A Hilton-Milner-type theorem and an intersection conjecture for signed sets

Peter Borg

Department of Mathematics, University of Malta
Msida MSD 2080, Malta
p.borg.02@cantab.net

Abstract

A family $\mathcal{A}$ of sets is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect (i.e. have at least one common element). $\mathcal{A}$ is said to be *centred* if there is an element common to all the sets in $\mathcal{A}$; otherwise, $\mathcal{A}$ is said to be *non-centred*. For any $r \in [n] := \{1, ..., n\}$ and any integer $k \geq 2$, let $\mathcal{S}_{n,r,k}$ be the family

$$\{\{(x_1, y_1), ..., (x_r, y_r)\} \colon x_1, ..., x_r \text{ are distinct elements of } [n],\ y_1, ..., y_r \in [k]\}$$

of *k-signed r-sets on* $[n]$. Let $m := \max\{0, 2r - n\}$. We establish the following Hilton-Milner-type theorems, the second of which is proved using the first:
(i) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty *cross-intersecting* (i.e. any set in $\mathcal{A}_1$ intersects any set in $\mathcal{A}_2$) sub-families of $\mathcal{S}_{n,r,k}$, then

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq \binom{n}{r}k^r - \sum_{i=m}^{r}\binom{r}{i}(k-1)^i\binom{n-r}{r-i}k^{r-i} + 1.$$

(ii) If $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$, $2 \leq r \leq n$, then

$$|\mathcal{A}| \leq \begin{cases} \binom{n-1}{r-1}k^{r-1} - \sum_{i=m}^{r-1}\binom{r}{i}(k-1)^i\binom{n-1-r}{r-1-i}k^{r-1-i} + 1 & \text{if } r < n; \\ k^r - (k-1)^r + r - 1 & \text{if } r = n. \end{cases}$$

We also determine the extremal structures. (ii) is a stability theorem that extends Erdős-Ko-Rado-type results proved by various authors. We then show that (ii) leads to further evidence for an intersection conjecture suggested by the author about general signed set systems.

Keywords: Extremal set theory, intersecting families, cross-intersecting families, signed sets.

1 Introduction

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose elements are sets themselves). Arbitrary sets and families are taken to be finite. $\mathbb{N}$ is the set of positive integers $\{1, 2, ...\}$. For $m, n \in \mathbb{N}$ with $m \leq n$, we denote $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ by $[m, n]$, and if $m = 1$ then we also write $[n]$; $[0]$ is taken to be the *empty set* $\emptyset$. For a set X, the *power set* $\{A \colon A \subseteq X\}$ of X is denoted by 2^X, and the *r-uniform* sub-family $\{Y \subseteq X \colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$. The *Cartesian product* $X \times Y$ of two sets X and Y is the set $\{(x, y) \colon x \in X, y \in Y\}$.

For a family $\mathcal{F}$, we represent the union of all sets in $\mathcal{F}$ by $U(\mathcal{F})$, and the size of a largest set in $\mathcal{F}$ by $\alpha_{\mathcal{F}}$. For any set V we denote the family $\{F \in \mathcal{F} \colon F \cap V \neq \emptyset\}$ by $\mathcal{F}(V)$. If $u \in U(\mathcal{F})$, then $\mathcal{F}(\{u\})$ is called a *star of* $\mathcal{F}$. Note that $\mathcal{F}(\{u\}) = \{F \in \mathcal{F} \colon u \in F\}$.

A family $\mathcal{F}_1$ is said to be *isomorphic* to a family $\mathcal{F}_2$ if there exists a bijection $\beta \colon U(\mathcal{F}_1) \to U(\mathcal{F}_2)$ such that, for any subset F of $U(\mathcal{F}_1)$, F is a member of $\mathcal{F}_1$ iff (if and only if) the set $\{\beta(i) \colon i \in F\}$ is a member of $\mathcal{F}_2$; we write $\mathcal{F}_1 \cong \mathcal{F}_2$.

A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. $\mathcal{A}$ is said to be *centred* if there is an element common to all the sets in $\mathcal{A}$ (i.e. $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$); otherwise, $\mathcal{A}$ is said to be *non-centred*. Note that if $\mathcal{A}$ is a centred sub-family of a family $\mathcal{F}$, then $\mathcal{A}$ is a sub-family of a star of $\mathcal{F}$ and is trivially intersecting.

Let $\mathcal{F}$ be a family. If either $U(\mathcal{F}) = \emptyset$ (the only case in which $\mathcal{F}$ has no intersecting sub-families) or one of the largest intersecting sub-families of $\mathcal{F}$ is a star $\mathcal{C}$ (i.e. no intersecting sub-family of $\mathcal{F}$ has more sets than $\mathcal{C}$), then we say that $\mathcal{F}$ has the *star property*. If either $U(\mathcal{F}) = \emptyset$ or all the largest intersecting sub-families of $\mathcal{F}$ are stars, then we say that $\mathcal{F}$ has the *strict* star property.

A classical result in extremal set theory is the Erdős-Ko-Rado (EKR) Theorem [12], which says that if $r \leq n/2$, then $\binom{[n]}{r}$ has the star property, i.e. the size of any intersecting sub-family of $\binom{[n]}{r}$ is at most the size $\binom{n-1}{r-1}$ of any star of $\binom{[n]}{r}$ (note that $\binom{[n]}{r}$ is intersecting if $n/2 < r \leq n$). There are various proofs of the EKR Theorem, two of which are particularly short and beautiful: Katona's proof [16], which featured an elegant argument known as the *cycle method*, and Daykin's proof [10] using another fundamental result known as the Kruskal-Katona Theorem [17, 18]. Many other EKR-type results were proved after the publication of [12]; the survey papers [11] and [13] are recommended. In particular, Hilton and Milner [14] extended the EKR Theorem by establishing the size of a largest non-centred intersecting sub-family of $\binom{[n]}{r}$ ($r \leq n/2$).

For $2 \leq r \leq n/2$ let $\mathcal{N}_{n,r}$ denote the non-centred intersecting sub-family $\{A \in \binom{[n]}{r} \colon 1 \in A,\ A \cap [2, r+1] \neq \emptyset\} \cup \{[2, r+1]\}$ of $\binom{[n]}{r}$.

Theorem 1.1 ([14]) *If $2 \leq r \leq n/2$ and $\mathcal{A}$ is a non-centred intersecting sub-family of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq |\mathcal{N}_{n,r}| = \binom{n-1}{r-1} - \binom{n-1-r}{r-1} + 1$.*

We now define *signed sets* and outline the EKR-type results relevant to the contributions in this paper.

Let X be an n-set $\{x_1, ..., x_n\}$. Let $y_1, ..., y_n \in \mathbb{N}$. We call the set $\{(x_1, y_1), ..., (x_n, y_n)\}$ a *k-signed n-set* if $\max\{y_i \colon i \in [n]\} \leq k$. For any integer $k \geq 1$, we define $\mathcal{S}_{X,k}$ to be the family of k-signed n-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1, y_1), ..., (x_n, y_n)\} \colon y_1, ..., y_n \in [k]\}.$$

We need to define $\mathcal{S}_{\emptyset,k} := \emptyset$. With a slight abuse of notation, for a family $\mathcal{F}$ we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

For the special case $\mathcal{F} = \binom{[n]}{r}$, the family $\mathcal{S}_{\mathcal{F},k}$ is also denoted by $\mathcal{S}_{n,r,k}$; so

$$\mathcal{S}_{n,r,k} = \left\{ \{(x_1, y_1), \dots, (x_r, y_r)\} \colon \{x_1, \dots, x_r\} \in \binom{[n]}{r}, y_1, \dots, y_r \in [k] \} \right\}.$$

Note that we therefore have that $\mathcal{S}_{n,n,k}$ and $\mathcal{S}_{[n],k}$ are the same family.

Berge discovered the star property of $\mathcal{S}_{[n],k}$, and Livingston showed that $\mathcal{S}_{[n],k}$ has the strict star property unless $k = 2$.

A well-known generalisation of the Berge-Livingston result was first stated by Meyer [20] and proved by Deza and Frankl [11].

Theorem 1.2 ([11]) *Let $r \leq n$ and $k \geq 2$. Then:*
(i) $\mathcal{S}_{n,r,k}$ has the star property;
(ii) $\mathcal{S}_{n,r,k}$ has the strict star property unless $r = n \geq 3$ and $k = 2$.

It is worth pointing out that the EKR problem for signed sets has attracted much attention and has been expressed in various equivalent formulations; an account of this is given in [2]. The 'signed sets' terminology was introduced in [7] (in which an alternative proof and an application of Theorem 1.2 are given) for a setting that can be re-formulated as $\mathcal{S}_{n,r,k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced in [3], the theme of which is the following general conjecture.

Conjecture 1.3 ([3]) *For any family $\mathcal{F}$ and any integer $k \geq 2$,*
(i) $\mathcal{S}_{\mathcal{F},k}$ has the star property;
(ii) $\mathcal{S}_{\mathcal{F},k}$ does not have the strict star property only if $k = 2$ and there exist three distinct elements x_1, x_2, x_3 of $U(\mathcal{F})$ such that $\mathcal{F}(\{x_1\}) = \mathcal{F}(\{x_2\}) = \mathcal{F}(\{x_3\})$ and $\mathcal{S}_{\mathcal{F},2}(\{(x_1, 1)\})$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.

The converse of (ii) is true, simply because $\{A \in \mathcal{S}_{\mathcal{F},2} \colon |A \cap \{(x_1, 1), (x_2, 1), (x_3, 1)\}| \geq 2\}$ is a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},2}$ that is as large as $\mathcal{S}_{\mathcal{F},2}(\{(x_1, 1)\})$. Obviously we cannot replace $k \geq 2$ by $k \geq 1$, because if $\mathcal{F}$ does not have the star property (for example, $\mathcal{F}$ is a non-star intersecting family such as $\binom{[3]}{2}$)), then neither does $\mathcal{S}_{\mathcal{F},1}$ (since $\mathcal{S}_{\mathcal{F},1}$ is a copy of $\mathcal{F}$). In the same paper this conjecture is proved for families $\mathcal{F}$ that are *compressed with respect to an element* $x^* \in U(\mathcal{F})$, i.e. families $\mathcal{F}$ such that if $x \in F \in \mathcal{F}$ and $x^* \notin F$, then $(F \backslash \{x\}) \cup \{x^*\} \in \mathcal{F}$; this generalises Theorem 1.2 since $\binom{[n]}{r}$ is compressed with respect to any element of $[n]$. Holroyd and Talbot [15] essentially proved the 'non-strict' part of the conjecture for uniform families (families whose sets are of the same size) that have the star property, and part (ii) of the conjecture for such families was then verified in [3]. In [5] the conjecture is proved for the case when k is sufficiently large, depending only on the size $\alpha_{\mathcal{F}}$ of a largest set in $\mathcal{F}$.

Theorem 1.4 ([5]) *Conjecture 1.3 is true if $k \geq \frac{1}{2}(\alpha_{\mathcal{F}} - 1){\alpha_{\mathcal{F}}}^2$.*

It is worth pointing out that Conjecture 1.3(i) has a striking resemblance to the well-known Chvátal Conjecture [8, 9], which is one of the central problems in extremal set theory. The Chvátal Conjecture says that for any family $\mathcal{F}$, $\bigcup_{F\in\mathcal{F}} 2^F$ has the star property. A conjecture generalising these two conjectures is suggested in [4].

2 Main results

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$. Sometimes a result for a pair of cross-intersecting families is needed as a stepping stone to a result for intersecting families. For example, in order to obtain Theorem 1.1, Hilton and Milner [14] proved the following result.

Theorem 2.1 (Hilton and Milner [14]) *Let $1 \leq r \leq n/2$, and let $\mathcal{A}_1, \mathcal{A}_2$ be non-empty cross-intersecting sub-families of $\binom{[n]}{r}$. Let $\mathcal{B}_1 := \{[r]\}$ and $\mathcal{B}_2 := \{A \in \binom{[n]}{r} \colon A \cap [r] \neq \emptyset\}$. Then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq |\mathcal{B}_1| + |\mathcal{B}_2| = \binom{n}{r} - \binom{n-r}{r} + 1.$$

In Section 4, we prove the following signed sets analogue of the above result for the purpose of obtaining our main result.

Theorem 2.2 *Let $\mathcal{A}_1, \mathcal{A}_2$ be non-empty cross-intersecting sub-families of $\mathcal{S}_{n,r,k}$. Let $m := \max\{0, 2r - n\}$. Then*

$$|\mathcal{A}_1| + |\mathcal{A}_2| \leq \binom{n}{r}k^r - \sum_{i=m}^{r} \binom{r}{i}(k-1)^i \binom{n-r}{r-i} k^{r-i} + 1.$$

Unless $r = n$ and $k = 2$, equality holds iff either $\mathcal{A}_i = \{A\}$ and $\mathcal{A}_{3-i} = \mathcal{S}_{n,r,k}(A)$ for some $i \in [2]$ and $A \in \mathcal{S}_{n,r,k}$ or $r = 2$ and $\mathcal{A}_1 = \mathcal{A}_2 \cong \mathcal{S}_{n,2,k}(\{(1,1)\})$.

A different proof of this result in a more general form has been simultaneously obtained by Wang and Zhang [21].

For $n, i \in \mathbb{N}$ with $n \geq 3$, let

$$N_{n,i} := \{(1, i+1)\} \cup ([2, n] \times [1]) = \{(1, i+1), (2, 1), (3, 1), ..., (n, 1)\}.$$

For $n \geq 3$ and $2 \leq r \leq n$, let $\mathcal{N}_{n,r,k}$ be the non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$ given by

$$\mathcal{N}_{n,r,k} := \begin{cases} \{A \in \mathcal{S}_{n,r,k} \colon (1,1) \in A,\ A \cap ([2, r+1] \times [1]) \neq \emptyset\} & \text{if } r < n; \\ \{A \in \mathcal{S}_{n,r,k} \colon (1,1) \in A,\ A \cap N_{n,1} \neq \emptyset\} \cup \{N_{n,1}, ..., N_{n,k-1}\} & \text{if } r = n. \end{cases}$$

For $3 \leq r \leq n$, let $\mathcal{T}_{n,r,k}$ be the *triangle family* $\{A \in \mathcal{S}_{n,r,k} \colon |A \cap ([3] \times [1])| \geq 2\}$. Note that $\mathcal{N}_{n,r,k}$ and $\mathcal{T}_{n,r,k}$ are non-centred intersecting families. In Section 5, we prove the following extension of Theorem 1.2 using Theorem 2.2.

Theorem 2.3 *Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$ with $n \geq 3$ and $2 \leq r \leq n$. Let $m := \max\{0, 2r-n\}$. Then*

$$|\mathcal{A}| \leq |\mathcal{N}_{n,r,k}| = \begin{cases} \binom{n-1}{r-1}k^{r-1} - \sum_{i=m}^{r-1}\binom{r}{i}(k-1)^i\binom{n-1-r}{r-1-i}k^{r-1-i} + 1 & \text{if } r < n; \\ k^r - (k-1)^r + r - 1 & \text{if } r = n. \end{cases}$$

Unless $r = n$ and $k = 2$, equality holds iff one of the following holds:
(i) $\mathcal{A} \cong \mathcal{N}_{n,r,k}$;
(ii) $r = 3$ and $\mathcal{A} \cong \mathcal{T}_{n,3,k}$;
(iii) $r = n = 4$ and $\mathcal{A} \cong \mathcal{T}_{4,4,k}$.

Note that this is an analogue of Theorem 1.1 for signed sets and an extension of Theorem 1.2.

In Section 6 we apply Theorem 2.3 to obtain the following improvement of Theorem 1.4.

Theorem 2.4 *Conjecture 1.3 is true if $k \geq \alpha_{\mathcal{F}}{}^2$.*

3 The compression operation

The proofs of Theorems 2.2 and 2.3 are based on the *compression* (also known as *shifting* or *pushing-up*) method used in [11] for the proof of Theorem 1.2. We again refer the reader to [13] for a survey on the applications of compression in extremal set theory.

For $(a,b) \in [n] \times [2,k]$, let $\Delta_{a,b}\colon 2^{\mathcal{S}_{2^{[n]},k}} \to 2^{\mathcal{S}_{2^{[n]},k}}$ be defined by

$$\Delta_{a,b}(\mathcal{A}) := \{\delta_{a,b}(A)\colon A \in \mathcal{A}, \delta_{a,b}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{a,b}(A) \in \mathcal{A}\},$$

where $\delta_{a,b}\colon \mathcal{S}_{2^{[n]},k} \to \mathcal{S}_{2^{[n]},k}$ is defined by

$$\delta_{a,b}(A) := \begin{cases} (A\backslash\{(a,b)\}) \cup \{(a,1)\} & \text{if } (a,b) \in A; \\ A & \text{otherwise.} \end{cases}$$

Note that $|\Delta_{a,b}(\mathcal{A})| = |\mathcal{A}|$. Another fundamental property of $\Delta_{a,b}$ is that $\Delta_{a,b}(\mathcal{A})$ is intersecting if $\mathcal{A}$ is intersecting. Moreover, the following holds, which is a special case of [5, Lemma 3.1].

Lemma 3.1 *Suppose $\mathcal{A} \subset \mathcal{S}_{2^{[n]},k}$ and $V \subseteq [n] \times [2,k]$ such that $(A \cap B)\backslash V \neq \emptyset$ for any $A, B \in \mathcal{A}$. Then $(C \cap D)\backslash(V \cup \{(a,b)\}) \neq \emptyset$ for any $C, D \in \Delta_{a,b}(\mathcal{A})$.*

Corollary 3.2 *Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{S}_{2^{[n]},k}$. Let*

$$\mathcal{A}^* := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{A}).$$

Then $A \cap B \cap ([n] \times [1]) \neq \emptyset$ for any $A, B \in \mathcal{A}^*$.

Proof. By repeated application of Lemma 3.1, $(A \cap B)\backslash([n] \times [2,k]) \neq \emptyset$ for any $A, B \in \mathcal{A}^*$. The result follows since $(A \cap B)\backslash([n] \times [2,k]) = A \cap B \cap ([n] \times [1])$. □

4 Non-empty cross-intersecting families of signed sets

This section is dedicated to the proof of Theorem 2.2, which requires the following lemma for the second part (the characterisation of the extremal structures).

Lemma 4.1 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be non-empty cross-intersecting sub-families of $\mathcal{S}_{n,2,k}$, where $(2,k) \neq (n,2)$. Suppose $\mathcal{A}_i \neq \Delta_{a,b}(\mathcal{A}_1) = \Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}(\{(c,d)\})$ for some $i \in [2]$ and $(c,d) \in [n] \times [k]$. Then $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{S}_{n,2,k}(\{(a,b)\})$.*

Proof. We may assume that $i = 1$. So there exists $A_1 \in \mathcal{A}_1 \backslash \Delta_{a,b}(\mathcal{A}_1)$ such that $\delta_{a,b}(A_1) \in \Delta_{a,b}(\mathcal{A}_1) \backslash \mathcal{A}_1$; let $A_1' := \delta_{a,b}(A_1)$. Thus, for some $(a_1, b_1) \in ([n] \backslash \{a\}) \times [k]$, $A_1 = \{(a,b), (a_1,b_1)\}$ and $(c,d) \in A_1' = \{(a,1), (a_1,b_1)\}$. If $(c,d) = (a_1,b_1)$ then $A_1 \in \mathcal{S}_{n,2,k}(\{(c,d)\})$, and hence $A_1 \in \Delta_{a,b}(\mathcal{A}_1)$, a contradiction. So $(c,d) = (a,1)$ and hence, by the assumptions of the lemma, $\Delta_{a,b}(\mathcal{A}_1) = \Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}(\{(a,1)\})$. Note that this implies that $|A \cap \{(a,1),(a,b)\}| = 1$ for all $A \in \mathcal{A}_1 \cup \mathcal{A}_2$. If there exists $A_2 \in \mathcal{A}_2$ such that $(a,1) \in A_2$, then, since $A_1 \cap A_2 \neq \emptyset$ (as $\mathcal{A}_1, \mathcal{A}_2$ are cross-intersecting), A_2 can only be A_1'. Together with $\Delta_{a,b}(\mathcal{A}_2) = \mathcal{S}_{n,2,k}(\{(a,1)\})$, this implies that $\mathcal{A}_2$ contains $\mathcal{B} := \mathcal{S}_{n,2,k}(\{(a,b)\}) \backslash \{A_1\}$. Given that $(2,k) \neq (n,2)$ (i.e. $k \geq 3$ if $n = 2$), for any $A \in \mathcal{S}_{n,2,k}(\{(a,1)\})$ there exists $B \in \mathcal{B}$ such that $A \cap B = \emptyset$. By the above, it follows that $\mathcal{A}_1 = \mathcal{S}_{n,2,k}(\{(a,b)\})$, which in turn forces $\mathcal{A}_2$ to be $\mathcal{S}_{n,2,k}(\{(a,b)\})$. □

Proof of Theorem 2.2. The result is trivial for $r = 1$. If $r = n$ and $k = 2$, then the result follows from the fact that for any $A := \{(x_1,y_1), ..., (x_n,y_n)\} \in \mathcal{S}_{n,n,2}$, the unique set in $\mathcal{S}_{n,n,2}$ that does not intersect A is $\{(x_1, 3-y_1), ..., (x_n, 3-y_n)\}$. We will therefore assume that $r \geq 2$ and $(r,k) \neq (n,2)$.

From $\mathcal{A}_1$ and $\mathcal{A}_2$ we construct an intersecting family $\mathcal{C}$ as follows. For each $i \in [2]$, we add the point $(n+1, i)$ to each set in $\mathcal{A}_i$ to obtain a new family $\mathcal{A}_i' = \{A \cup \{(n+1,i)\} \colon A \in \mathcal{A}_i\}$. Then we take $\mathcal{C}$ to be the sub-family $\mathcal{A}_1' \cup \mathcal{A}_2'$ of $\mathcal{S}_{n+1,r+1,k}$. $\mathcal{C}$ is intersecting because for any A and B in $\mathcal{C}$, $(n+1,i) \in A \cap B$ if $A, B \in \mathcal{A}_i'$ for some $i \in [2]$, and A and B intersect on $[n] \times [k]$ if $A \in \mathcal{A}_i'$ and $B \in \mathcal{A}_j'$ with $i \neq j$ (since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cross-intersecting). Let $\mathcal{D}$ be the family obtained by applying all the compressions $\Delta_{a,b}$ with $(a,b) \in [n] \times [2,k]$ to $\mathcal{C}$ in the order $\Delta_{1,2}, ..., \Delta_{1,k}, ..., \Delta_{n,2}, ..., \Delta_{n,k}$, i.e.

$$\mathcal{D} := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{C}).$$

We now remove the points $(n+1,1)$ and $(n+1,2)$ from the sets in the family $\mathcal{D}$ to obtain the two families $\mathcal{B}_1 := \{D \backslash \{(n+1,1)\} \colon (n+1,1) \in D \in \mathcal{D}\}$ and $\mathcal{B}_2 := \{D \backslash \{(n+1,2)\} \colon (n+1,2) \in D \in \mathcal{D}\}$, which are therefore sub-families of $\mathcal{S}_{n,r,k}$. By an argument similar to that of Corollary 3.2, Lemma 3.1 yields

$$D_1 \cap D_2 \cap (([n] \times [1]) \cup \{(n+1,1),(n+1,2)\}) \neq \emptyset \text{ for any } D_1, D_2 \in \mathcal{D}$$

(i.e. pairwise intersection is not contained in $[n] \times [2,k]$ since all compressions $\Delta_{a,b}$ with $(a,b) \in [n] \times [2,k]$ were applied), and hence

$$B_1 \cap B_2 \cap ([n] \times [1]) \neq \emptyset \text{ for any } B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2. \tag{1}$$

For each $i \in [2]$, choose a set A_i^* in $\mathcal{A}_i$ and let $B_i^* := \{(a,1) : a \in [n], A_i^* \cap (\{a\} \times [k]) \neq \emptyset\}$ (i.e. B_i^* is the set obtained by replacing each point (a,b) in A_i^* by the point $(a,1)$); it is easy to see that $B_i^* \in \mathcal{B}_i$, because any point $(a,b) \in [n] \times [k]$ in A_i^* with $b \geq 2$ is a point in $A_i^* \cup \{n+1, i\} \in \mathcal{D}$ that was shifted to $(a,1)$ via the compression $\Delta_{a,b}$.

Let X be the n-set $[n] \times [1]$. In view of (1), we just need to focus on the intersection of the sets with X. For each $i \in [2]$, let $\mathcal{B}_i^{(q)} := \{B \in \mathcal{B}_i : |B \cap X| = q\}$, $\mathcal{X}_i := \{B \cap X : B \in \mathcal{B}_i\}$, $\mathcal{X}_i^{(q)} := \{A \in \mathcal{X}_i : |A| = q\}$. Let R be the r-set $[r] \times [1]$. For each $q \in [r]$, let $\mathcal{E}^{(q)} := \{A \in \binom{X}{q} : A \cap R \neq \emptyset\}$ and $w_q := |\mathcal{S}_{n-q,r-q,k-1}| = \binom{n-q}{r-q}(k-1)^{r-q}$. By (1), for each $i \in [2]$, $\bigcup_{q=1}^{r} \mathcal{B}_i^{(q)}$ is a partition of $\mathcal{B}_i$. So we have

$$|\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{C}| = |\mathcal{D}| = |\mathcal{B}_1| + |\mathcal{B}_2| = \sum_{q=1}^{r}(|\mathcal{B}_1^{(q)}| + |\mathcal{B}_2^{(q)}|) \leq \sum_{q=1}^{r}(|\mathcal{X}_1^{(q)}| + |\mathcal{X}_2^{(q)}|)w_q \qquad (2)$$

and

$$\sum_{q=1}^{r} |\mathcal{E}^{(q)}|w_q + 1 = |\mathcal{S}_{n,r,k}(R)| + 1 = \binom{n}{r}k^r - \sum_{i=m}^{r}\binom{r}{i}(k-1)^i\binom{n-r}{r-i}k^{r-i} + 1. \qquad (3)$$

Let $1 \leq p \leq \min\{r, n/2\}$. If $\mathcal{X}_1^{(p)} \neq \emptyset$ and $\mathcal{X}_2^{(p)} \neq \emptyset$, then, by Theorem 2.1, we have $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq \binom{n}{p} - \binom{n-p}{p} + 1$, and hence $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq |\mathcal{E}^{(p)}| + 1$ with equality only if $p = r$. Now, without loss of generality, suppose $\mathcal{X}_2^{(p)} = \emptyset$. Then $|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| = |\mathcal{X}_1^{(p)}| \leq |\{A \in \binom{X}{p} : A \cap B_2^* \neq \emptyset\}| \leq |\mathcal{E}^{(p)}|$ with the first inequality resulting from (1).

Therefore, we have just shown that

$$1 \leq p \leq \min\{r, n/2\} \;\Rightarrow\; |\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}| \leq \begin{cases} |\mathcal{E}^{(p)}| & \text{if } p < r; \\ |\mathcal{E}^{(p)}| + 1 & \text{if } p = r. \end{cases} \qquad (4)$$

If $r \leq n/2$, then the upper bound in the theorem is immediate from (2), (3) and (4).

Suppose $r > n/2$. Set $w_0 := 0$, $\mathcal{E}^{(0)} := \mathcal{X}_1^{(0)} := \mathcal{X}_2^{(0)} := \emptyset$. Consider

$$n - r \leq p \leq \lfloor n/2 \rfloor.$$

Then,

$$|\mathcal{E}^{(n-p)}| = \binom{n}{n-p}, \quad |\mathcal{E}^{(p)}| = \begin{cases} \binom{n}{p} & \text{if } p \geq n-r+1; \\ \binom{n}{p} - 1 & \text{if } p = n-r. \end{cases} \qquad (5)$$

Also, an easy calculation yields

$$w_p \geq w_{n-p} \text{ with strict inequality if } p < n/2. \qquad (6)$$

By (1), for any $A \in \mathcal{X}_i^{(p)}$ and $B \in \mathcal{X}_{3-i}^{(n-p)}$, we cannot have $A = X \backslash B$; hence

$$|\mathcal{X}_i^{(p)}| + |\mathcal{X}_{3-i}^{(n-p)}| \leq \binom{n}{n-p}. \qquad (7)$$

Let $c_p := \left(|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|\right) w_p$. If n is odd, then

$$\begin{aligned}
&\sum_{q=n-r}^{r} \left(|\mathcal{X}_1^{(q)}| + |\mathcal{X}_2^{(q)}|\right) w_q = \sum_{q=n-r}^{r} c_q = \sum_{p=n-r}^{\lfloor n/2 \rfloor} (c_p + c_{n-p}) \\
&\le \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left((|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|)w_p + \left(2\binom{n}{n-p} - (|\mathcal{X}_1^{(p)}| + |\mathcal{X}_2^{(p)}|)\right) w_{n-p} \right) \quad \text{(by (7))} \\
&\le \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left(|\mathcal{E}^{(p)}|w_p + \left(2\binom{n}{n-p} - |\mathcal{E}^{(p)}|\right) w_{n-p} \right) \quad \text{(by (4), (6))} \\
&= \sum_{p=n-r}^{\lfloor n \rfloor/2} \left(|\mathcal{E}^{(p)}|w_p + \begin{cases} |\mathcal{E}^{(n-p)}|w_{n-p} & \text{if } p \ge n-r+1 \\ (|\mathcal{E}^{(n-p)}|+1)w_{n-p} & \text{if } p = n-r \end{cases} \right) \quad \text{(by (5))} \\
&= w_r + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q = 1 + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q. \qquad (8)
\end{aligned}$$

Similarly, if n is even, then

$$\begin{aligned}
&\sum_{q=n-r}^{r} \left(|\mathcal{X}_1^{(q)}| + |\mathcal{X}_2^{(q)}|\right) w_q = \sum_{q=n-r}^{r} c_q = \sum_{p=n-r}^{n/2-1} (c_p + c_{n-p}) + c_{n/2} \\
&\le \left(1 + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q - |\mathcal{E}^{(n/2)}|w_{n/2}\right) + \left(|\mathcal{X}_1^{(n/2)}| + |\mathcal{X}_2^{(n/2)}|\right) w_{n/2} \\
&\le 1 + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q - |\mathcal{E}^{(n/2)}|w_{n/2} + \binom{n}{n/2} w_{n/2} \quad \text{(by (4))} \\
&= 1 + \sum_{q=n-r}^{r} |\mathcal{E}^{(q)}|w_q \quad \text{(by (5))}. \qquad (9)
\end{aligned}$$

We know that if $n - r \ge 2$, then (4) holds for $p = 1, ..., n - r - 1$. Together with (2), (3), (8) and (9), this gives us the desired upper bound for $|\mathcal{A}_1| + |\mathcal{A}_2|$.

Now suppose the upper bound is attained. Then $|\mathcal{X}_1^{(1)}| + |\mathcal{X}_2^{(1)}| = |\mathcal{E}^{(1)}| = r$ if $n - r \ge 2$ (by (4)), and the same holds by (6), (8) and (9) if $n - r \le 1$. We may assume that $|\mathcal{X}_1^{(1)}| \ge |\mathcal{X}_2^{(1)}|$. By (1), for each $i \in [2]$, each set in $\mathcal{X}_i$ intersects each set in $\mathcal{X}_{3-i}$, and hence each single-element set in $\mathcal{X}_i^{(1)}$ must be contained in the intersection of the sets in $\mathcal{X}_{3-i}$.

Suppose $\mathcal{X}_2^{(1)} = \emptyset$. Then, since $|\mathcal{X}_1^{(1)}| + |\mathcal{X}_2^{(1)}| = r$, we have $|\mathcal{X}_1^{(1)}| = r$, and hence B_2^* is the only set in $\mathcal{X}_2$. So $\mathcal{A}_2 = \{A_2^*\}$ and $\mathcal{A}_1 \subseteq \mathcal{S}_{n,r,k}(A_2^*)$. Since $|\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{S}_{n,r,k}(R)| + 1$, $\mathcal{A}_1 = \mathcal{S}_{n,r,k}(A_2^*)$.

Suppose $\mathcal{X}_2^{(1)} \ne \emptyset$ instead. Then $\mathcal{X}_1^{(1)} = \mathcal{X}_2^{(1)} = \{x\}$ for some $x \in X$, and hence $r = 2$ since $|\mathcal{X}_1^{(1)}| + |\mathcal{X}_2^{(1)}| = r$. By (1), $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{S}_{n,2,k}(\{x\})$. Since $|\mathcal{B}_1| = |\mathcal{B}_2| = |\mathcal{A}_1| + |\mathcal{A}_2| = |\mathcal{S}_{n,2,k}([2] \times [1])| + 1 = 2|\mathcal{S}_{n,2,k}(\{x\})|$, we actually have $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{S}_{n,2,k}(\{x\})$. It follows by Lemma 4.1 that $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{S}_{n,2,k}(\{(a, b)\})$ for some $(a, b) \in [n] \times [k]$. □

5 Non-centred intersecting families of signed sets

This section is dedicated to the proof of Theorem 2.3. We first prove a set of lemmas to ensure that in the proof of Theorem 2.3 we may work with a non-centred intersecting family $\mathcal{A} \subset \mathcal{S}_{n,r,k}$ that is invariant under any compression $\Delta_{a,b}$. The really important lemma is the first one, and the others will only be used for the extremal cases of Theorem 2.3.

Lemma 5.1 *Let $a \in [n]$, $b \in [2,k]$ and $(r,k) \neq (n,2)$. Suppose $\mathcal{A}$ is a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$ and $\Delta_{a,b}(\mathcal{A})$ is centred. Then $|\mathcal{A}| < |\mathcal{N}_{n,r,k}|$.*

Proof. Since $\mathcal{A}$ is non-centred and $\Delta_{a,b}(\mathcal{A})$ is centred, we clearly have $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{n,r,k}(\{(a,1)\})$ and hence $\mathcal{A} = \mathcal{A}(\{(a,1),(a,b)\})$. Thus, the families $\mathcal{A}_1 := \mathcal{A}(\{(a,1)\}) \backslash \mathcal{A}(\{(a,b)\})$ and $\mathcal{A}_2 := \mathcal{A}(\{(a,b)\}) \backslash \mathcal{A}(\{(a,1)\})$ are non-empty since $\mathcal{A}$ is non-centred. It follows that $\mathcal{A}_1' := \{A \backslash \{(a,1)\} \colon A \in \mathcal{A}_1\}$ and $\mathcal{A}_2' := \{A \backslash \{(a,b)\} \colon A \in \mathcal{A}_2\}$ are non-empty cross-intersecting sub-families of a family isomorphic to $\mathcal{S}_{n-1,r-1,k}$. By Theorem 2.2,

$$|\mathcal{A}_1'| + |\mathcal{A}_2'| \leq |\mathcal{S}_{n-1,r-1,k}([r-1] \times [1])| + 1 < |\mathcal{N}_{n,r,k}|.$$

Since $|\mathcal{A}| = |\mathcal{A}(\{(a,1),(a,b)\})| = |\mathcal{A}_1'| + |\mathcal{A}_2'|$, the result follows. □

Lemma 5.2 *Let $a \in [n]$, $b \in [2,k]$ and $(r,k) \neq (n,2)$. Suppose $\mathcal{A}$ is an intersecting sub-family of $\mathcal{S}_{n,r,k}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{N}_{n,r,k}$. Then $\mathcal{A} \cong \mathcal{N}_{n,r,k}$.*

Proof. We may assume without loss of generality that, for some $k_1, k_2 \in [k]$,

$$\Delta_{a,b}(\mathcal{A}) = \{A \in \mathcal{S}_{n,r,k} \colon (1,k_1) \in A,\, A \cap ([2, \min\{r+1,n\}] \times \{k_2\}) \neq \emptyset\} \cup \mathcal{B}$$

where $\mathcal{B}$ consists only of the set $[2, r+1] \times \{k_2\}$ if $r < n$, and $\mathcal{B}$ consists of the sets $N_i' := \{(1,i)\} \cup ([2,n] \times \{k_2\})$, $i = 1, ..., n$, if $r = n$ (note that in this case $\mathcal{B} \cap (\Delta_{a,b}(\mathcal{A}) \backslash \mathcal{B}) = \{N_{k_1}'\}$). Let $\mathcal{N}'$ be the family on the right-hand side of the equality above, and let $N := [2, \min\{r+1,n\}] \times \{k_2\}$.

Since $\Delta_{a,b}(\mathcal{A}) \neq \mathcal{A}$, there exists $A^* \in \mathcal{A} \backslash \Delta_{a,b}(\mathcal{A})$ such that $\delta_{a,b}(A^*) \in \Delta_{a,b}(\mathcal{A}) \backslash \mathcal{A}$. Taking $A' := \delta_{a,b}(A^*)$, we therefore have $(a,1) \in A' \in \mathcal{N}'$. Suppose $r+1 < a \leq n$. Then, by definition of $\mathcal{N}'$, the set $(A' \backslash \{(a,1)\}) \cup \{(a,b)\}$ is also in $\mathcal{N}'$ (i.e. $A^* \in \mathcal{N}'$), but this contradicts $A^* \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$. So $a \leq \min\{r+1, n\}$.

Let $\mathcal{A}_1 := \mathcal{A}(\{(1,1)\})$, $\mathcal{A}_2 := \mathcal{A}(\{(1,b)\})$, $\mathcal{A}_0 := \mathcal{A} \backslash (\mathcal{A}_1 \cup \mathcal{A}_2)$. Let $\mathcal{A}_1' := \{A \backslash \{(1,1)\} \colon A \in \mathcal{A}_1\}$ and $\mathcal{A}_2' := \{A \backslash \{(1,b)\} \colon A \in \mathcal{A}_2\}$.

Case I: $r < n$. So $\mathcal{N}' = \{A \in \mathcal{S}_{n,r,k}(\{(1,k_1)\}) \colon A \cap N \neq \emptyset\} \cup \{N\}$.

Consider first $a = 1$. By $(a,1) \in A' \in \mathcal{N}' = \Delta_{a,b}(\mathcal{A})$ and the definition of $\mathcal{N}'$, we then have $k_1 = 1$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{N\}$ and $A^* \in \mathcal{A}_2$. Suppose $\mathcal{A}_1 \neq \emptyset$. Then, $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are non-empty and cross-intersecting sub-families of a family isomorphic to $\mathcal{S}_{n-1,r-1,k}$. By Theorem 2.2, we obtain $|\mathcal{A}_1'| + |\mathcal{A}_2'| \leq |\{A \in \mathcal{S}_{n-1,r-1,k} \colon A \cap ([r-1] \times [1]) \neq \emptyset\}| + 1$, and hence $|\mathcal{A}| < |\mathcal{N}_{n,r,k}|$, which is a contradiction as $|\mathcal{A}| = |\Delta_{a,b}(\mathcal{A})| = |\mathcal{N}'| = |\mathcal{N}_{n,r,k}|$. So $\mathcal{A}_1 = \emptyset$ and hence $\mathcal{A} = \mathcal{A}_2 \cup \{N\} \subseteq \mathcal{S}_{n,r,k}(\{(1,b)\}) \cup \{N\}$. Since $\mathcal{A}$ is intersecting, $\mathcal{A} \subseteq \mathcal{N}'' := \{A \in \mathcal{S}_{n,r,k}(\{(1,b)\}) \colon A \cap N \neq \emptyset\} \cup \{N\} \cong \mathcal{N}'$. Since $|\mathcal{A}| = |\mathcal{N}'|$, $\mathcal{A} = \mathcal{N}''$.

Now consider $2 \leq a \leq r+1$. Suppose $k_2 \neq 1$. Since $(a,1) \in A' \in \mathcal{N}'$, we then get $A' \neq N$, $(1,k_1) \in A' \cap A^*$, $|A^* \cap N| \geq |A' \cap N| > 0$, and hence $A^* \in \mathcal{N}'$, a contradiction. So $k_2 = 1$.

Let $N' := (N\backslash\{(a,1)\}) \cup \{(a,b)\}$. Since $N \in \mathcal{N}' = \Delta_{a,b}(\mathcal{A})$ and $N' \notin \mathcal{N}'$, we clearly have $\mathcal{A} \subset \mathcal{S}_{n,r,k}(\{(1,k_1)\}) \cup \{M\}$ where $M \in \{N, N'\}$ and $M \in \mathcal{A}$. Since $\mathcal{A}$ is intersecting and $|\mathcal{A}| = |\Delta_{a,b}(\mathcal{A})| = |\mathcal{N}_{n,r,k}|$, $\mathcal{A} = \{A \in \mathcal{S}_{n,r,k}(\{(1,k_1)\})\colon A \cap M \neq \emptyset\} \cup \{M\}$. (Since $\mathcal{A} \neq \mathcal{N}'$, $M = N'$).

Case II: $r = n$. So $\mathcal{N}' = \{A \in \mathcal{S}_{n,r,k}(\{(1,k_1)\})\colon A \cap N \neq \emptyset\} \cup \{N_1', ..., N_k'\}$ and $N_i' = \{1,i\} \cup N$, $i = 1, ..., n$. Since $(r,k) \neq (n,2)$, $k \geq 3$.

Consider first $a = 1$. Suppose $k_1 \neq 1$. Then, since $(1,1) = (a,1) \in A' \in \mathcal{N}'$, we must have $A' = N_1'$ and hence $A^* = N_b'$, a contradiction to $A^* \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$. So $k_1 = 1$. Thus, since $\Delta_{1,b}(\mathcal{A}) = \mathcal{N}'$, we clearly have $\mathcal{A}_0 = \{N_i'\colon i \in [k]\backslash\{1,b\}\}$, $\mathcal{A}_1' \cup \mathcal{A}_2' = \mathcal{S}_{\binom{[2,n]}{n-1},k}(N)$ (i.e. $\mathcal{A}_1' \cup \mathcal{A}_2' = \mathcal{S}_{\mathcal{F},k}(N)$, $\mathcal{F} = \binom{[2,n]}{n-1}$), $\mathcal{A}_1' \cap \mathcal{A}_2' = \{N\}$, $|\mathcal{A}_1'| + |\mathcal{A}_2'| = |\mathcal{S}_{n-1,n-1,k}([n-1] \times [1])| + 1$. By Theorem 2.2, it follows that, for some $j \in [2]$, $\mathcal{A}_j' = \{N\}$ and $\mathcal{A}_{3-j}' = \mathcal{S}_{\binom{[2,n]}{n-1},k}(N)$. Thus, since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, either $\mathcal{A} = \{A \in \mathcal{S}_{n,n,k}(\{(1,b)\})\colon A \cap N \neq \emptyset\} \cup \{N_i'\colon i \in [k]\} \cong \mathcal{N}'$ or $\mathcal{A} = \mathcal{N}'$; since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{N}'$, the former holds.

If $2 \leq a \leq n$ then, by the same argument for the corresponding sub-case $2 \leq a \leq r+1$ of Case I, $N = [2,n] \times [1]$ and $\mathcal{A} = \{A \in \mathcal{S}_{n,n,k}(\{(1,k_1)\})\colon A \cap ((N\backslash\{(a,1)\}) \cup \{a,b\}) \neq \emptyset\} \cup \{(N\backslash\{(a,1)\}) \cup \{(1,i),(a,b)\}\colon i \in [k]\} \cong \mathcal{N}_{n,n,k}$. □

Lemma 5.3 *Let $a \in [n]$, $b \in [2,k]$ and $k \geq 3$. Suppose $\mathcal{A}$ is an intersecting sub-family of $\mathcal{S}_{n,r,k}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{T}_{n,r,k}$. Then $\mathcal{A} \cong \mathcal{T}_{n,r,k}$.*

Proof. We may assume without loss of generality that $\Delta_{a,b}(\mathcal{A}) = \mathcal{T}' := \{A \in \mathcal{S}_{n,r,k}\colon |A \cap T| \geq 2\}$ where $T := [3] \times \{k'\}$, $k' \in [k]$. Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A})$, there exists $A_1 \in \mathcal{A}\backslash\Delta_{a,b}(\mathcal{A})$ such that $\delta_{a,b}(A_1) \in \Delta_{a,b}(\mathcal{A})\backslash\mathcal{A}$. Let $A_2 := \delta_{a,b}(A_1)$. Since $A_2 \in \mathcal{T}'$, $|A_2 \cap T| \geq 2$. Thus, since $(a,1) \in A_2$, we have $(a,1) \in T$ because otherwise we get $|A_1 \cap T| \geq |A_2 \cap T| = 2$ contradicting $A_1 \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{T}'$. So $a \in [3]$, $k' = 1$ and $\mathcal{T}' = \mathcal{T} := \mathcal{T}_{n,r,k}$. We may assume that $a = 1$.

Let $T' := T\backslash\{(1,1)\} = \{(2,1),(3,1)\}$. We prove the lemma by showing that $\mathcal{A} = \mathcal{T}'' := \{A \in \mathcal{S}_{n,r,k}\colon |A \cap (\{1,b\} \cup T')| \geq 2\}$. Indeed, let A^* be an arbitrary set in $\mathcal{A}$. Since $\Delta_{1,b}(\mathcal{A}) = \mathcal{T}$, $1 \leq |A^* \cap T'| \leq 2$. If $|A^* \cap T'| = 2$ then it is immediate that $A^* \in \mathcal{T}''$. Suppose $|A^* \cap T'| = 1$. Let $B \in \binom{T'}{1}\backslash\{A_1 \cap T', A^* \cap T'\}$ (note that $|A_1 \cap T'| \leq 1$ as otherwise we get $A_1 \in \mathcal{T}'$ contradicting $A_1 \notin \Delta_{a,b}(\mathcal{A}) = \mathcal{T}'$). So $B \cap A_1 = B \cap A^* = \emptyset$. Let $j \in [2,3]$ such that $(j,1)$ is the unique member of B, and let $C := [2,r]\backslash\{j\}$. For each $c \in C$, choose $k_c \in [k]$ such that $(c,k_c) \notin A_1 \cup A^*$ (note that this is possible since $k \geq 3$). Let $A_3 := \{(1,b)\} \cup B \cup \{(c,k_c)\colon c \in C\}$, $A_4 := \delta_{1,b}(A_3)$. Since $A_4 \in \mathcal{T} = \Delta_{1,b}(\mathcal{A})$, at least one of A_3 and A_4 is in $\mathcal{A}$. Since $\mathcal{A}$ is intersecting and $A_4 \cap A_1 = \emptyset$, $A_4 \notin \mathcal{A}$. So $A_3 \in \mathcal{A}$. Suppose $(1,b) \notin A^*$; then $A^* \cap A_3 = \emptyset$, which contradicts $\mathcal{A}$ intersecting. So $(1,b) \in A^*$, and hence we again obtain $A^* \in \mathcal{T}''$. We have therefore shown that $\mathcal{A} \subseteq \mathcal{T}''$. Since $|\mathcal{A}| = |\Delta_{1,b}(\mathcal{A})| = |\mathcal{T}| = |\mathcal{T}''|$, $\mathcal{A} = \mathcal{T}''$. □

Lemma 5.4 *Let $n > 3$ and $a \in [n]$. Suppose $\mathcal{A}$ is an intersecting sub-family of $\mathcal{S}_{n,3,2}$ and $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) \cong \mathcal{T}_{n,3,2}$. Then $\mathcal{A} \cong \mathcal{T}_{n,3,2}$.*

Proof. By the argument in the proof of Lemma 5.4, we may assume that $a = 1$ and $\Delta_{1,b}(\mathcal{A}) = \mathcal{T} := \mathcal{T}_{n,3,2}$. Let $\mathcal{A}_1 := \mathcal{A}(\{(1,1)\})$, $\mathcal{A}_1' := \{A\backslash\{(1,1)\}\colon A \in \mathcal{A}_1\}$, $\mathcal{A}_2 := \mathcal{A}(\{(1,b)\})$, $\mathcal{A}_2' := \{A\backslash\{(1,b)\}\colon A \in \mathcal{A}_2\}$, $\mathcal{A}_0 := \mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2)$. Let $Z := [2,3] \times [1]$. It is to check that having $\Delta_{1,b} = \mathcal{T}$ implies $\mathcal{A}_0 = \{A \in \mathcal{S}_{n,3,2} \colon A \cap \{(1,1),(1,b)\} = \emptyset,\ Z \subset A\}$, $\mathcal{A}_1' \cup \mathcal{A}_2' = \mathcal{S}_{\mathcal{F},2}(Z)$ where $\mathcal{F} = \binom{[2,n]}{2}$, $\mathcal{A}_1' \cap \mathcal{A}_2' = \{Z\}$, $|\mathcal{A}_1'| + |\mathcal{A}_2'| = |\mathcal{S}_{\mathcal{F},2}(Z)| + 1 = |\mathcal{S}_{n-1,2,2}([2] \times [1])| + 1$. Since $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are cross-intersecting (as $\mathcal{A}$ is intersecting), it follows by Theorem 2.2 that, for some $j \in [2]$, $\mathcal{A}_j' = \{Z\}$ and $\mathcal{A}_{3-j}' = \mathcal{S}_{\mathcal{F},2}(Z)$. Thus, since $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, either $\mathcal{A} = \{A \in \mathcal{S}_{n,3,2} \colon A \cap (\{(1,b)\} \cup Z) \neq \emptyset\} \cong \mathcal{T}$ or $\mathcal{A} = \mathcal{T}$. (Since $\mathcal{A} \neq \Delta_{a,b}(\mathcal{A}) = \mathcal{T}$, the former holds). □

Proof of Theorem 2.3. The result is trivial for $r = 2$ because a non-centred intersecting family of sets of size 2 can only be of the form $\{\{a,b\},\{a,c\},\{b,c\}\}$. The case when $k = 2$ and $r = n$ is also easy because for any set $A := \{(x_1,y_1),...,(x_n,y_n)\} \in \mathcal{S}_{n,n,2}$, the unique set in $\mathcal{S}_{n,n,2}$ that does not intersect A is $\{(x_1, 3-y_1),...,(x_n, 3-y_n)\}$; thus, the size of an intersecting sub-family of $\mathcal{S}_{n,n,2}$ is at most $2^{n-1} = |\mathcal{N}_{n,n,2}|$. So we now assume that $r \geq 3$ and $(r,k) \neq (n,2)$.

Let $\mathcal{N} := \mathcal{N}_{n,r,k}$. Since $\mathcal{N}$ is a non-centred intersecting sub-family of $\mathcal{S}_{n,r,k}$, we may assume that

$$|\mathcal{A}| \geq |\mathcal{N}|. \tag{10}$$

Let $\mathcal{A}^* := \Delta_{n,k} \circ ... \circ \Delta_{n,2} \circ ... \circ \Delta_{1,k} \circ ... \circ \Delta_{1,2}(\mathcal{A})$. So $|\mathcal{A}^*| = |\mathcal{A}|$. By Corollary 3.2, $\mathcal{A}^*$ is intersecting. By (10) and Lemma 5.1, $\mathcal{A}^*$ is non-centred. By Lemmas 5.2–5.4, if $\mathcal{A}^*$ is isomorphic to one of $\mathcal{N}$, $\mathcal{T}_{n,3,k}$ and $\mathcal{T}_{4,4,k}$, then so is $\mathcal{A}$. We may therefore assume that $\mathcal{A} = \mathcal{A}^*$. Taking $X := [n] \times [1]$, Corollary 3.2 then gives us

$$A_1 \cap A_2 \cap X \neq \emptyset \text{ for any } A_1, A_2 \in \mathcal{A}. \tag{11}$$

Define $\mathcal{A}^{(q)} := \{A \in \mathcal{A} \colon |A \cap X| = q\}$ and $\mathcal{A}_X^{(q)} := \{A \cap X \colon A \in \mathcal{A}^{(q)}\}$. Define $\mathcal{N}^{(q)}$ and $\mathcal{N}_X^{(q)}$ similarly. Define w_q as in the proof of Theorem 2.2. We have

$$|\mathcal{A}^{(q)}| \leq |\mathcal{A}_X^{(q)}| w_q, \quad |\mathcal{N}^{(q)}| = |\mathcal{N}_X^{(q)}| w_q, \tag{12}$$

$$|\mathcal{A}| = \sum_{p=1}^{r} |\mathcal{A}^{(p)}|, \quad |\mathcal{N}| = \sum_{p=1}^{r} |\mathcal{N}^{(p)}|. \tag{13}$$

In view of (12) and (13), it is easy to check that $|\mathcal{T}_{n,3,k}| = |\mathcal{N}_{n,3,k}|$ and $|\mathcal{T}_{4,4,k}| = |\mathcal{N}_{4,4,k}|$. Thus, it remains to show that equality holds in (10) and that $\mathcal{A}$ satisfies one of parts (i), (ii) and (iii) of the theorem.

Let $\mathcal{A}_X := \{A \cap X \colon A \in \mathcal{A}\} = \bigcup_{p=1}^{r} \mathcal{A}_X^{(p)}$. Since $\mathcal{A}$ is non-centred, it follows by (11) that

$$\mathcal{A}_X \text{ is non-centred.} \tag{14}$$

An immediate implication of (14) is that

$$\mathcal{A}_X^{(1)} = \emptyset = \mathcal{N}_X^{(1)}. \tag{15}$$

Consider $2 \leq p \leq \min\{r, n/2\}$. If $\mathcal{A}_X^{(p)}$ is non-centred, then, by Theorem 1.1, we have $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_{n,p}|$, and hence $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|$; note that $|\mathcal{N}_{n,p}| = |\mathcal{N}_X^{(p)}|$ if $p = r$, and if $p < r$, then, since $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_{n,p}|$, $|\mathcal{A}_X^{(p)}| < |\mathcal{N}_X^{(p)}|$ unless $p = 2$, $\mathcal{A}_X^{(p)} \cong \binom{[3]\times[1]}{2}$, and either $r = 3$

or $r = 4 = n$. Now suppose $\mathcal{A}_X^{(p)}$ is centred and $x \in \bigcap_{A \in \mathcal{A}_X^{(p)}} A$. By (14), there exists $B \in \mathcal{A}_X$ such that $x \notin B$. Thus, by (11), $\mathcal{A}_X^{(p)} \subseteq \{A \in \binom{X}{p} : x \in A,\, A \cap B \neq \emptyset\}$ and hence $|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|$ with equality only if $p < r$ and $\mathcal{A}_X^{(p)} \cong \mathcal{N}_X^{(p)}$.

Therefore, we have shown that

$$|\mathcal{A}_X^{(p)}| \leq |\mathcal{N}_X^{(p)}|, \quad p = 1, ..., \min\{r, \lfloor n/2 \rfloor\}, \tag{16}$$

and that

$$\begin{aligned} &p \leq \min\{r, n/2\},\ p < r,\ |\mathcal{A}_X^{(p)}| = |\mathcal{N}_X^{(p)}|,\ \mathcal{A}_X^{(p)} \not\cong \mathcal{N}_X^{(p)} \\ &\Rightarrow\ p = 2,\ \min\{r, n-1\} = 3,\ \mathcal{A}_X^{(p)} = \binom{T}{2} \text{ for some } T \in \binom{X}{3}. \end{aligned} \tag{17}$$

Case I: $r \leq n/2$ (so $n \geq 6$ as $r \geq 3$). Then, by (10), (12) and (13), we have equalities in (16). By (17), it follows that either $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$ or $r = 3$ and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ for some $T \in \binom{X}{3}$.

Suppose $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$. Then, for some $x \in X$ and $A^* \in \binom{X \backslash \{x\}}{r}$, $\mathcal{A}_X^{(2)} = \{A \in \binom{X}{2} : x \in A, A \cap A^* \neq \emptyset\}$. Let $\mathcal{B} := \{B \in \mathcal{S}_{n,r,k} : x \in B,\, B \cap A^* \neq \emptyset\} \cup \{A^*\}$. Clearly, for any $C \in \mathcal{S}_{n,r,k} \backslash \mathcal{B}$ there exists $A \in \mathcal{A}_X^{(2)}$ such that $A \cap C = \emptyset$; thus, by (11), $\mathcal{A} \subseteq \mathcal{B}$. Since $\mathcal{B} \cong \mathcal{N}$, it follows by (10) that $\mathcal{A} = \mathcal{B}$.

Now suppose $r = 3$ and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ for some $T \in \binom{X}{3}$. Let $\mathcal{T}' := \{A \in \mathcal{S}_{n,3,k} : |A \cap T| \geq 2\}$. Clearly, for any $C \in \mathcal{S}_{n,r,k} \backslash \mathcal{T}'$ (i.e. $|C \cap T| \leq 1$) there exists $A \in \mathcal{A}_X^{(2)}$ such that $A \cap C = \emptyset$; thus, by (11), $\mathcal{A} \subseteq \mathcal{T}'$. Since $|\mathcal{T}'| = |\mathcal{N}_{n,3,k}|$, it follows by (10) that $\mathcal{A} = \mathcal{T}'$. So $\mathcal{A} \cong \mathcal{T}_{n,3,k}$.

Case II: $r > n/2$. Suppose $n = r = 3$. Then, since $\mathcal{A}_X^{(3)} = \emptyset$ or $\mathcal{A}_X^{(3)} = \{X\}$, it clearly follows by (14) and (15) that $\mathcal{A}_X^{(2)} \cong \binom{[3] \times [1]}{2}$. By the argument in Case I, $\mathcal{A} \subseteq \mathcal{T}_{3,3,k}$. Since $\mathcal{T}_{3,3,k} = \mathcal{N}_{3,3,k}$, it follows by (10) that $\mathcal{A} = \mathcal{N}_{3,3,k}$.

Now suppose $n \geq 4$. Consider $n - r \leq p \leq n/2$. Note that

$$w_p \geq w_{n-p} \text{ with strict inequality if } p < n/2. \tag{18}$$

By (11), for any $A \in \mathcal{A}_X^{(p)}$ and $B \in \mathcal{A}_X^{(n-p)}$, we cannot have $A = X \backslash B$; therefore

$$|\mathcal{A}_X^{(p)}| + |\mathcal{A}_X^{(n-p)}| \leq \binom{n}{n-p} = |\mathcal{N}_X^{(p)}| + |\mathcal{N}_X^{(n-p)}|. \tag{19}$$

We have

$$\begin{aligned} |\mathcal{A}^{(p)}| + |\mathcal{A}^{(n-p)}| &= |\mathcal{A}_X^{(p)}| w_p + |\mathcal{A}_X^{(n-p)}| w_{n-p} \\ &\leq |\mathcal{A}_X^{(p)}| w_p + \left(\binom{n}{n-p} - |\mathcal{A}_X^{(p)}| \right) w_{n-p} && \text{(by (19))} \\ &\leq |\mathcal{N}_X^{(p)}| w_p + \left(\binom{n}{n-p} - |\mathcal{N}_X^{(p)}| \right) w_{n-p} && \text{(by (16), (18))} \\ &= |\mathcal{N}_X^{(p)}| w_p + |\mathcal{N}_X^{(n-p)}| w_{n-p} = |\mathcal{N}^{(p)}| + |\mathcal{N}^{(n-p)}|. \end{aligned} \tag{20}$$

Thus, if n is odd, then

$$\sum_{q=n-r}^{r} |\mathcal{A}^{(q)}| = \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left(|\mathcal{A}^{(p)}| + |\mathcal{A}^{(n-p)}|\right) \leq \sum_{p=n-r}^{\lfloor n/2 \rfloor} \left(|\mathcal{N}^{(p)}| + |\mathcal{N}^{(n-p)}|\right) = \sum_{q=n-r}^{r} |\mathcal{N}^{(q)}|,$$

and if n is even, then

$$\begin{aligned}\sum_{q=n-r}^{r} |\mathcal{A}^{(q)}| &= \sum_{p=n-r}^{n/2-1} \left(|\mathcal{A}^{(p)}| + |\mathcal{A}^{(n-p)}|\right) + |\mathcal{A}^{(n/2)}| \\ &\leq \sum_{p=n-r}^{n/2-1} \left(|\mathcal{N}^{(p)}| + |\mathcal{N}^{(n-p)}|\right) + |\mathcal{N}^{(n/2)}| \quad \text{(by (16), (20))} \\ &= \sum_{q=n-r}^{r} |\mathcal{N}^{(q)}|.\end{aligned}$$

Therefore,

$$\sum_{q=n-r}^{r} |\mathcal{A}^{(q)}| \leq \sum_{q=n-r}^{r} |\mathcal{N}^{(q)}|. \tag{21}$$

Suppose $n - r \leq 1$. Then, by (13) and (21), we have $|\mathcal{A}| \leq |\mathcal{N}|$ with equality only if the inequalities giving rise to (21) are equalities. By (10), this is indeed the case. It follows by (16) and (18) (and $n \geq 4$) that we particularly have $|\mathcal{A}_X^{(2)}| = |\mathcal{N}_X^{(2)}|$ (see (20)). By (17), either $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$ or $\min\{r, n-1\} = 3$ and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ for some $T \in \binom{X}{3}$. By the argument in Case I, it follows that $\mathcal{A} \cong \mathcal{N}$ or $\mathcal{A} \cong \mathcal{T}_{n,r,k}$, and the latter holds only if $r = 3$ or $n = r = 4$.

Finally, suppose $n - r \geq 2$. By (12) and (16), $\sum_{q=1}^{n-r-1} |\mathcal{A}^{(q)}| \leq \sum_{q=1}^{n-r-1} |\mathcal{N}^{(q)}|$ with equality only if equality holds in (16) for $p = 1, ..., n-r-1$. We also have $\sum_{q=n-r}^{r} |\mathcal{A}^{(q)}| \leq \sum_{q=n-r}^{r} |\mathcal{N}^{(q)}|$ (by (21)) with equality only if equality holds in (20) for $p = n-r, ..., \lfloor n/2 \rfloor$. Together with (13), these summations yield $|\mathcal{A}| \leq |\mathcal{N}|$. By (10), $|\mathcal{A}| = |\mathcal{N}|$. Since we thus have equality in (16) for $p = 2$, it follows by (17) that either $\mathcal{A}_X^{(2)} \cong \mathcal{N}_X^{(2)}$ or $r = 3$ (note that $n - 1 > 3$ as $n - r \geq 2$ and $r \geq 3$) and $\mathcal{A}_X^{(2)} \cong \binom{T}{2}$ for some $T \in \binom{X}{3}$. As above, this yields $\mathcal{A} \cong \mathcal{N}$ or $\mathcal{A} \cong \mathcal{T}_{n,3,k}$. □

6 Intersecting systems of signed sets

Here we prove Theorem 2.4 using Theorem 2.3 with $r = n$.

For any set X in a family $\mathcal{F}$ and any sub-family $\mathcal{A}$ of $\mathcal{S}_{\mathcal{F},k}$, we define

$$\mathcal{A}_X := \mathcal{A} \cap \mathcal{S}_{X,k}.$$

The following two lemmas are important ingredients.

Lemma 6.1 *For any $n \geq 0$ and $k \geq 3$, let $b_{n,k} \in \mathbb{N}$ such that the size of a largest non-centred intersecting sub-family of $\mathcal{S}_{[n],k}$ is not greater than $b_{n,k}$. Let $\mathcal{F}$ be a family, and let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. Then $|\mathcal{A}_X| \leq b_{|X|,k}$ for all $X \in \mathcal{F}$.*

Proof. Let $X \in \mathcal{F}$. If $\mathcal{A}_X$ is non-centred, then the result is immediate. So suppose $\mathcal{A}_X$ is centred, i.e. $|\bigcap_{A \in \mathcal{A}_X} A| \geq 1$.

Case 1: $|\bigcap_{A \in \mathcal{A}_X} A| = 1$. Let (x, y) be the unique member of $\bigcap_{A \in \mathcal{A}_X} A$. Since $\mathcal{A}$ is non-centred, there exists $A^* \in \mathcal{A}$ such that $(x, y) \notin A^*$. Let $A' := A^* \cap U(\mathcal{S}_{X,k}) = A^* \cap (X \times [k])$, and choose $A'' \in \mathcal{S}_{X,k}$ such that $(x, y) \notin A''$ and $A' \subset A''$. So $\mathcal{A}_X \cup \{A''\}$ is a non-centred sub-family of $\mathcal{S}_{X,k}$, and it is also intersecting because every set in $\mathcal{A}_X$ must intersect A^* on some element of A'. Therefore, $|\mathcal{A}_X| \leq b_{|X|,k} - 1$.

Case 2: $|\bigcap_{A \in \mathcal{A}_X} A| \geq 2$. Let $I := \bigcap_{A \in \mathcal{A}_X} A$, and let $(x_1, y_1), ..., (x_{|I|}, y_{|I|})$ be the distinct elements of I. Since I is a subset of a signed set, $x_1, ..., x_{|I|}$ are distinct. If $|I| = |X|$ then I is the unique member of $\mathcal{A}_X$ (as the sets in $\mathcal{A}_X$ are of size $|X|$), so suppose $|I| < |X|$. Let $x^* \in X \backslash \{x_1, ..., x_{|I|}\}$. It is easy to see that, given that $k \geq 3$, we can choose two sets $A_1, A_2 \in \mathcal{S}_{X,k}$ such that $A_1 \cap I = \{(x_1, y_1)\}$, $A_2 \cap I = \{(x_2, y_2)\}$ and $A_1 \cap A_2 = \{(x^*, 1)\}$. So $\mathcal{A}_X \cup \{A_1, A_2\}$ is a non-centred intersecting sub-family of $\mathcal{S}_{X,k}$, and hence $|\mathcal{A}_X| \leq b_{n,k} - 2$. $\Box$

Lemma 6.2 *Let $\mathcal{F}$ be a family, and let $\mathcal{A}$ be a sub-family of $\mathcal{S}_{\mathcal{F},k}$. Let Z be a subset of $U(\mathcal{S}_{\mathcal{F},k})$ such that $A \cap Z \neq \emptyset$ for all $A \in \mathcal{A}$. Then there exists $x \in U(\mathcal{F})$ such that*

$$|\mathcal{A}| \leq |Z| \sum_{F \in \mathcal{F}(\{x\})} |\mathcal{A}_F|.$$

Proof. We have $\mathcal{A} = \bigcup_{C \in \binom{Z}{1}} \mathcal{A}(C)$. Choose $C^* \in \binom{Z}{1}$ such that $|\mathcal{A}(C)| \leq |\mathcal{A}(C^*)|$ for all $C \in \binom{Z}{1}$. We then have

$$|\mathcal{A}| = |\bigcup_{C \in \binom{Z}{1}} \mathcal{A}(C)| \leq \sum_{C \in \binom{Z}{1}} |\mathcal{A}(C)| \leq |Z||\mathcal{A}(C^*)|.$$

Obviously $C^* = \{(x, y)\}$ for some $x \in U(\mathcal{F})$ and $y \in [k]$. So $\mathcal{A}(C^*) = \bigcup_{F \in \mathcal{F}(\{x\})} \mathcal{A}_F$. Hence the result. $\Box$

Proof of Theorem 2.4. Let $\mathcal{A}$ be a non-centred intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. $\mathcal{A}$ cannot have a set A of size 1, because otherwise each set in $\mathcal{A}$ must contain A, and hence $\mathcal{A}$ is centred. So $|A| \geq 2$ for all $A \in \mathcal{A}$, and hence we can assume that

$$|F| \geq 2 \text{ for all } F \in \mathcal{F}.$$

Suppose $\mathcal{A}$ has two distinct sets A and B of size 2 such that $A, B \in \mathcal{S}_{F,k}$ for some $F \in \mathcal{F}$. Then, since A and B intersect, we have $A = \{(x_1, y_1), (x_2, y_2)\}$ and $B = \{(x_1, y_1), (x_2, y_2')\}$, where $\{x_1, x_2\} = F$, $y_1, y_2, y_2' \in [k]$ and $y_2 \neq y_2'$. So every set in $\mathcal{A}$ contains (x_1, y_1) because it intersects both A and B, and it cannot contain both (x_2, y_2) and (x_2, y_2') (since it is a signed set); however, this contradicts the assumption that $\mathcal{A}$ is non-centred. Therefore,

$$F \in \mathcal{F}, |F| = 2, \mathcal{A}_F \neq \emptyset \;\Rightarrow\; |\mathcal{A}_F| = 1. \tag{22}$$

It clearly follows that if $\alpha_{\mathcal{F}} = 2$, then $\mathcal{A} = \{(x_1, y_1), (x_2, y_2)\}, \{(x_1, y_1), (x_3, y_3)\}, \{(x_2, y_2), (x_3, y_3)\}$ for some distinct $x_1, x_2, x_3 \in U(\mathcal{F})$ such that $\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\} \in \mathcal{F}$, and hence the centred sub-family $\{\{(x_1,y_1),(x',y')\}\colon (x',y')\in(\{x_2\}\times[k])\cup(\{x_3\}\times[k])\}$ of $\mathcal{S}_{\mathcal{F},k}$ is larger than $\mathcal{A}$. This actually proves Conjecture 1.3 for the case $\alpha_{\mathcal{F}}\leq 2$.

We now consider families $\mathcal{F}$ with $\alpha_{\mathcal{F}}\geq 3$ and $k\geq k_0:={\alpha_{\mathcal{F}}}^2$.

Consider $n\geq 3$. For each $i\in[1,n-1]$, let $\mathcal{N}'_i:=\{A\in\mathcal{S}_{n,n,k}\colon (1,1),(i+1,1)\in A\}$. Let $\mathcal{N}'_n:=\{N_{n,1},...,N_{n,k-1}\}$, where $N_{n,1},...,N_{n,k-1}$ are as defined in Section 2. Clearly $\mathcal{N}_{n,n,k}=\bigcup_{i=1}^n\mathcal{N}'_i$ and hence $|\mathcal{N}_{n,n,k}|\leq\sum_{i=1}^n|\mathcal{N}'_i|=(n-1)k^{n-2}+(k-1)<nk^{n-2}$. By Theorem 2.3 with $r=n$, we can take $b_{n,k}=|\mathcal{N}_{n,n,k}|$ in Lemma 6.1, and hence $|\mathcal{A}_F|\leq|\mathcal{N}_{n,n,k}|<|F|k^{|F|-2}$ for all $F\in\mathcal{F}$ with $|F|=n$. Also, by (22), we again have $|\mathcal{A}_F|<|F|k^{|F|-2}$ for all $F\in\mathcal{F}$ with $|F|=2$.

Let $B\in\mathcal{A}$. Since each $A\in\mathcal{A}$ intersects B, Lemma 6.2 tells us that $|\mathcal{A}|\leq|B|\sum_{F\in\mathcal{F}(\{x\})}|\mathcal{A}_F|$ for some $x\in U(\mathcal{F})$. Let $\mathcal{C}:=\mathcal{S}_{\mathcal{F},k}(\{(x,1)\})$. We have

$$\begin{aligned}|\mathcal{A}|&\leq\alpha_{\mathcal{F}}\sum_{F\in\mathcal{F}(\{x\})}|\mathcal{A}_F|<\alpha_{\mathcal{F}}\sum_{F\in\mathcal{F}(\{x\})}|F|k^{|F|-2}\leq\sum_{F\in\mathcal{F}(\{x\})}{\alpha_{\mathcal{F}}}^2k^{|F|-2}\\&=\sum_{F\in\mathcal{F}(\{x\})}k_0k^{|F|-2}\leq\sum_{F\in\mathcal{F}(\{x\})}k^{|F|-1}=\sum_{F\in\mathcal{F}(\{x\})}|\mathcal{C}_F|=|\mathcal{C}|.\end{aligned}$$

Since $\mathcal{C}$ is a centred sub-family of $\mathcal{S}_{\mathcal{F},k}$, the result follows. □

References

[1] C. Berge, Nombres de coloration de l'hypergraphe h-parti complet, in: Hypergraph Seminar (Columbus, Ohio 1972), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, 13-20.

[2] P. Borg, Intersecting families of sets and permutations: a survey, in: Advances in Mathematics Research (A.R. Baswell Ed.), Volume 16, Nova Science Publishers, Inc., 2011, to appear, http://arxiv.org/abs/1106.6144.

[3] P. Borg, Intersecting systems of signed sets, Electron. J. Combin. 14 (2007) #R41.

[4] P. Borg, On Chvátal's conjecture and a conjecture on families of signed sets, European J. Combin. 32 (2011) 140–145.

[5] P. Borg, On t-intersecting families of signed sets and permutations, Discrete Math. 309 (2009) 3310-3317.

[6] P. Borg, On cross-intersecting uniform sub-families of hereditary families, Electron. J. Combin. 17 (2010) R60.

[7] B. Bollobás and I. Leader, An Erdős-Ko-Rado theorem for signed sets, Comput. Math. Appl. 34 (1997) 9-13.

[8] V. Chvátal, Unsolved Problem No. 7, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974.

[9] V. Chvátal, Intersecting families of edges in hypergraphs having the hereditary property, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974, pp. 61–66.

[10] D.E. Daykin, Erdős-Ko-Rado from Kruskal-Katona, J. Combin. Theory Ser. A 17 (1974) 254-255.

[11] M. Deza and P. Frankl, Erdős-Ko-Rado theorem - 22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419-431.

[12] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 12 (1961) 313-320.

[13] P. Frankl, The shifting technique in extremal set theory, in: Combinatorial Surveys (C. Whitehead, Ed.), Cambridge Univ. Press, London/New York, 1987, 81-110.

[14] A.J.W. Hilton and E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 18 (1967) 369-384.

[15] F.C. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, Discrete Math. 293 (2005) 165-176.

[16] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, J. Combin. Theory Ser. B 13 (1972) 183-184.

[17] G.O.H. Katona, A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, Akadémiai Kiadó (1968) 187-207.

[18] J.B. Kruskal, The number of simplices in a complex, in: Mathematical Optimization Techniques, University of California Press, Berkeley, California, 1963, pp. 251-278.

[19] M.L. Livingston, An ordered version of the Erdős-Ko-Rado Theorem, J. Combin. Theory Ser. A 26 (1979) 162-165.

[20] J.-C. Meyer, Quelques problèmes concernant les cliques des hypergraphes k-complets et q-parti h-complets, in: Hypergraph Seminar (Columbus, Ohio 1972), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, 127-139.

[21] J. Wang and H. Zhang, Nontrivial independent sets of bipartite graphs and cross-intersecting families, J. Combin. Theory Ser. A 120 (2013) 129–141.