984, 1985, 1986a, b, c, d), finite element and complex variable boundary element method computer models of freezing and thawing soils in algid climates were developed for two- and three-dimensional problems, including impacts from groundwater flow regimes and with isothermal phase change in moist soils. These models are based on the dominating effects of the phase change process in the heat transport budget and have been applied to problems involving soil water phase change and water-ice interface phase change effects in long-term simulations.

For the systems modeling approach contemplated for the subject mathematical formulation, the total estimate of freezing and thawing latent heat effects may be taken as an area weighted sum of the various individual components composing the problem domain. This approach assumes a subdivision of the problem domain into subareas (or modeling cells), each subarea j having area A_j, a corresponding latent heat of fusion value La_j, and a measure of the frozen material subject to phase change, I_j. Assuming that all subarea latent heat parameters and subarea I_j can be related to reference values denoted by La and I, respectively, enables the governing heat transport equation to be simplified. An example of such an approach of area averaging properties includes the Thiessen polygon method used for determining global temperature in Anagnostopoulos et al. (2010). By using such an area averaging approach, the subarea j latent heat of fusion, La_j can be related to a reference value by

$$La_j = c_j * La \tag{3}$$

where La = reference latent heat of fusion; and c_j = a constant.

Similarly, the subarea j frozen material, $I_j(t)$, can be related to a reference monitored frozen material, $I(t)$, by

$$I_j(t) = d_j * I(t) \tag{4}$$

where d_j = a constant; and $I(t)$ = reference measure of frozen material as a function of time. (In the previous, subareas and relevant properties are to be calibrated in the final lumped parameter system formulation to be discussed subsequently).

Using the previous relationships, the latent heat effects related to changes in the various $I_j(t)$ functions established throughout the various subareas composing the problem domain (i.e., the cryosphere) are given by

$$\begin{aligned} La_1 * \frac{dI_1(t)}{dt} + La_2 * \frac{dI_2(t)}{dt} + \ldots + La_n * \frac{dI_n(t)}{dt} \\ = c_1 * La * d_1 * \frac{dI(t)}{dt} + c_2 * La * d_2 * \frac{dI(t)}{dt} + \ldots + c_n * La * d_n * \frac{dI(t)}{dt} \\ = \left(\sum_{j=1}^{n} c_j * d_j\right) * La * \frac{dI(t)}{dt} = r * La * \frac{dI(t)}{dt} \end{aligned} \tag{5}$$

where La and $I(t)$ = the appropriate reference values as utilized in the previous development; and r = a constant defined in Eq. (5). Examination of ice volume estimates (e.g., Dyurgerov and Meier 1999) indicates that glacial volume changes follow a pattern of slow growth and rapid melting. To model this phenomenon, Imbrie et al. (1980, 1984, 2011) use a mathematical model that describes each of these two timing processes (see Imbrie et al. Fig. 4). Such a model response applied to transitions between glacials and interglacials brings modeling estimates into closer agreement with the shape of the ice volume plot versus time, such as developed from proxy data using $d^{18}O$. Other such relationships may be used, including using power relationships on the variable $I(t)$.

Heat Transport Feedback Response Formulation

The heat transport effects that impact the problem domain include numerous components, including but by no means limited to solar and internal heat within the planet and circulation of the atmosphere and oceans with corresponding interface and convective heat transport effects, among others. All of these components interface and integrate together, and the combined systems model is represented by the term [Heat]. (The notation [Heat] is used to indicate all the various components of heat transport that effect phase change in the cryosphere. A full mathematical formulation of the total heat budget can be found in Bitz and Marshall (2012), among others). [Heat] includes all the thermal balance elements of the heat budget, leading to the magnitude term, $H(t)$. The function $H(t)$ is determined by calibration to known information about the appropriate temperature $T(t)$ function from historic data. In GCMs, the various individual components and interrelationships between components that assemble to form [Heat] are modeled or continue to be enhanced with new additions of component submodels. In the current approach, $H(t)$ is developed by examining what $H(t)$ did in the past with respect to the known information of the $T(t)$ function. For example, the ensemble of components that form the underpinnings of [Heat], leading to $H(t)$, resulted in a cycle over the last nearly two million years of glacials and interglacials, and therefore, that cycle period is continued in the current model (although other $H(t)$ functions may be used instead). This modeling approach of building $H(t)$, based on what $H(t)$ did in the past, may sidestep many complexities that are involved in assembling approximations of the various subprocesses and feedbacks that form the climate and associated boundary conditions. Further, such an $H(t)$ as used in this paper can still be modified to represent changes in the environment and then used within the proposed model for predictions of the conceptual model output, namely, the $I(t)$ function. Inferences can then be drawn that relate to the climate outside of the cryosphere, given the effects on the cryosphere. The resulting model construct can be used as model verification for GCMs, under suitable boundary and initial conditions, for the effects on the cryosphere, such as the Arctic and Antarctic regions.

In the proposed model, a portion of $H(t)$ is returned by means of various feedback responses, such as solar heat returned by snow cover, among other effects, and the net amount of heat lost to the heat balance relationship caused by these feedback responses is represented in the model as a proportion of the $H(t)$ function by the proportion factor, k, so that

$$\sum \text{returned heat components} = k^* \frac{I(t)}{I_0} * H(t), \tag{6}$$

where k = function of several factors representing the components of heat feedback responses; and I = defined by Eq. (4). There are several components in the heat transport process that result in incoming heat not completely utilized in the phase

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change process, but portions of that heat are reflected or returned back into the atmosphere or water, such as albedo (for example, Bitz 2012). The heat feedback model component addresses heat losses from the heat budget model that are not already included in the integrated processes leading to the temperature estimate data itself. Assuming that the system model initiates with the representative extent, I_0 (where I_0 is a representative initial condition value of the total measure of frozen material, such as a local maximum), of $I(t)$ throughout the problem domain, then the ratio, $I(t)/I_0$, provides an estimate of the relative proportion of $I(t)$ versus the representative extent, I_0. Such a ratio is assumed to include the effects of changing aerial coverage and relevant domain perimeter lengths where heat feedback responses depend on such measures. Other relationships may be used instead of the proposed ratio, $I(t)/I_0$, and such enhancements may be directly inserted into the *EXCEL* computer program (available by request from first listed author of current paper). (For example, an explicit finite difference model of the proposed conceptual model was also prepared as part of the current investigation, where the influence of $I(t)$ on the return heat feedback component is modeled by $[I(t)/I_0]^p,\ p > 0$.) Relationships between ice volume surface areas versus ice volume have been examined by several researchers, and a power law type of relationship is found to model this functional relationship (Raper and Braithwaite 2009; Wigley and Raper 2005). Such a power law formulation can be readily inserted into the finite difference model.

Heat Budget System Model Formulation

The mathematical model approach used is to develop a simple model of the heat budget by using ice volume as the target variable. Similar to the modeling approach of Imbrie et al. (1980, 1984, 2011), a simple model structure is developed that attempts to capture the essential features of more complex models in a class of simple models.

Combining the previous relationships, the proposed lumped parameter formulation is

$$H(t) = k^*[I(t)/I_0] * H(t) - r^* La^* \frac{dI(t)}{dt}. \tag{7}$$

To develop a formulation for the forcing function, $H(t)$ (i.e., the nonhomogeneous term on the right side of the equals sign in the previously described ordinary differential equation), the model is required to simulate the glacial and interglacial periods of the Pleistocene period that preceded the current Holocene period. Data that represents temperature and ice volume characteristics of the Pleistocene glacial and interglacials are found in several papers (for example, Petit et al. 1999, 2001). The literature describes use of other sources of evidence of the Pleistocene era, including analysis of dust and ocean sediment, among other approaches. Some of the more frequently presented data types seen in the literature are based on isotope ratio d^{18}O, deuterium and estimates of temperature based on isotope ratio dD, such as shown in Figs. 1–4. [Temperature is estimated after correction for sea-water isotopic composition (Bintanja et al. 2005) and for ice sheet elevation on EDC3 age scale (Loulergue et al. 2007)]. Plots of these data strongly suggest that periodic cycles may exist that explain the underpinnings of these data.

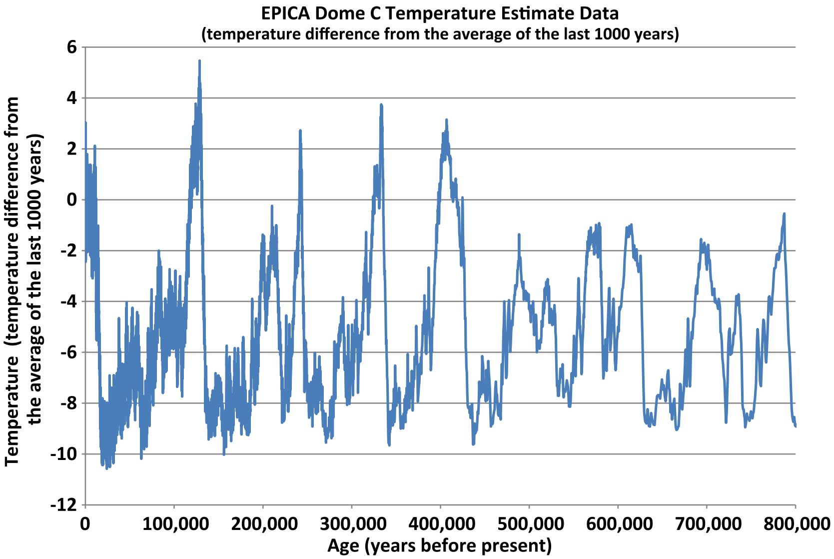

Fig. 1. EPICA Dome C temperature estimate data [data from Jouzel et al. (2007b)]

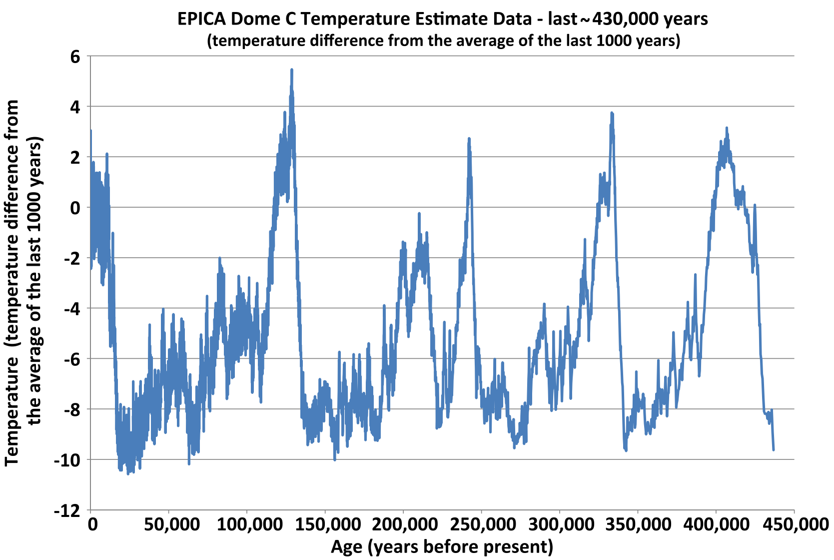

Fig. 2. EPICA Dome C temperature estimate data—last ~430,000 years [data from Jouzel et al. (2007b)]

For initial model development, considered first is a basic two-cycle periodic relationship to describe the heat term of the previously lumped parameter analog, given by

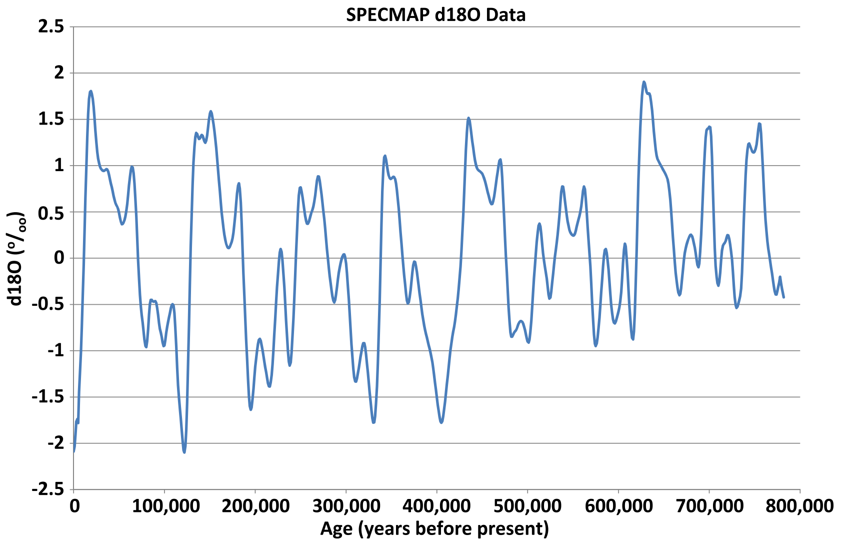

Fig. 3. Cyclic pattern of ice volume as estimated from d18O [data from Imbrie et al. (1984)]

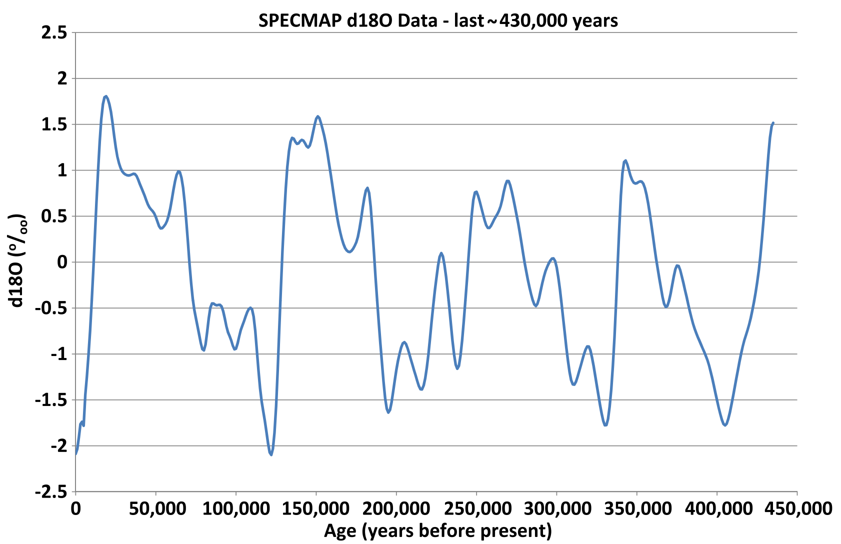

Fig. 4. Cyclic pattern of ice volume as estimated from d18O—last ~430,000 years [data from Imbrie et al. (1984)]

$$H(t) = H_0 + a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \sin(\omega_1 t) + a_4 \sin(\omega_2 t) \qquad (8)$$

where H_0 = mean value of the magnitude of [Heat]; and constants a_n and ω_n (timing period lengths) are determined by calibration of the model to the Pleistocene glacials and interglacials. In Eq. (8), a two-cycle formulation is initially considered, not only for demonstration purposes of the conceptual mathematical model, but for subsequent assessment of model convergence in outcome estimates as the number of cycles is increased. Later in this paper, more complex forcing function formulations are developed. Implicit in the previous formulation of Eq. (8) is the assumption that the available Pleistocene period data represents the various cumulative and interdependent relationships that compose the planet's numerous heat transport effects over the last two million or so years. Consequently, under the previous assumption, the calibrated model may apply to the future under the conditions that the assumptions are applicable, similar to a null hypothesis formulation.

Eqs. (7) and (8) are combined to form the proposed initial systems model,

$$\begin{aligned} &H_0 + a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \sin(\omega_1 t) + a_4 \sin(\omega_2 t) \\ &\quad = k^* \left[\frac{I(t)}{I_0}\right] * H(t) - r^* La^* \frac{dI(t)}{dt} \end{aligned} \qquad (9)$$

where the set S of parameters [used in the mathematical model of Eq. (9)] is given by

$$S = \{H_0, a_n, \omega_n, k, I_0, La, r\}. \qquad (10)$$

S contains the parameters defined as H_0 : temperature average over the time domain; a_n = Fourier series coefficients [a_j where in (7), $j = 1, 2, \ldots, 4$]; ω_n = Fourier series time cycle; k = calibration coefficient dealing with heat feedback system elements; I_0 = simulation initial volumetric ice content of the cryosphere, preferably valued at a relative maximum at start of simulation; La = water latent heat of fusion; and r = calibration coefficient relating the area-averaged volumetric latent heat of fusion to that of water (La) for all types of frozen material modeled, including (but not limited to) ice, snow, and soil water. Parameters in S may be calibrated to the glacial and interglacial data of temperature and cryosphere ice volume (for which the isotope data serves as proxies) of the Pleistocene assuming that the variation of $I(t)$ seen in such data is explained by the variation in the magnitude of [Heat], although perhaps a time lag (i.e., the average time period between causal temperature and the resulting total ice volume) may exist between $H(t)$ and $I(t)$ (for example, Berger et al. 1993). For the two-cycle model, the timing parameters ω_1 and ω_2, may be estimated from these data by considerations of the timing of the observed cyclic periods. The corresponding latent heat parameter, La, is a reference value representative of subarea or cell latent heat values throughout the problem domain. Such a reference value may be the value for pure water. The parameter r, however, represents the myriad of individual proportional constants involved in Eq. (5) and may be estimated as a lumped parameter by calibration of the solution of Eq. (9) to historic data of frozen material extent, $I(t)$. The lumped parameter combination $r^* La$ necessarily includes phase change effects for a variety of conditions and aerial extents, as seen in Eq. (5). (For simplicity, the product $r^* La$

will be hereafter denoted simply by La'). The magnitude of the various heat components used in the basic formulation of the forcing function, $H(t)$, is represented in the model by lumped parameters H_0, a_n, and ω_n, which may be calibrated to the historic data, such as those considered subsequently. The timing parameters, ω_n, can be calibrated to the cyclic pattern of ice volume estimates, such as found in Petit et al. (1999, 2001).

Estimates of the initial model cycle lengths for the $H(t)$ function of Eq. (8) are $\omega_1 = (2\pi/100{,}000)$ and $\omega_2 = (2\pi/41{,}000)$. This result is consistent with the analysis of Earth's obliquity and associated ice-age cycles discussed in Liu (1992). Other explanations of the noted cycle length and other cycles are the subject of other publications (for example, Petit et al. 1999, 2001). In Berger et al. (1993), a lag between the timing of the Earth's obliquity cycles and the timing of the proxy data commonly used for analysis of frozen material or ice volume is suggested as caused by the response time of consequential effects that add to or substantially cause changes in frozen material volume.

The proposed conceptual model operates on a set of parameters that have ranges suggested in the literature. The following text reviews some of the parameter ranges that are used in this study:

1. ***La***—Representative value of latent heat of fusion: A value of approximately 334 kJ/kg of pure water is used. This value may serve as an upper bound for La, with reductions to La considered in the model during calibration of the model and also for impact analysis from environmental changes. Adjustments to La caused by local environmental impacts are made in the associated factors used in the previous mathematical model development.
2. $\boldsymbol{I_0}$—Representative value of frozen material in the cryosphere: In the proposed conceptual model, only a representative value is used, which may be given on a unit area basis among other approaches. During the maximal glacial extent of the Pleistocene period, estimates of land area covered by glaciers are in the range of approximately 27 to over 30% (Ruddiman 2001) with thickness in locations of several thousand feet. (After model normalization, the representative value of I_0 becomes the model normalized value of 1.0, and model estimates of ice volume are in relative proportion with respect to the initial value and timing assumed.)
3. ***k***—The heat feedback component of the proposed conceptual model assumes that the relevant coefficients for the various involved heat transport elements are less than 1.0.
4. $\boldsymbol{\omega_n}$—This set of timing parameters enables different hypotheses to be tested as an underlying cause of the apparent cyclic nature of climatic trends during the Pleistocene period. Examples of such proposed explanations can be found in works by Imbrie et al. (1980, 1984, 2011), Lui (1992), Lisiecki et al. (2005a, b), and Kirkby et al. (2004).

Several of the parameters of the mathematical model, i.e., $\{H_0, \alpha_n, \omega_n, k, I_o, La'\}$ are estimated by calibration to Pleistocene period trends as described subsequently. The $H(t)$-related parameters for the basic model, H_0, a_n, and ω_n, are calibrated to the assumed representation or proxy of the $H(t)$ function. For example, the Vostok temperature data or the EPICA DOME temperature data may be used. (It is contemplated subsequently that other forcing function formulations may be used, and therefore, other parameters may be involved that are calibrated to a selected proxy data.) That is, it may be assumed that when transformed into a normalized $N(0, 1)$ form [i.e., as used in the current paper, the standard $N(0, 1)$ random variable transformation], the normalized $H(t)$ function may be approximated by the selected temperature proxy data also transformed into the same normalized $N(0, 1)$ form (see, for example, Imbrie et al. 1980, 1984, 2011). Further details about such normalization processes and correlation between selected data types are provided subsequently.

Data Types and Use of Proxy Relationships for Model Formulation

The solution to the initial basic conceptual model of Eq. (9) and the more complex extended models presented subsequently relate an input forcing function, $H(t)$, to the output $I(t)$ function. The input function is assumed to follow the trends of data reported in the literature, such as ice core temperature estimates (Jouzel et al. 2007b; Kawamura et al. 2007b; Petit et al. 2001). The objective of this paper is not to subscribe to a particular type of data proxy but instead to provide a modeling construct that may be useful in linking or defining relationships between the selected types of proxy data. Under assumed natural conditions, the output function, $I(t)$, is assumed to also follow the data trends reported in the literature that relate frozen material measure to proxy data such as $d18O$, among others. Several papers show analyses describing linkages between selected data types over time by combining the various realizations and then presenting hypotheses that describe observed similarities and differences between the realizations. In the current paper, the authors instead provide a linkage through the mathematical formulation described by Eq. (7) and its extensions demonstrated subsequently.

Because normalized data are used, the forcing function, $H(t)$, is linearly related to the selected temperature data proxy. That is, for some constants a_1 and a_2, and assuming $H(t) = a_1 + a_2 T(t)$, then when transformed into normalized $N(0, 1)$ form, both functions $H(t)$ and $T(t)$ plot identically. Consequently, one can work directly with the $N(0, 1)$ normalized form of a selected $T(t)$ realization. Further, the estimated temperature $T(t)$ is typically given as a linear function of the proxy data, dD (for example, Petit et al. 1999, 2001; Jouzel et al. 1987, 1993, 1996). Consequently, the model formulation necessarily is using a linear transform of dD as the forcing function, and therefore, the $N(0, 1)$ transform of the available dD source data serves to describe the forcing function.

Under assumed natural conditions, the output function, $I(t)$, is assumed to also follow the trends reported in the literature that relate frozen material measure to proxy data, such as $d18O$, among others. Some papers suggest that ice volume estimates may be developed as a linear function of $d18O$ (e.g., Petit et al. 1999, 2001; Zachos et al. 2001) and other papers suggest that ice volume estimates may be developed by using a nonlinear function of $d18O$ (e.g., Mix and Ruddiman 1984). However, as shown in Figs. 3 and 4, the ice volume estimates presented by many authors are simply trackings of an identified set of $d18O$ data. Assuming that the relationship is linear for the range of conditions contemplated in the modeling formulation, the $N(0, 1)$ transform of the available $d18O$ source data describes the ice volume trends during the Pleistocene time period. (If the relationship is nonlinear, then the $N(0,1)$ transform of the $d18O$ source data may require further considerations, such as including other statistical moments or using another choice of transformation.) In the current paper, $d18O$ data (Kawamura et al. 2007a; Petit et al. 2001) is used as a proxy for representing the trends of ice volume, which becomes the target for the calibration of the conceptual model formulation, as discussed subsequently.

From the previous discussion, it is seen that the general solution to the conceptual model operates on estimates of the magnitude of heat, $H(t)$, assumed to be a linear function of temperature. In turn, these are assumed to be a linear function of dD and produce an output of ice measure that is calibrated to estimates of ice volume. Ice volume is assumed to be a linear function of $d18O$ (as stated previously, the development can consider a nonlinear transformation of $d18O$ or other proxy data). Hence, the model solution is a mapping (i.e., a relationship between the two measured variables considered) from dD to $d180$.

The proxy data types considered in this paper include, but are not necessarily limited to, the sources listed in Table 1. Each type of proxy shown in bold in Table 1 was collected and processed by partitioning the study time domain into 1,000-year intervals [the selection of a 1,000-year partition is consistent with the partition size used in Imbrie et al. (1980, 1984, 2011) in a study of phase-space modeling for Pleistocene ice volume]. The resulting partition was modified to better fit the source data relative maximum and minimum data to capture the respective peak and minimal values. The modified partitioned data were then transformed by using the standard normal $N(0,1)$ transformation. In such a normalized form, calibration of the forcing $H(t)$ function to a selected proxy and calibration of the governing PDE solution $I(t)$ is made simpler, because fewer parameters are involved. Correlation plots of various normalized proxy types are shown in Fig. 5(a–c). From Fig. 5, it is seen that use of such proxy representations, as found in the literature, may be considerably impacted by the apparent unremarkable correlation, which may result in large confidence level bands if a stochastic modeling approach is integrated into the conceptual model. The correlation plots shown in Fig. 5 suggest possible linear trends between the considered variables, which indicate that use of the $N(0,1)$ transformation for normalization of the considered variables is not inappropriate.

From the previous figures, it is observed that the different data sources for a particular variable, such as estimated temperature or estimated ice volume, do not agree well. Consequently, any model using such a data source as input (or output for calibration) would be impacted by such differences. Assuming that the assemblage of identical data types are equally likely realizations of a stochastic process, then the forcing function may be modeled as a set of input realizations (i.e., estimated temperature), each equally likely, and the model then calibrated to each of the assembled output realizations (i.e., estimated ice volume or measure), each equally likely.

Initial Conceptual Model Solutions for Simplifications of Eq. (7)

Some simplifications of the initial basic model governing PDE are briefly presented.

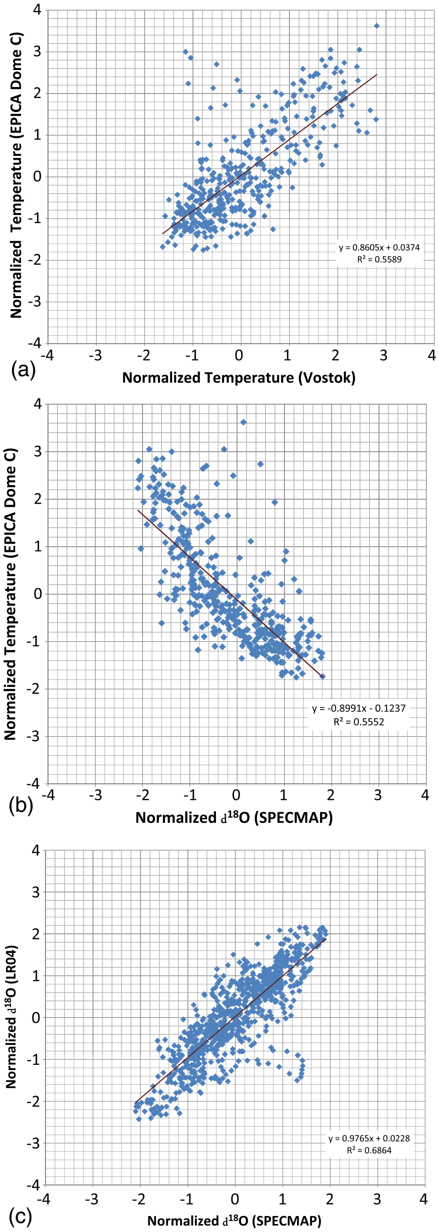

Fig. 5. Input-output correlations: (a) correlation plot of normalized temperature estimates (EPICA Dome C) versus normalized temperature estimates (Vostok); (b) correlation plot of normalized temperature estimates (EPICA Dome C) versus normalized d18O (SPECMAP); (c) correlation plot of normalized d18O (LR04) versus normalized d18O (SPECMAP)

If the magnitude of total heat, $H(t)$, is assumed to be the constant value, H_0, on the right hand side of Eq. (7), then the conceptual model is a nonhomogeneous first-order differential equation with constant coefficients. (The effect of this linearizing assumption is to cause warming periods to be warmer and cooling periods to be cooler, causing estimates of phase change to be amplified.) Possible cases of interest include:

1. $\boldsymbol{k = 0}$

   If $k = 0$, then there is no returned heat, and $[\text{Heat}] = -La'^{*}(dI(t)/dt)$. Assuming $[\text{Heat}] = H_0$, where H_0 is the so-called sun constant, the solution is

$$I(t) = (-H_0/La') * t + I_0 \tag{11}$$

   where I_0 = representative initial condition value of the total measure of frozen material. This solution says that for heat input nearly constant, the frozen material measure should decrease linearly to zero.

2. $\boldsymbol{k > 0}$

   If $k > 0$, and again, assuming $[\text{Heat}] = H_0$, with initial condition $I(t = 0) = I_0$, then a first-order ordinary differential equation (ODE) results with solution

$$I(t) = \frac{I_0}{k} * \left[1 - \exp\left(\frac{k^* H_0}{La'^* I_0} t\right) + k^* \exp\left(\frac{k^* H_0}{La'^* I_0} t\right)\right] \tag{12}$$

   with parameters as defined in Eq. (10).

For the solution of Eq. (9), where the magnitude $H(t)$ of heat is given by Eq. (8), a linearization of the heat feedback term may be made by

$$k^*[I(t)/I_0] * H(t) = p^* I(t) \tag{13}$$

where p = representative mean constant value for the function $kH(t)/I_0$ over the range of values under consideration in the analysis. Combining Eqs. (9) and (13) gives another conceptual model

$$\begin{aligned} &H_0 + a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \sin(\omega_1 t) + a_4 \sin(\omega_2 t) \\ &\quad = p^* I(t) - La'^* \frac{dI(t)}{dt} \end{aligned} \tag{14}$$

which has solution

$$I(t) = \{p^4[H_0 - e^{pt/La'}(a_1 + a_2 + H_0 - I_0 p) + a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \sin(\omega_1 t) + a_4 \sin(\omega_2 t)] + La'[p^3(-e^{pt/La'}\{a_3\omega_1 + a_4\omega_2\} + a_3\omega_1 \cos\{\omega_1 t\} + a_4\omega_2 \cos\{\omega_2 t\} - a_1\omega_1 \sin\{\omega_1 t\} - a_2\omega_2 \sin\{\omega_2 t\}) + La'(p^2\{H_0[\omega_1^2 + \omega_2^2] - e^{pt/La'}[(a_2 + H_0 - I_0 p)\omega_1^2 + (a_1 + H_0 - I_0 p)\omega_2^2] + \omega_2^2[a_1 \cos(\omega_1 t) + a_3 \sin(\omega_1 t)] + \omega_1^2[a_2 \cos(\omega_2 t) + a_4 \sin(\omega_2 t)]\} + \omega_1\omega_2 La'\{a_3 p\omega_2 \cos[\omega_1 t] + a_4 p\omega_1 \cos[\omega_2 t] - a_1 p\omega_2 \sin[\omega_1 t] - a_2 p\omega_1 \sin[\omega_2 t] + H_0\omega_1\omega_2 La' - e^{pt/La'}[p(a_4\omega_1 + a_3\omega_2) + (H_0 - I_0 p)\omega_1\omega_2 La']\})]\}/\{p[p^2 + \omega_1^2(La')^2] \times [p^2 + \omega_2^2(La')^2]\} \quad (15)$$

where the initial condition is set as $I(t = 0) = I_0$.

Model Parameter Ranges

The conceptual models shown previously (and the more complex conceptual models presented subsequently) involve modeling parameters that have nonnegative values, and several of them have upper bounds. For example, the latent heat for pure water may be assumed to be the upper bound of the parameter La, as used in the model construct leading to Eq. (5) with lumped parameter r having value less than or equal to one. The initial condition value of $I(t)$, I_0, however, is not known but may be estimated based on efforts to calibrate model outcome to known information regarding the glacials during the Pleistocene period.

Calibration of Model Parameters

As mentioned previously, Figs. 1–4 strongly suggest that there are cyclic patterns to temperature and ice volume records. Each of these plots provides information as to the function $I(t)$ regarding the variation and timing of variations in the function, $I(t)$. Assuming that a relationship exists between the magnitude of available heat, $H(t)$, and the dependent variable of the conceptual model, $I(t)$, then cyclic or other trends evident in the previous figures may be assumed to be generally applicable to the trends of the function $H(t)$. Issues regarding lag in time between $H(t)$ and $I(t)$ can be hypothesized and incorporated into the conceptual models but are not considered further in the current development. [For example, such lag time considerations may be modeled as a translation in time in the model response outcome, $I(t)$, to a prescribed input, $H(t)$.]

Normalization of Eq. (9)

The number of calibration parameters in the governing basic model formulation of Eq. (9) can be reduced by the following substitutions (several of these normalization parameters are used in the more complex extended model formulations presented subsequently):

$$i(t) = \frac{I(t)}{I_0} \quad (16a)$$

$$h(t) = \frac{H(t)}{H_0} \quad (16b)$$

$$r' = r^* La^* \frac{I_o}{H_0} = La'^* \frac{I_0}{H_0} \quad (16c)$$

$$k' = k \quad (16d)$$

where in Eq. (16*d*), a place holder is retained for future considerations. Combining Eqs. (7) and (16), the governing formulation becomes

$$h(t) = k'^* i(t)^* h(t) - r'^* \frac{di(t)}{dt} \quad (17)$$

where the parameter set for the basic two-cycle model is reduced to S' given by

$$S' = \{H_0, a_n, \omega_n, k', r'\} \quad (18)$$

In the substitutions of Eq. (16), $i(t)$ represents the proportion of the initial measure of frozen material in the cryosphere domain, and $h(t)$ represents the proportion of the representative mean value of the magnitude of [Heat]. Consequently, the initial condition would be $i(t = 0) = 1$. It is noted that in Eq. (17), it may be that $i(t) > 1$. In addition, if the initial condition is established at the timing of a glacial maximum, then the corresponding $i(t = 0) = 1$ condition still applies, and a lower bound for the value of $i(t)$ during the Pleistocene period may correspond to the current glacial conditions, which is estimated to have a corresponding value of approximately $i(t) = 0.35$ based on current ocean water levels versus ocean water levels during the glacial maximum (Ruddiman 2001).

For the initial basic conceptual model, where $H(t)$ is assumed to be the constant H_0 in the right-hand side of Eq. (9), with the left-hand side modified according to Eq. (16), that now the equivalent statement for that scenario is $h(t) = 1$ on the right-hand side of Eq. (17) giving, with the left-hand side modified according to Eq. (16)

$$1 + [a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) + a_3 \sin(\omega_1 t) + a_4 \sin(\omega_2 t)]/H_0 = k'^* i(t) - r'^* \frac{di(t)}{dt} \quad (19)$$

which for normalized boundary condition $i(t = 0) = 1$, has the solution

$$
\begin{aligned}
i(t) = \{ & [H_0 - e^{tk'/r'}(a_1 + a_2 + H_0) + a_1\cos(\omega_1 t) + a_2\cos(\omega_2 t) + a_3\sin(\omega_1 t) \\
& + a_4\sin(\omega_2 t)] \times [k']^4 + e^{tk'/r'} H_0 [k']^5 + [H_0(\omega_1^2 + \omega_2^2) - e^{tk'/r'}(\{a_2 + H_0\}\omega_1^2 \\
& + \{a_1 + H_0\}\omega_2^2) + \omega_2^2(a_1\cos\{\omega_1 t\} + a_3\sin\{\omega_1 t\}) + \omega_1^2(a_2\cos\{\omega_2 t\} \\
& + a_4\sin\{\omega_2 t\})] \times [(k')^2(r')^2] - [-1 + e^{tk'/r'}] H_0 \omega_1^2 \omega_2^2 [r']^4 \\
& + \omega_1\omega_2 k'[r']^3[a_3\omega_2\cos(\omega_1 t) + a_4\omega_1\cos(\omega_2 t) - a_1\omega_2\sin(\omega_1 t) - a_2\omega_1\sin(\omega_2 t) \\
& - e^{tk'/r'}(a_4\omega_1 + a_3\omega_2 - H_0\omega_1\omega_2 r')] + [k']^3 r'[a_3\omega_1\cos(\omega_1 t) + a_4\omega_2\cos(\omega_2 t) \\
& - a_1\omega_1\sin(\omega_1 t) - a_2\omega_2\sin(\omega_2 t) + e^{tk'/r'}(-a_3\omega_1 - a_4\omega_2 \\
& + H_0\{\omega_1^2 + \omega_2^2\} r')]\} / \{H_0 k'[(k')^2 + \omega_1^2 (r')^2] \times [(k')^2 + \omega_2^2 (r')^2]\} \qquad (20)
\end{aligned}
$$

Computational Models for Eqs. (17) and (20)

Computer models of Eq. (20) were developed by using the computer programs Mathematica and *EXCEL*. Additionally, another *EXCEL* computer model was developed that numerically approximates Eq. (17) by a finite difference, mid-time step advancement scheme (FDM). The FDM model was verified by applying it to the solution of the linearized version of Eq. (17), as seen in Eq. (20) and comparing results of the FDM model to the Mathematica model. Inherent in both modeling approaches is the simplification that the feedback responses within the thermal budget correlate to the function $i(t)$, which is the ratio of $I(t)$ to the initial value, I_0. Other feedback relationships that express the correlation between feedback to the magnitude of frozen material, $I(t)$, can be readily substituted into the FDM model. Because exact mathematical solutions are used for subsequent assessment, uncertainty in modeling results caused by the usual computational issues regarding long-term simulations and cumulative computational error are eliminated.

Demonstrations

Demonstration 1: Two-Cycle Model Approximation of Possible Impact to Pleistocene Period Cryosphere Sequence of Glacials and Interglacials due to Changes in Conceptual Model Parameter Values

In Demonstration 1, a two-cycle Fourier Series partial sum (fitted to the proxy for temperature) is used as the forcing function in the conceptual mathematical model. Fig. 6 shows a plot of the two-cycle Fourier Series partial sum versus the target data (see Figs. 1 and 2). In Figs. 6–11, time is shown in clock-time rather than before present time. The modeling results, using the program Mathematica and the two-cycle input realization (of Fig. 6), are shown in Fig. 7, where model output is plotted versus the estimate of ice volume from $d18O$ (see Figs. 3 and 4).

Demonstration 2: Six-Cycle Model Approximation of h(t) in Eq. (20)

In Demonstration 2, a six-cycle Fourier Series partial sum (fitted to the proxy for temperature) is used as the forcing function in the conceptual mathematical

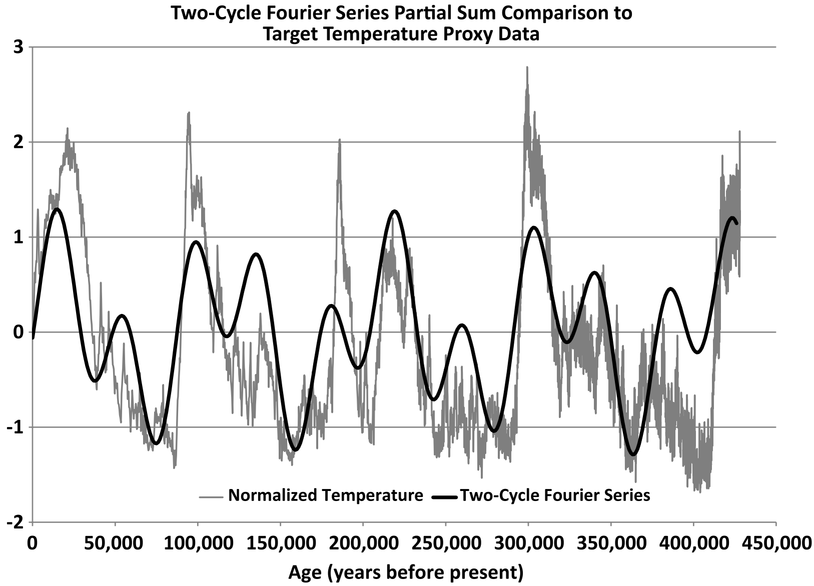

Fig. 6. Two-cycle Fourier series partial sum comparison to target temperature proxy data

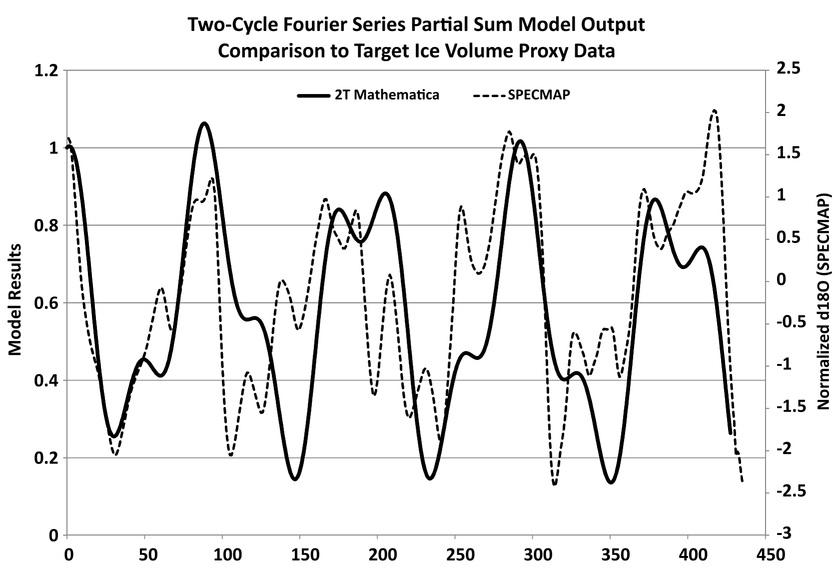

Fig. 7. Two-cycle Fourier series partial sum model output comparison to target ice volume proxy data

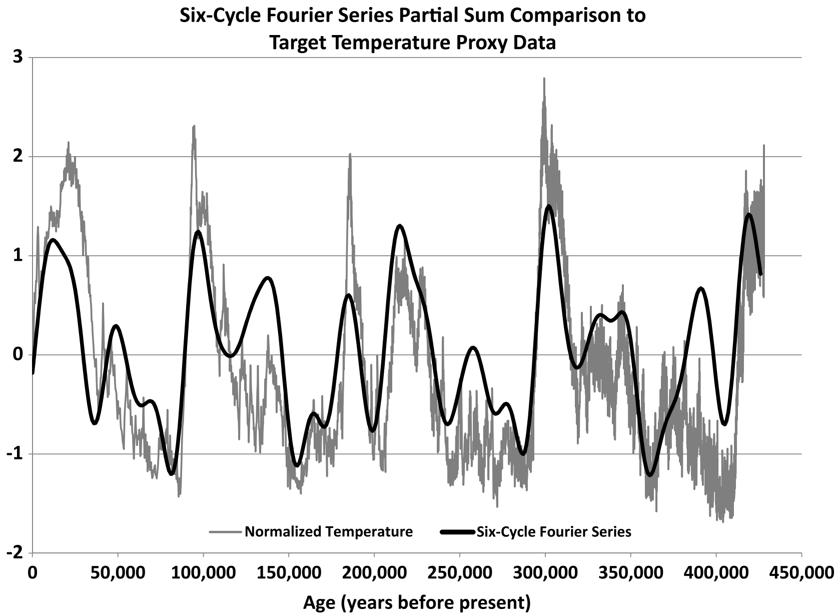

Fig. 8. Six-cycle Fourier series partial sum comparison to target temperature proxy data

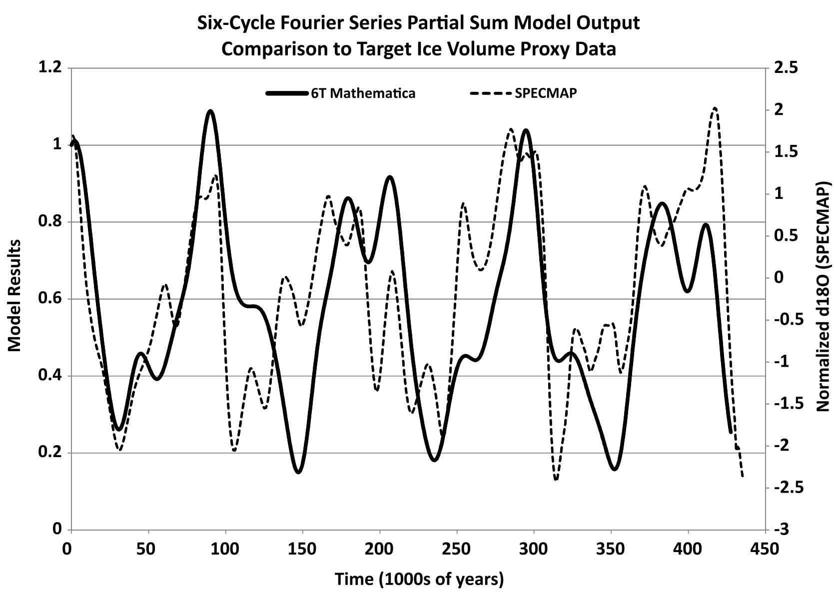

Fig. 9. Six-cycle Fourier series partial sum model output comparison to target ice volume proxy data

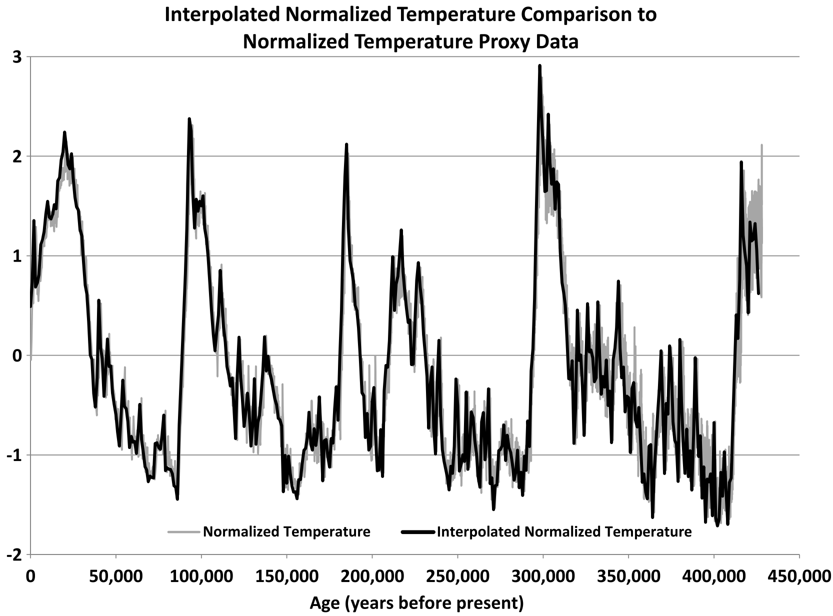

Fig. 10. Interpolated normalized temperature comparison to normalized temperature proxy data

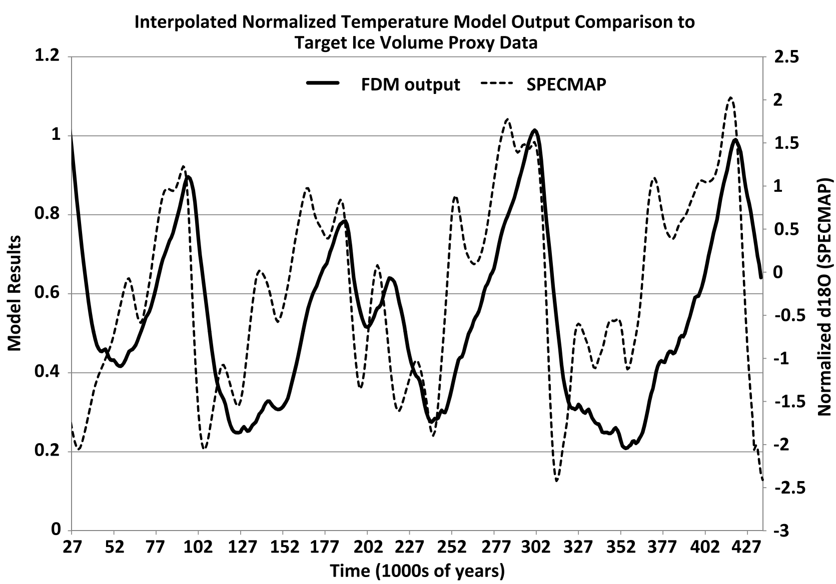

Fig. 11. Interpolated normalized temperature model output comparison to target ice volume proxy data

model. Fig. 8 shows a plot of the six-cycle Fourier Series partial sum versus the target data (see Figs. 1 and 2). The modeling results, using program Mathematica and the six-cycle input realization (of Fig. 8), are shown in Fig. 9, where model output is plotted versus the estimate of ice volume from $d18O$ (see Figs. 3 and 4).

Demonstration 3: Complex Model (Fdm) Formulation of h(t) in Eq. (20)

The linear version of the FDM analog is applied directly to the normalized proxy data of Figs. 1 and 2 as the forcing function with model estimates (see Fig. 10). From Eq. (9), the linear version is developed by setting $H(t) = H_0$ in the model feedback component. Otherwise, the nonlinear version is obtained. Using the linear version of the FDM model, modeling results are shown in Fig. 11, where model output is plotted versus the estimate of ice volume from $d18O$ (see Figs. 3 and 4).

It is apparent that numerous other test situations can be simulated by the developed models. One strength of the modeling approach is the ability to modify model parameters directly and develop estimates of changes in the cryosphere ice volume. The models can also be used to predict future cryosphere ice volume estimates based on continuation of the developed Fourier Series partial sum forcing functions, assuming such forcing functions may be applicable. Resolution of whether the current Holocene period is an independent age or if it is a continuation of the cyclical pattern of glacials and interglacials characteristic of the Pleistocene period is not a goal of the current paper.

Discussion and Conclusions

The previous demonstrations show three different modeling levels of complexity in describing the total heat forcing function used in the governing mathematical description of heat transport effects in the cryosphere seen in Eq. (7). Increasing levels of complexity for the forcing function are developed by use of: (a) a two-cycle Fourier Series partial sum fitted to normalized proxy data representing temperature, (b) a six-cycle Fourier Series partial sum fitted to the same normalized proxy data, and (c) direct use of the normalized proxy data operated upon by a central-difference time advancement finite difference method (FDM) analog. The third model considered (i.e., the FDM model) is developed as a linear model of the return-heat component of the governing mathematical formulation. The two Fourier-Series-based analogs are exactly solved by use of the computer program Mathematica, and then, those exact solutions are used for subsequent modeling purposes and also for validating the FDM analog for the same input conditions.

The two forcing functions developed by Fourier Series partial sums of two- and six-cycles, fitted to the temperature proxy data, can also be used to estimate future ice volume impacts caused by small changes in model parameters. By changing modeling parameter values, climate change impacts on the cryosphere may be estimated. Fundamental to the use of these forcing functions is the assumption of applicability of these forcing functions into the future. This fundamental issue

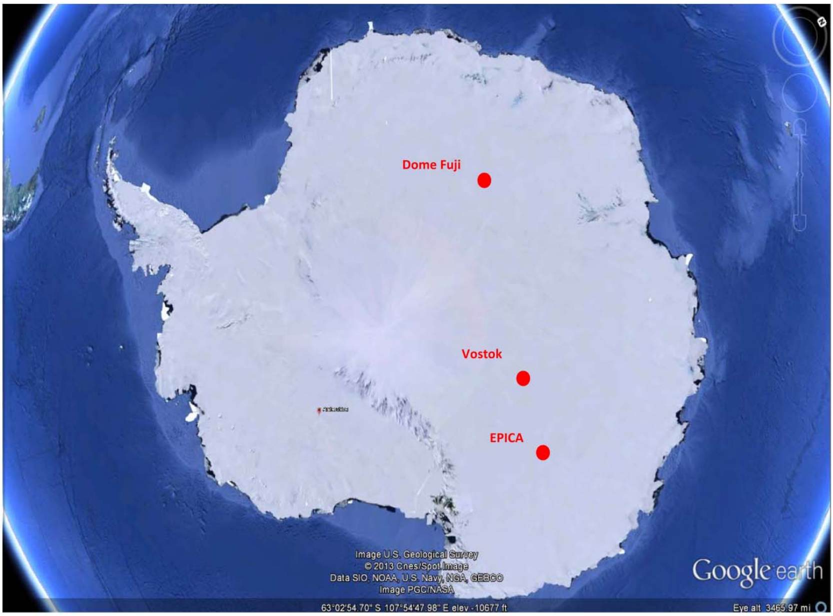

Fig. 12. Map of Dome Fuji, Vostok, and EPICA locations (Image © Google, U.S. Geological Survey, © Cnes/Spot Image, Data SIC, NOAA, U.S. Navy, NGA, GEBCO, Image PCG/NASA)

(that is, whether or not the Pleistocene period has truly ended with the last glacial period and that the Holocene period is not a continuation of the Pleistocene) is not addressed in the current paper. Notwithstanding, the mathematical underpinnings of the proposed mathematical model and its input forcing function still provide information toward answering questions of the what if type.

Further research is needed in a variety of areas involved with the many components of the current work presented. (Map of Dome Fuji, Vostok, and EPICA locations shown in Fig. 12.) However, a goal of the paper was to build a model framework starting with a more simplistic modeling assemblage and to assess the modeling results in their explanation of target model output. Because model input/output is assessed by comparison and use of proxy data for temperature and ice volume, such assessment is only as good as the proxy data are in representing the true values of temperature and ice volume. It is noted that another FDM model was developed that allows variable model parameters throughout the simulation. This variable parameter model can be calibrated to the targeted ice volume outcome with better fitting results than shown in the previous figures.

The linear form of the modeling approach presented is a first-order ordinary differential equation with constant coefficients, which is solvable by the usual methods, including computer solutions, such as using the program Mathematica. Consequently, the modeling approach is suitable for use by students that have experience with ordinary differential equations and provides a good example application of mathematical modeling to problems in the physical sciences, increasing awareness of use of mathematical tools in problem solving.

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