

Chaotic Examples in Low-Dimensional Topology

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Abstract. Our earlier paper provided an introduction to basic ideas relating topological techniques to chaos theory. In the present paper we provide additional details on a number of these techniques. We go into more detail on properties of inverse limits related to chaos. In particular, we provide a detailed outline of the result of Jubran on producing a chaotic embedding of the Whitehead continuum. This paper was partially motivated by a talk given by the second author at the 8th international summer school and conference *Chaos 2011: Let's Face Chaos Through Nonlinear Dynamics* (CAMTP, University of Maribor, Slovenia, 26 June - 10 July 2011).

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1. INTRODUCTION

After summarizing the topics covered in our previous paper on topology and chaos [12], we review and outline the standard topological results on inverse limit spaces that are useful in the study of chaos.

Dynamical systems have invariant sets and induced maps on these invariant sets that can often be viewed as maps on inverse limit spaces. These techniques are then used to show how Jubran, Garity and Schori [10, 11] produced a chaotic embedding of the Whitehead continuum in R^3 .

Our previous paper [12] covered the basic examples of the squaring map from the unit circle to itself and of the dyadic solenoid. Some of the properties of inverse limits discussed in [12] are gone into in more detail in the present paper.

A basic reference for the topological terms we use is Munkres' book on topology [19]. A more comprehensive text such as Kuratowski's [18] gives details on some of the inverse limit concepts discussed. Two books that provide a good introduction to dynamical systems are Devaney's text and Falconer's text [8, 9]. Another excellent reference for the topology of chaos is Gilmore's and Lefranc's text [14]. Schori's paper [20] is a good reference for a focus on inverse limits.

2. DYNAMICAL SYSTEMS AND CHAOS

See [9] for an introduction to dynamical systems. For completeness, we review the basic terminology. For a subset D of R^n and a continuous function $f : D \rightarrow D$, the iterates $f, f^2, f^3, \dots, f^k, \dots$ form a *dynamical system* on D where $f^k = f \circ f \circ \dots \circ f$ is

the composition of f with itself k times. We use the word map to refer to a continuous function and use the symbol \cong to represent homeomorphism. A closed subset A of D is an *attractor* for this system if $f(A) = A$ and if for each point $p \in D$, the distance between the iterates $f^k(p)$ and A converges to 0. The *orbit* of a point p is the set consisting of the iterates $f^k(p)$. The point p is *periodic* if $f^k(p) = p$ for some k .

For an attractor A as above, the restriction of f to A , $f|_A$ is *chaotic* if the following three conditions hold:

1. The orbit of some point p in A is dense in A ;
2. The periodic points of f in A are dense in A ; and
3. The map f is *sensitive to initial conditions* on A . That is, there is a $\delta > 0$ such that for every $p \in A$, and for every neighborhood U of p , there is a point $x \in U$ and an iterate $f^k(x)$ of x so that the distance between $f^k(p)$ and $f^k(x)$ is greater than δ .

For compact A , the first condition can be replaced by:

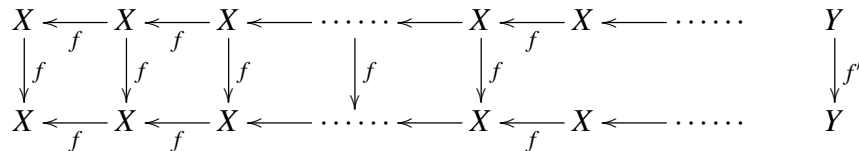
- (1a) Topological transitivity, that is, for each pair of neighborhoods U and V there is an iterate f^k such that $f^k(V) \cap U \neq \emptyset$.

See section 1.8 of Devaney [8] for a discussion. In 1992, Banks, Brooks, Cairns, Davis, and Stacey [1] showed that (1a) and (2) imply (3).

3. TOPOLOGICAL TECHNIQUES

3.1. Properties of Inverse Limits

Setting: Even though inverse limits are later defined for maps between sequences of spaces, we focus here on the situation most applicable to *dynamics*. That is, the case in which each space and each map is the same. Let $f : X \rightarrow X$ be a surjective map from a compact metric space to itself. Let (X, f) be the associated *inverse system* with *inverse limit* Y . Recall that Y is the subspace of the product $\prod_{i=1}^{\infty} X_i$ (where each $X_i \cong X$), consisting of points $(x_1, x_2, \dots, x_n, \dots)$ with the property that for each $n \geq 2$, $f(x_n) = x_{n-1}$. Let f' be the associated *shift map* from Y to Y given by $f'(x_1, x_2, \dots, x_n, \dots) = (f(x_1), f(x_2), \dots, f(x_n), \dots) = (f(x_1), x_1, \dots, x_{n-1}, \dots)$.



The following results capture the relationship between the dynamics on X and the dynamics on Y . See any of the standard references such as [8, 18, 20] for more details on these standard results. For completeness, we include an outline of the proof.

Theorem 3.1. *The following properties hold for the setting described above.*
P1: A basis for the topology on Y consists of sets of the form $p_i^{-1}(U)$ where U is open in X and p_i is the canonical projection onto the i -th coordinate.

P2: Y is a compact metric space.

P3: f' is a homeomorphism.

P4: If a is a periodic point of f of order n , then $a' = (\overline{a = f^n(a), f^{n-1}(a), \dots, f(a), \dots})$ is a periodic point of f' of order n .

P5: If the periodic points are dense in X , then they are dense in Y .

P6: If p has a dense orbit in X , then there is a point with dense orbit in Y .

P7: If f is topologically transitive, so is f' .

Proof:

P1: Let $U = \prod_{i=1}^k U_i \times \prod_{i=k+1}^{\infty} X$ be a standard basis element for $\prod_{i=1}^{\infty} X$. Let $U' = U \cap Y$ be the corresponding basis element of Y . Let $W = \bigcap_{i=1}^k (f^{k-1})^{-1}(U_i)$. Then $U' = p_k^{-1}(W)$. Conversely, every $p_i^{-1}(U)$ results from a basis element of $\prod_{i=1}^{\infty} X$ intersected with Y .

P2: It suffices to show that Y is closed in $\prod_{i=1}^{\infty} X$. This follows from the continuity of f and the fact that a sequence in $\prod_{i=1}^{\infty} X$ converges if and only if each component sequence converges.

P3: If $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ are distinct points in Y , then $f'(x) = (f(x_1), x_1, x_2, x_3, \dots)$ and $f'(y) = (f(y_1), y_1, y_2, y_3, \dots)$ are distinct. So f' is one-to-one. If $x = (x_1, x_2, x_3, \dots)$ is an arbitrary point of Y , and x_0 is chosen so that $f(x_0) = x_1$, then $f'(x_0, x_1, x_2, \dots) = x$, so f' is surjective. Since f' is continuous and Y is compact, the result follows. Note that $(f')^{-1}(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$.

P4: It follows directly from the definition of f' that $(f')^n(a') = a'$.

P5: Let $U' = p_i^{-1}(U)$ be a basis element for Y . Choose a such that $f^n(a) = a$ and $a \in U$. Let $x = (x_n)_{n=1}^{\infty}$ be the point in Y defined by $x_j = f^k(a)$ where $k = (i - j) \bmod n$. Then x is a periodic point of f' in U' .

P6: Let a be a point in X with a dense orbit under f . Recursively define a point $x = (x_n)_{n=1}^{\infty}$ in Y by letting $x_1 = a$. Having defined x_1, \dots, x_{k-1} so that $f(x_i) = x_{i-1}$, let x_k be chosen so that $f(x_k) = x_{k-1}$. The point so defined has a dense orbit in Y .

P7: Let $U' = p_i^{-1}(U)$ and $V' = p_j^{-1}(V)$ be arbitrary basic open sets in Y . Choose k so that $f^k(V) \cap f^{-(j-i)}(U) \neq \emptyset$. Then $(f')^k(V') \cap U' \neq \emptyset$ as required. ■

The next corollary follows immediately from the preceding results.

Corollary 3.2. *Let $f : X \rightarrow X$ be as in the setting above. If f is chaotic, so is $f' : Y \rightarrow Y$.*

The advantage of this point of view is that the chaotic map f' is a homeomorphism whereas the original map f need not be (and in common examples it is not).

For later reference, we give the general definition of inverse limit here. Let a system of spaces A_1, A_2, A_3, \dots and maps $f_i : A_{i+1} \rightarrow A_i$ be given. The *inverse limit* of this system, denoted

$$\lim_{\leftarrow} (A_i, f_i)$$

is the subset of the topological product of spaces $\prod_{i=1}^{\infty} A_i$ consisting of points (a_1, a_2, a_3, \dots) with $f_i(a_{i+1}) = a_i$ for all i .

3.2. A Chaotic Embedding of the Whitehead Continuum

Consider the torus T_1 embedded in the torus T_0 as shown below.

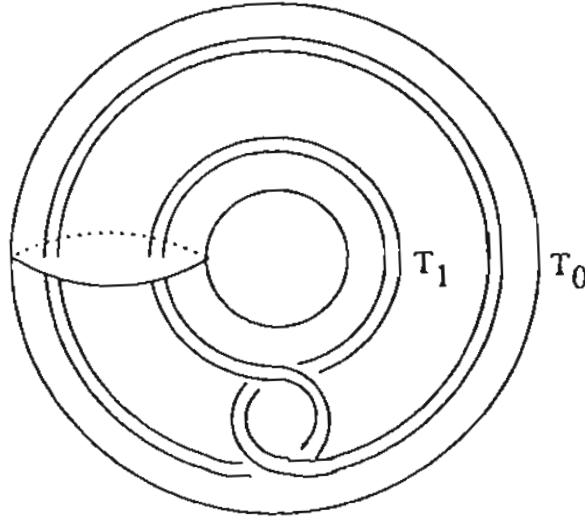


FIGURE 1. The Whitehead Link

There is a homeomorphism $h : R^3 \rightarrow R^3$ taking T_0 onto T_1 . Then $h(T_1) = h^2(T_0)$ is a self-linked torus contained in the interior of $T_1 = h(T_0)$. Assume $T_0, h(T_0), h^2(T_0), \dots$ are constructed efficiently to force the 1-dimensionality of their intersection $W = \bigcap_{i=0}^{\infty} h^i(T_0)$. Then W is called the *Whitehead continuum* [22]. We refer to h as a *Whitehead map*. The following results along with a history can be found in Jubran's thesis, [16].

- W is a cell-like noncellular subset of R^3 ;
- R^3/W is not homeomorphic to R^3 ;
- $(R^3/W) \times R^1$ is homeomorphic to R^4 ;
- W is homeomorphic to the Knaster continuum K ; and
- The embedding of K onto W in R^3 is inequivalent to the standard embedding of K in $R^2 \times \{0\} \subset R^3$.

Let $h : R^3 \rightarrow R^3$ be an arbitrary Whitehead map. Let $\Lambda = \bigcap_{i=0}^{\infty} h^i(T_0)$.

While Λ is a local attractor for h , it can be shown that the restriction of h to Λ is not necessarily chaotic in general. (See [16].)

What goes wrong is the failure of topological transitivity: there is an open set U of Λ that gets mapped into itself under h .

3.3. Modifying the Construction:

Our goal here is to describe a modification of the above construction to produce an embedding of the Whitehead continuum in R^3 in a chaotic manner. See [10] and [16] and for more complete details.

Let $T = S^1 \times D$ be such that $T \subset B_3$ where D is a 2-cell and B_3 is a 3-cell. The construction proceeds via the following steps:

Step 1: There is a map $H : B_3 \rightarrow B_3$ that is a *near homeomorphism*, i.e. there is a sequence of homeomorphisms $H_{r_i} : B_3 \rightarrow B_3$ converging uniformly to H . In addition, each H_{r_i} is a Whitehead map.

Step 2: There exists a homeomorphism $F : \lim_{\leftarrow} (B_3, H) \rightarrow \lim_{\leftarrow} (B_3, H_{r_i})$ such that $F(\lim_{\leftarrow} (T_1, H)) = \lim_{\leftarrow} (T_1, H_{r_i})$.

Step 3: Taking S^1 to be the quotient space of $[0, 1]$ generated by identifying the endpoints $\{0\}$ and $\{1\}$, then the restriction of H to $S^1 \times \{0\}$, where $\{0\}$ is the center of D , is the function $h : S^1 \rightarrow S^1$ defined by

$$h(x) = \begin{cases} 2x, & \text{for } 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

which is known to be chaotic.

Step 4: The set $\lim_{\leftarrow} (T, H)$ is a local attractor for $\hat{H} : \lim_{\leftarrow} (B_3, H) \rightarrow \lim_{\leftarrow} (B_3, H)$.

Step 1 discussion:

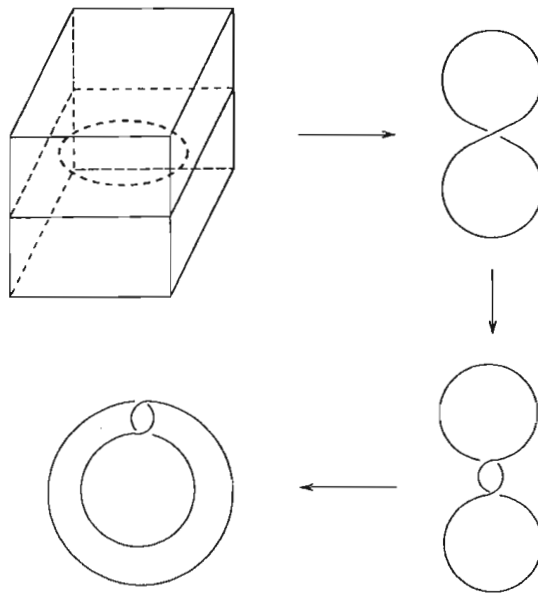


FIGURE 2. Steps in the Whitehead Construction

As a first step, imagine the homeomorphism G from R^3 to itself that takes T to a Whitehead link in T arising as follows. (See Figure 2 for an illustration.) Twist a flexible 3-cell B in such a way that the boundary stays fixed and the interior is twisted and so that a top view of $S^1 \subset \text{Int}(B^2)$ goes through the following stages:

- A half twist is introduced;
- Another half twist is introduced; and
- The top loop is folded down over the bottom loop which produces the desired self-linking.

We now define three *pseudo-isotopies* P_t^1, P_t^2 and P_t^3 .

The map P_1^1 shrinks the solid torus $G(T)$ to $G(S^1)$ leaving $G(S^1)$ fixed. The map P_1^2 eliminates the self-linking of $G(S^1)$. The map P_1^3 shrinks the torus T to its core S^1 .

Define the near-homeomorphism $H : B_3 \rightarrow B_3$ by $H = P_1^3 \circ P_1^2 \circ P_1^1 \circ G$. (See Figure 3 for an illustration.) The map H is the one needed previously to construct the chaotic embedding of the Whitehead continuum.

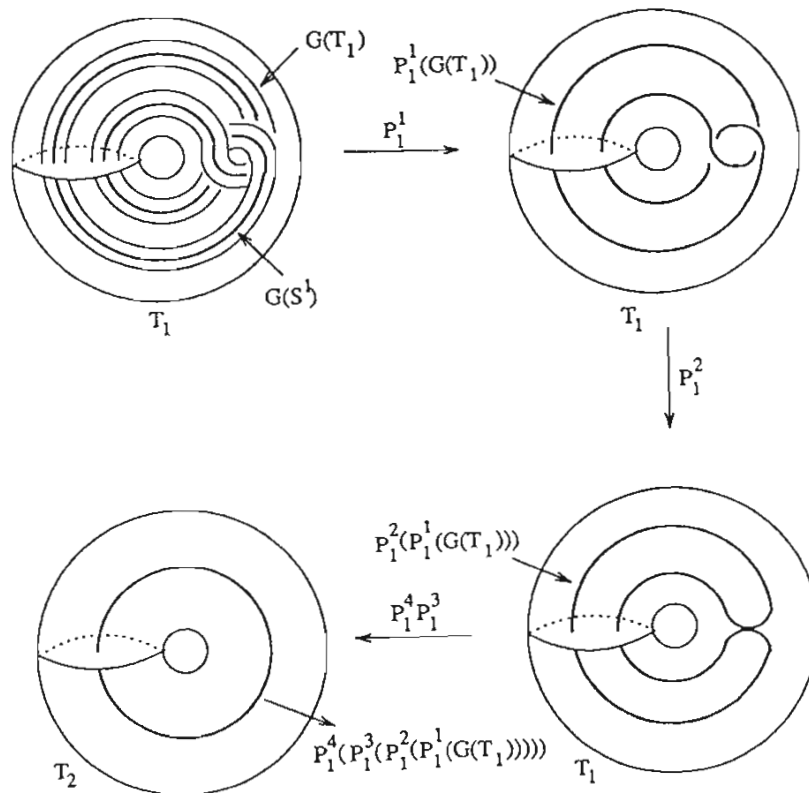


FIGURE 3. Steps in the Near-homeomorphism Construction

Step 2 discussion:

Note that (2) implies that $\lim_{\leftarrow}(T, H)$ is embedded in $\lim_{\leftarrow}(B_3, H)$ just as the standard Whitehead continuum is embedded in B_3 .

The homeomorphism $F : \lim_{\leftarrow}(B_3, H) \rightarrow \lim_{\leftarrow}(B_3, H_{r_i})$ such that $F(\lim_{\leftarrow}(T_1, H)) = \lim_{\leftarrow}(T_1, H_{r_i})$ is obtained by a technical result on inverse limits showing that the two inverse limits are homeomorphic. See [16] for details.

$$\begin{array}{ccccccc} (B^3, T) \xleftarrow{H_{r_1}} (B^3, T) \xleftarrow{H_{r_2}} (B^3, T) \cdots & \cdots & (B^3, T) \xleftarrow{H_{r_n}} (B^3, T) & \cdots & (B^3, W) & & \\ & & & & \cong \downarrow F & & \\ (B^3, T) \xleftarrow{H} (B^3, T) \xleftarrow{H} (B^3, T) \cdots & \cdots & (B^3, T) \xleftarrow{H} (B^3, T) & \cdots & (B^3, W') & & \end{array}$$

Step 3 discussion:

W' obtained as an inverse limit of T under the map H has an induced map $H' : W' \rightarrow W'$ as in the section on inverse limit properties. The key point of Step 3 is that this induced map on the inverse limit is chaotic. This follows from the homeomorphism between the inverse limit of the spaces T and the inverse limit of the spaces S^1 and from the fact that the map on the inverse limit of the spaces S^1 is chaotic.

$$\begin{array}{ccccccc} (B^3, T) \xleftarrow{H} (B^3, T) \xleftarrow{H} (B^3, T) \cdots & \cdots & (B^3, T) \xleftarrow{H} (B^3, T) \cdots & & (B^3, W') & & \\ & & & & \cong \downarrow F' & & \\ (B^3, S^1) \xleftarrow{H} (B^3, S^1) \xleftarrow{H} (B^3, S^1) \cdots & \cdots & (B^3, S^1 T) \xleftarrow{H} (B^3, S^1 T) \cdots & & (B^3, W') & & \end{array}$$

Thus (3) implies that H restricted to $\lim_{\leftarrow}(T, H)$ is chaotic and thus produces the chaotic embedding of the Whitehead continuum.

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