Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 892420, 10 pages
doi:10.1155/2012/892420

Research Article

New Exact Solutions to the KdV-Burgers-Kuramoto Equation with the Exp-Function Method

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Received 9 February 2012; Accepted 19 March 2012

Academic Editor: Khalida Inayat Noor

Based on the characteristics of the truncated Painlevé expansion method and the Exp-function method, new generalized solitary wave solutions are constructed for the KdV-Burgers-Kuramoto equation, which cannot be directly constructed from the Exp-function method. This work highlights the power of the Exp-function method in providing generalized solitary wave solutions of different physical structures.

1. Introduction

The investigation of the exact solutions of nonlinear partial differential equations plays an important role in mathematics, physics, and other applied science areas. In recent years, a variety of powerful methods has been proposed and analyzed to construct the explicit exact solutions to nonlinear evolution equations. Among these methods, the Exp-function method proposed by He and Wu [1] is particularly notable in its power and applicability in solving nonlinear problems, and it has been successfully applied to many kinds of nonlinear partial differential equations [2–17]. All of these applications verified that the Exp-function method is a straightforward, efficient, and versatile technique for finding generalized solitary, periodic, and rational solutions for nonlinear evolution equations as well as for revealing intriguing characteristics of various inner-wave interactions.

The KdV-Burgers-Kuramoto equation [18, 19] is

$$u_t + \nu u u_x + \mu u_{xxx} + \alpha u_{xx} + \gamma u_{xxxx} = 0, \tag{1.1}$$

where ν, μ, α, and γ are constants. This equation is an important mathematical model arising in many different physical contexts to describe many phenomena which are simultaneously involved in nonlinearity, dissipation, dispersion, and instability, especially at the description of turbulence processes. Since the solutions possess their actual physical application [20], various effective methods have been applied to construct the exact solutions of the KdV-Burgers-Kuramoto equation. These methods include the tanh function method [21–23], the homogeneous balance method [24], and the generalized F-expansion method [20]. As a consequence, it is still a significant and interesting task to search for new explicit exact solutions for nonlinear evolution equations. In the present work we extend the Exp-function method in combination with the truncated Painlevé expansion method [25, 26] to obtain new nontrivial exact solitary wave solutions to the KdV-Burgers-Kuramoto equation (1.1). As a result, we have found some new exact solitary wave solutions to (1.1) in the case where the Exp-function method provides trivial solutions only, which will be investigated in more detail in the following section.

2. Solitary Wave Solutions by the Exp-Function Method

Using the transformation $u = u(\eta)$, $\eta = kx + \omega t$, (1.1) becomes the ordinary differential equation:

$$\omega u' + k\nu u u' + k^2 \alpha u'' + k^3 \mu u''' + k^4 \gamma u'''' = 0. \tag{2.1}$$

According to the Exp-function method [1], we assume that the solution of (2.1) can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-d}^{c} a_n e^{n\eta}}{\sum_{m=-q}^{p} b_m e^{m\eta}} = \frac{a_{-d} e^{-d\eta} + \cdots + a_c e^{c\eta}}{b_{-q} e^{-q\eta} + \cdots + b_p e^{p\eta}}, \tag{2.2}$$

where c, d, p, and q are positive integers, a_n and b_m are unknown constants, which are to be determined later.

By balancing the linear term of the highest order in (2.1) with the highest order nonlinear term, we get $c+3p = 2(p+c)$, and, thus, $p = c$. Similarly, balancing the linear term of the lowest order in (2.1) with the lowest order nonlinear term yields $q = d$. Here, the values of c and d can be freely chosen [1]. With the help of Maple, we have obtained only trivial solutions of (1.1) for the cases where (i) $p = c = 1$ and $d = q = 1$, (ii) $p = c = 1$ and $d = q = 2$, and (iii) $p = c = 2$ and $d = q = 1$, but one new solitary wave solution in the case where $p = c = 2$ and $d = q = 2$, which is given by

$$u(x,t) = \frac{a_2 e^{2\eta} + a_2 b_1 e^{\eta} + a_2 b_0 + a_2 b_{-1} e^{-\eta}}{e^{2\eta} + b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}}, \tag{2.3}$$

where $\eta = kx + \omega t$, k and ω are arbitrary constants, and the coefficients a_2, b_{-1}, b_0, b_1 are free parameters. We expect that the other cases for the values of c and d will produce nontrivial solitary wave solutions to (1.1). In the next section we will consider finding new exact solitary wave solutions to (1.1) in the case where the Exp-function method provides trivial solutions only, particularly when $p = c = 1$ and $q = d = 1$.

3. New Solitary Wave Solutions to the KdV-Burgers-Kuramoto Equation

Let us consider the KdV-Burgers-Kuramoto equation defined by (1.1). The truncated Painlevé expansion method will be adopted to find a dependent variable transformation for finding solitary wave solutions to the considered equation. Substituting $u(x,t)$ in the form

$$u(x,t) = \frac{u_0(x,t)}{f(x,t)^r} + \frac{u_1(x,t)}{f(x,t)^{r-1}} + \cdots + u_r(x,t) \tag{3.1}$$

into (1.1) and finding the order of the pole $r = 2$ for the solution $u(x,t)$ and the functions $u_0(x,t), u_1(x,t), \dots, u_r(x,t)$, we obtain the dependent variable transformation

$$u(x,t) = \frac{60\gamma}{\nu}\frac{\partial^3 \ln f}{\partial x^3} + \frac{15\alpha}{\nu}\frac{\partial^2 \ln f}{\partial x^2} + \left(\frac{240\mu}{76\nu} - \frac{15\alpha^2}{76\nu\gamma}\right)\frac{\partial \ln f}{\partial x}, \tag{3.2}$$

where $\gamma\nu \neq 0$. Throughout the remainder of this paper we assume that γ and μ satisfy the condition $\gamma\nu \neq 0$.

Substitution of this transformation into the KdV-Burgers-Kuramoto equation (1.1) yields a rather complicated system of nonlinear equations in f for which we seek the solution form of the Exp-function method [1]:

$$f(x,t) = \frac{\sum_{n=-c}^{d} a_n e^{n\eta}}{\sum_{m=-p}^{q} b_m e^{m\eta}}, \tag{3.3}$$

where $\eta = kx + \omega t$.

For the sake of simplicity, we choose $p = c = 1$ and $q = d = 1$. Based on the consideration in (3.2) and (3.3), in this paper we assume that (1.1) admits a generalized solitary wave solution of the form

$$u(x,t) = \frac{g(x,t)}{76\nu\gamma\left(b_1 e^{\eta} + b_0 + b_{-1}e^{-\eta}\right)^3\left(a_1 e^{\eta} + a_0 + a_{-1}e^{-\eta}\right)^3}, \tag{3.4}$$

where

$$\begin{aligned} g(x,t) = {} & A_5 e^{5\eta} + A_4 e^{4\eta} + A_3 e^{3\eta} + A_2 e^{2\eta} + A_1 e^{\eta} + A_0 \\ & + B_1 e^{-\eta} + B_2 e^{-2\eta} + B_3 e^{-3\eta} + B_4 e^{-4\eta} + B_5 e^{-5\eta}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
A_0 = -15\Big(&-12\alpha^2 a_{-1}^2 b_1^2 b_0 a_0 - 6\alpha^2 a_{-1} b_1^2 b_{-1} a_0^2 + 6384\gamma^2 k^2 a_1^2 b_0^2 a_{-1} b_{-1} - 6\alpha^2 a_{-1} b_1 b_0^2 a_0^2 \\
&+ 14592\gamma^2 k^2 a_1^2 b_{-1}^2 a_{-1} b_1 - 684\gamma k a_1^2 b_0^2 \alpha b_{-1} a_{-1} + 1824\gamma^2 k^2 a_1^2 b_0 b_{-1}^2 a_0 \\
&- 6384\gamma^2 k^2 a_1 b_0^2 a_{-1}^2 b_1 - 684\gamma k a_1 b_0^2 \alpha a_{-1}^2 b_1 - 2\alpha^2 a_{-1}^3 b_1^3 + 6384\gamma^2 k^2 a_1 b_{-1}^2 a_0^2 b_1 \\
&+ 684\gamma k a_0^2 b_1 \alpha b_{-1}^2 a_1 + 684\gamma k a_0^2 b_{-1} \alpha b_1^2 a_{-1} + 456\gamma k a_0^3 b_1 \alpha b_{-1} b_0 \\
&- 1824\gamma^2 k^2 a_{-1}^2 b_1^2 b_0 a_0 - 456\gamma k a_1 b_0^3 \alpha a_{-1} a_0 + 192\mu\gamma a_{-1}^2 b_1^2 b_0 a_0 - 6\alpha^2 a_{-1}^2 b_1^2 b_{-1} a_1 \\
&+ 12\alpha^2 a_1^2 b_0 b_{-1}^2 a_0 + 6\alpha^2 a_0^2 b_1 b_{-1}^2 a_1 - 1824\gamma^2 k^2 a_0^2 b_{-1} b_0^2 a_1 + 96\mu\gamma a_{-1} b_1 b_0^2 a_0^2 \\
&- 6384\gamma^2 k^2 a_{-1} b_1^2 a_0^2 b_{-1} - 96\mu\gamma a_{-1} b_0^2 b_{-1} a_1^2 - 32\,\mu\gamma a_1^3 b_{-1}^3 - 96\mu\gamma a_1^2 b_1 b_{-1}^2 a_{-1} \\
&- 192\mu\gamma a_1^2 b_0 b_{-1}^2 a_0 + 96\mu\gamma a_{-1}^2 b_1^2 b_{-1} a_1 + 96\mu\gamma a_{-1}^2 b_1 b_0^2 a_1 \\
&+ 1824\gamma^2 k^2 a_{-1} b_1 b_0^2 a_0^2 + 32\mu\gamma a_{-1}^3 b_1^3 - 96\mu\gamma a_1 b_1 b_{-1}^2 a_0^2 - 96\mu\gamma a_1 b_0^2 b_{-1} a_0^2 \\
&+ 96\mu\gamma a_{-1} b_1^2 b_{-1} a_0^2 - 14592\gamma^2 k^2 a_1 b_{-1} a_{-1}^2 b_1^2 + 2\alpha^2 a_1^3 b_{-1}^3 + 6\alpha^2 a_1^2 b_1 b_{-1}^2 a_{-1} \\
&+6\alpha^2 a_{-1} b_0^2 b_{-1} a_1^2 - 6\alpha^2 a_{-1}^2 b_1 b_0^2 a_1 + 6\alpha^2 a_0^2 b_{-1} b_0^2 a_1\Big) k, \\
A_1 = -15\Big(&9\alpha^2 a_{-1} b_0 b_{-1}^2 a_1^2 + 9\alpha^2 a_0^2 b_{-1}^2 b_0 a_1 + 96\mu\gamma a_{-1}^2 b_1 b_0 b_{-1} a_1 + 6\alpha^2 a_{-1} b_1 b_{-1}^2 a_1 a_0 \\
&- 9\alpha^2 a_{-1}^2 b_1 b_0^2 a_0 + \alpha^2 a_0^3 b_1 b_{-1}^2 - 6\alpha^2 a_{-1}^2 b_1 b_0 b_{-1} a_1 - 9\alpha^2 a_{-1}^2 b_1^2 b_{-1} a_0 + \alpha^2 a_0^3 b_{-1} b_0^2 \\
&- 5\alpha^2 a_{-1}^3 b_1^2 b_0 + 6\alpha^2 a_{-1} b_0^2 b_{-1} a_1 a_0 - \alpha^2 a_{-1}^2 b_0^3 a_1 + 5\alpha^2 a_0 b_{-1}^3 a_1^2 - 6\alpha^2 a_{-1} b_1 b_0 b_{-1} a_0^2 \\
&- \alpha^2 a_{-1} b_0^3 a_0^2 + 80\mu\gamma a_{-1}^3 b_1^2 b_0 + 304\gamma k a_0^3 b_1 \alpha b_{-1}^2 - 456\gamma k a_1 b_0 \alpha a_{-1}^2 b_1 b_{-1} \\
&- 1824\gamma^2 k^2 a_1 b_0^2 a_{-1} a_0 b_{-1} + 76\alpha\gamma k a_0^3 b_{-1} b_0^2 + 456\gamma k a_1 b_{-1}^2 \alpha a_{-1} b_1 a_0 + 304\gamma^2 k^2 a_{-1} b_0^3 a_0^2 \\
&- 304\gamma^2 k^2 a_0^3 b_{-1} b_0^2 + 16\mu\gamma a_{-1} b_0^3 a_0^2 - 16\mu\gamma a_0^3 b_{-1} b_0^2 - 76\alpha\gamma k a_{-1} b_0^3 a_0^2 \\
&+ 96\mu\gamma a_{-1} b_1 b_0 b_{-1} a_0^2 - 16\mu\gamma a_0^3 b_1 b_{-1}^2 - 912\gamma k a_1 b_0^2 \alpha a_{-1} b_{-1} a_0 - 304\gamma^2 k^2 a_{-1}^3 b_1^2 b_0 \\
&- 14592\gamma^2 k^2 a_1 b_{-1} a_{-1}^2 b_1 b_0 + 8208\gamma^2 k^2 a_1^2 b_{-1}^2 a_{-1} b_0 + 14592\gamma^2 k^2 a_1 b_{-1}^2 a_{-1} b_1 a_0 \\
&- 96\mu\gamma a_{-1} b_1 b_{-1}^2 a_1 a_0 + 1824\gamma^2 k^2 a_{-1} b_0 a_0^2 b_1 b_{-1} + 2432\gamma^2 k^2 a_0^3 b_1 b_{-1}^2 \\
&+ 144\mu\gamma a_{-1}^2 b_1^2 b_{-1} a_0 + 684\gamma k a_{-1}^2 b_1^2 \alpha b_{-1} a_0 - 144\mu\gamma a_1 b_0 b_{-1}^2 a_0^2 + 144\mu\gamma a_{-1}^2 b_1 b_0^2 a_0 \\
&- 80\mu\gamma a_1^2 b_{-1}^3 a_0 - 96\mu\gamma a_{-1} b_0^2 b_{-1} a_1 a_0 + 16\mu\gamma a_{-1}^2 b_0^3 a_1 - 144\mu\gamma a_{-1} b_0 b_{-1}^2 a_1^2 \\
&+ 912\gamma k a_{-1} b_1 \alpha a_0^2 b_{-1} b_0 + 76\gamma k a_{-1}^3 b_1^2 \alpha b_0 - 8208\gamma^2 k^2 a_{-1}^2 b_1^2 a_0 b_{-1} - 304\gamma k a_1 b_0^3 \alpha a_{-1}^2 \\
&-76\gamma k a_1^2 b_{-1}^3 \alpha a_0 + 304\gamma^2 k^2 a_1^2 b_{-1}^3 a_0 - 684\gamma k a_1^2 b_0 \alpha b_{-1}^2 a_{-1} - 2432\gamma^2 k^2 a_1 b_0^3 a_{-1}^2\Big) k, \\
A_2 = -15\Big(&-12\alpha^2 a_{-1}^2 b_1 b_0 b_{-1} a_0 - 4\alpha^2 a_{-1}^3 b_1 b_0^2 - 4\alpha^2 a_{-1}^3 b_1^2 b_{-1} + 12\alpha^2 a_{-1} b_0 b_{-1}^2 a_1 a_0 \\
&- 2\alpha^2 a_{-1}^2 b_0^3 a_0 + 2\alpha^2 a_0^3 b_{-1}^2 b_0 + 4\alpha^2 a_1^2 b_{-1}^3 a_{-1} + 912\alpha\gamma k a_{-1}^2 b_1 b_0 b_{-1} a_0 + 4\alpha^2 a_0^2 b_{-1}^3 a_1
\end{aligned}$$

$$
\begin{aligned}
&- 304\gamma^2 k^2 a_{-1}^2 b_0^3 a_0 + 304\gamma^2 k^2 a_{-1}^3 b_1 b_0^2 + 2432\gamma^2 k^2 a_1^2 b_{-1}^3 a_{-1} - 76\gamma k a_0^2 b_{-1}^3 \alpha a_1 \\
&+ 76\gamma k a_{-1}^3 b_1 \alpha b_0^2 - 304\gamma k a_1^2 b_{-1}^3 \alpha a_{-1} - 684\gamma k a_1 b_0^2 \alpha a_{-1}^2 b_{-1} + 8208\gamma^2 k^2 a_{-1} b_1 a_0^2 b_{-1}^2 \\
&- 32\mu\gamma a_0^3 b_{-1}^2 b_0 - 304\gamma^2 k^2 a_1 b_{-1}^3 a_0^2 + 64\mu\gamma a_{-1}^3 b_1 b_0^2 - 1824\gamma^2 k^2 a_{-1}^2 b_1 b_0 a_0 b_{-1} \\
&- 76\gamma k a_{-1}^2 b_0^3 \alpha a_0 - 912\alpha\gamma k a_{-1} b_0 b_{-1}^2 a_1 a_0 - 64\mu\gamma a_1 b_{-1}^3 a_0^2 + 32\mu\gamma a_{-1}^2 b_0^3 a_0 \\
&+ 304\gamma k a_{-1}^3 b_1^2 \alpha b_{-1} + 304\gamma^2 k^2 a_0^3 b_{-1}^2 b_0 + 76\gamma k a_0^3 b_{-1}^2 \alpha b_0 + 64\mu\gamma a_{-1}^3 b_1^2 b_{-1} \\
&- 64\mu\gamma a_1^2 b_{-1}^3 a_{-1} - 192\mu\gamma a_{-1} b_0 b_{-1}^2 a_1 a_0 + 192\mu\gamma a_{-1}^2 b_1 b_0 b_{-1} a_0 - 2432\gamma^2 k^2 a_{-1}^3 b_1^2 b_{-1} \\
&\left. + 1824\gamma^2 k^2 \mathrm{a}_1 b_{-1}^2 a_{-1} b_0 a_0 + 684\gamma k a_{-1} b_1 \alpha a_0^2 b_{-1}^2 - 8208\gamma^2 k^2 a_1 b_{-1} a_{-1}^2 b_0^2 \right) k,
\end{aligned}
$$

$$
\begin{aligned}
A_3 = -15\Big(&\alpha^2 a_0^3 b_{-1}^3 - \alpha^2 a_{-1}^3 b_0^3 - 456\gamma k a_1 b_{-1}^3 \alpha a_0 a_{-1} + 3\alpha^2 a_{-1}^2 b_0 b_{-1}^2 a_1 \\
&+ 48\mu\gamma a_{-1}^2 b_1 b_{-1}^2 a_0 - 96\mu\gamma a_1 b_{-1}^3 a_0 a_{-1} + 16\mu\gamma a_{-1}^3 b_0^3 - 48\mu\gamma a_{-1} b_0 b_{-1}^2 a_0^2 \\
&+ 96\mu\gamma a_{-1}^3 b_1 b_0 b_{-1} + 6\alpha^2 a_0 b_{-1}^3 a_1 a_{-1} + 1824\gamma^2 k^2 a_{-1} b_0 a_0^2 b_{-1}^2 \\
&- 1824\gamma^2 k^2 a_{-1}^2 b_0^2 a_0 b_{-1} - 48\mu\gamma a_{-1}^2 b_0 b_{-1}^2 a_1 - 3\alpha^2 a_{-1}^2 b_0^2 b_{-1} a_0 - 16\mu\gamma a_0^3 b_{-1}^3 \\
&- 684\gamma k a_1 b_{-1}^2 \alpha a_{-1}^2 b_0 + 684\gamma k a_{-1}^2 b_1 \alpha a_0 b_{-1}^2 + 48\mu\gamma a_{-1}^2 b_0^2 b_{-1} a_0 \\
&+ 6384\gamma^2 k^2 a_{-1}^2 b_1 a_0 \mathrm{b}_{-1}^2 - 6384\gamma^2 k^2 a_1 b_{-1}^2 a_{-1}^2 b_0 + 456\gamma k a_{-1}^3 b_1 \alpha b_0 b_{-1} \\
&+ 3\alpha^2 a_{-1} b_0 b_{-1}^2 a_0^2 - 6\alpha^2 a_{-1}^3 b_1 b_0 b_{-1} - 3\alpha^2 a_{-1}^2 b_1 b_{-1}^2 a_0 \Big) k,
\end{aligned}
$$

$$
\begin{aligned}
A_4 = -15\Big(&-2432\gamma^2 k^2 a_1 b_{-1}^3 a_{-1}^2 - 32\mu\gamma a_1 b_{-1}^3 a_{-1}^2 + 2432\gamma^2 k^2 a_{-1}^3 b_1 b_{-1}^2 \\
&+ 32\mu\gamma a_{-1}^3 b_1 b_{-1}^2 + 304\alpha\gamma k a_{-1}^3 b_1 b_{-1}^2 - 2\alpha^2 a_{-1}^3 b_0^2 b_{-1} + 2\alpha^2 a_0^2 b_{-1}^3 a_{-1} \\
&- 304\gamma^2 k^2 a_{-1}^3 b_0^2 b_{-1} + 76\gamma k a_{-1}^3 b_0^2 \alpha b_{-1} + 304\gamma^2 k^2 a_0^2 b_{-1}^3 a_{-1} - 76\gamma k a_0^2 b_{-1}^3 \alpha a_{-1} \\
&+ 2\alpha^2 a_1 b_{-1}^3 a_{-1}^2 - 32\mu\gamma a_0^2 b_{-1}^3 a_{-1} - 2\alpha^2 a_{-1}^3 b_1 b_{-1}^2 + 32\mu\gamma a_{-1}^3 b_0^2 b_{-1} \\
&- 304\alpha\gamma k a_1 b_{-1}^3 a_{-1}^2 \Big) k,
\end{aligned}
$$

$$
\begin{aligned}
A_5 = -15\Big(&76\alpha\gamma k a_{-1}^3 b_0 b_{-1}^2 - 76\alpha\gamma k a_0 b_{-1}^3 a_{-1}^2 + \alpha^2 a_0 b_{-1}^3 a_{-1}^2 + 16\mu\gamma a_{-1}^3 b_0 b_{-1}^2 \\
&+ 304\gamma^2 k^2 a_{-1}^3 b_0 b_{-1}^2 - 304\gamma^2 k^2 a_0 b_{-1}^3 a_{-1}^2 - 16\mu\gamma a_0 b_{-1}^3 a_{-1}^2 - \alpha^2 a_{-1}^3 b_0 b_{-1}^2 \Big) k,
\end{aligned}
$$

$$
\begin{aligned}
B_1 = -15\Big(&-96\mu\gamma a_1^2 b_1 b_0 b_{-1} a_{-1} - 16\mu\gamma a_1^2 b_0^3 a_{-1} + 304\gamma^2 k^2 a_0^3 b_1 b_0^2 + 9\alpha^2 a_1^2 b_1 b_{-1}^2 a_0 \\
&- 9\alpha^2 a_{-1} b_1^2 b_0 a_0^2 + 684\gamma k a_1^2 b_{-1}^2 \alpha b_1 a_0 + 6\alpha^2 a_1^2 b_1 b_0 b_{-1} a_{-1} - 9\alpha^2 a_{-1}^2 b_1^2 b_0 a_1 \\
&- \alpha^2 a_0^3 b_1 b_0^2 + 8208\gamma^2 k^2 a_1^2 b_{-1}^2 a_0 b_1 - 304\gamma^2 k^2 a_1 b_0^3 a_0^2 - 80\mu\gamma a_1^3 b_0 b_{-1}^2 \\
&+ \alpha^2 a_1^2 b_0^3 a_{-1} + \alpha^2 a_1 b_0^3 a_0^2 + 6\alpha^2 a_1 b_1 b_0 b_{-1} a_0^2 + 9\alpha^2 a_1^2 b_0^2 b_{-1} a_0 - 6\alpha^2 a_{-1} b_1 b_0^2 a_1 a_0 \\
&- 5\alpha^2 a_0 b_1^3 a_{-1}^2 - \alpha^2 a_0^3 b_1^2 b_{-1} - 6\alpha^2 a_{-1} b_1^2 b_{-1} a_1 a_0 + 5\alpha^2 a_1^3 b_0 b_{-1}^2 \\
&- 912\gamma k a_1 b_0^2 \alpha a_{-1} b_1 a_0 + 144\mu\gamma a_{-1}^2 b_1^2 b_0 a_1 + 304\gamma k a_0^3 b_{-1} \alpha b_1^2 - 144\mu\gamma a_1^2 b_1 b_{-1}^2 a_0
\end{aligned}
$$

$$-76\alpha\gamma k a_1 b_0^3 a_0^2 - 16\mu\gamma a_1 b_0^3 a_0^2 + 16\mu\gamma a_0^3 b_1 b_0^2 + 2432\gamma^2 k^2 a_1^2 b_0^3 a_{-1} + 76\alpha\gamma k a_0^3 b_1 b_0^2$$
$$-304\gamma^2 k^2 a_{-1}^2 b_1^3 a_0 + 96\mu\gamma a_{-1} b_1^2 b_{-1} a_1 a_0 - 96\mu\gamma a_1 b_1 b_0 b_{-1} a_0^2 + 80\mu\gamma a_{-1}^2 b_1^3 a_0$$
$$+144\mu\gamma a_{-1} b_1^2 b_0 a_0^2 + 912\gamma k a_1 b_{-1}\alpha a_0^2 b_1 b_0 - 76\gamma k a_{-1}^2 b_1^3 \alpha a_0 + 456\gamma k a_1 b_{-1}\alpha a_0 b_1^2 a_{-1}$$
$$+1824\gamma^2 k^2 a_1 b_0^2 a_{-1} a_0 b_1 - 1824\gamma^2 k^2 a_1 b_0 a_0^2 b_1 b_{-1} + 76\gamma k a_1^3 b_0 \alpha b_{-1}^2 - 304\gamma k a_1^2 b_0^3 \alpha a_{-1}$$
$$-456\gamma k a_1^2 b_0 \alpha b_1 b_{-1} a_{-1} - 684\gamma k a_1 b_0 \alpha a_{-1}^2 b_1^2 + 14592\gamma^2 k^2 a_1^2 b_0 b_{-1} a_{-1} b_1$$
$$-14592\gamma^2 k^2 a_1 b_{-1} a_{-1} b_1^2 a_0 - 144\mu\gamma a_1^2 b_0^2 b_{-1} a_0 + 96\mu\gamma a_{-1} b_1 b_0^2 a_1 a_0 + 16\mu\gamma a_0^3 b_1^2 b_{-1}$$
$$\left.-2432\gamma^2 k^2 a_0^3 b_1^2 b_{-1} + 304\gamma^2 k^2 a_1^3 b_0 b_{-1}^2 - 8208\gamma^2 k^2 a_1 b_0 a_{-1}^2 b_1^2\right) k,$$

$$B_2 = -15\left(-4\alpha^2 a_0^2 b_1^3 a_{-1} - 2\alpha^2 a_0^3 b_1^2 b_0 - 4\alpha^2 a_{-1}^2 b_1^3 a_1 + 4\alpha^2 a_1^3 b_1 b_{-1}^2 + 4\alpha^2 a_1^3 b_0^2 b_{-1}\right.$$
$$+1824\gamma^2 k^2 a_1^2 b_0 b_{-1} a_0 b_1 + 64\mu\gamma a_{-1} b_1^3 a_0^2 - 8208\gamma^2 k^2 a_1 b_{-1} a_0^2 b_1^2 + 12\alpha^2 a_1^2 b_1 b_0 b_{-1} a_0$$
$$+2\alpha^2 a_1^2 b_0^3 a_0 - 12\alpha^2 a_{-1} b_1^2 b_0 a_1 a_0 - 912\alpha\gamma k a_{-1} b_1^2 b_0 a_1 a_0 - 304\gamma^2 k^2 a_1^3 b_0^2 b_{-1}$$
$$-76\gamma k a_0^2 b_1^3 \alpha a_{-1} - 1824\gamma^2 k^2 a_1 b_0 a_{-1} b_1^2 a_0 + 304\gamma^2 k^2 a_1^2 b_0^3 a_0 + 8208\gamma^2 k^2 a_1^2 b_0^2 a_{-1} b_1$$
$$+304\gamma^2 k^2 a_{-1} b_1^3 a_0^2 + 304\gamma k a_1^3 b_{-1}^2 \alpha b_1 + 76\gamma k a_1^3 b_0^2 \alpha b_{-1} - 192\mu\gamma a_1^2 b_1 b_0 b_{-1} a_0$$
$$+912\alpha\gamma k a_1^2 b_1 b_0 b_{-1} a_0 - 76\gamma k a_1^2 b_0^3 \alpha a_0 + 192\mu\gamma a_{-1} b_1^2 b_0 a_1 a_0 - 304\gamma k a_{-1}^2 b_1^3 \alpha a_1$$
$$+64\mu\gamma a_{-1}^2 b_1^3 a_1 - 32\mu\gamma a_1^2 b_0^3 a_0 - 64\mu\gamma a_1^3 b_0^2 b_{-1} - 64\mu\gamma a_1^3 b_1 b_{-1}^2 - 304\gamma^2 k^2 a_0^3 b_1^2 b_0$$
$$+76\gamma k a_0^3 b_1^2 \alpha b_0 - 2432\gamma^2 k^2 a_{-1}^2 b_1^3 a_1 + 684\gamma k a_1 b_{-1} \alpha a_0^2 b_1^2 + 2432\gamma^2 k^2 a_1^3 b_{-1}^2 b_1$$
$$\left.-684\gamma k a_1^2 b_0^2 \alpha b_1 a_{-1} + 32\mu\gamma a_0^3 b_1^2 b_0\right) k,$$

$$B_3 = -15\left(\alpha^2 a_1^3 b_0^3 - \alpha^2 a_0^3 b_1^3 + 3\alpha^2 a_1^2 b_1 b_0^2 a_0 - 3\alpha^2 a_0^2 b_1^2 b_0 a_1 + 3\alpha^2 a_1^2 b_1^2 b_{-1} a_0 - 48\mu\gamma a_1^2 b_1 b_0^2 a_0\right.$$
$$+96\mu\gamma a_{-1} b_1^3 a_1 a_0 - 16\mu\gamma a_1^3 b_0^3 - 48\mu\gamma a_1^2 b_1^2 b_{-1} a_0 - 3\alpha^2 a_1^2 b_1^2 b_0 a_{-1} + 6\alpha^2 a_1^3 b_1 b_0 b_{-1}$$
$$-96\mu\gamma a_1^3 b_1 b_0 b_{-1} - 6384\gamma^2 k^2 a_1^2 b_{-1} a_0 b_1^2 - 456\gamma k a_{-1} b_1^3 \alpha a_0 a_1 - 6\alpha^2 a_0 b_1^3 a_1 a_{-1}$$
$$-1824\gamma^2 k^2 a_1 b_0 a_0^2 b_1^2 + 456\gamma k a_1^3 b_0 \alpha b_1 b_{-1} + 1824\gamma^2 k^2 a_1^2 b_0^2 a_0 b_1 - 684\gamma k a_1^2 b_0 \alpha a_{-1} b_1^2$$
$$+684\gamma k a_1^2 b_{-1} \alpha a_0 b_1^2 + 48\mu\gamma a_1^2 b_1^2 b_0 a_{-1} + 6384\gamma^2 k^2 a_1^2 b_0 a_{-1} b_1^2 + 48\mu\gamma a_1 b_1^2 b_0 a_0^2$$
$$\left.+16\mu\gamma a_0^3 b_1^3\right) k,$$

$$B_4 = -15\left(-304\alpha\gamma k a_{-1} b_1^3 a_1^2 + 304\alpha\gamma k a_1^3 b_1^2 b_{-1} - 32\mu\gamma a_1^3 b_1^2 b_{-1} + 32\mu\gamma a_0^2 b_1^3 a_1\right.$$
$$-304\gamma^2 k^2 a_0^2 b_1^3 a_1 + 2432\gamma^2 k^2 a_{-1} b_1^3 a_1^2 - 2432\gamma^2 k^2 a_1^3 b_1^2 b_{-1} - 32\mu\gamma a_1^3 b_1 b_0^2$$
$$+32\mu\gamma a_{-1} b_1^3 a_1^2 + 76\gamma k a_1^3 b_0^2 \alpha b_1 - 76\gamma k a_0^2 b_1^3 \alpha a_1 - 2\alpha^2 a_0^2 b_1^3 a_1 - 2\alpha^2 a_{-1} b_1^3 a_1^2$$
$$\left.+304\gamma^2 k^2 a_1^3 b_0^2 b_1 + 2\alpha^2 a_1^3 b_1^2 b_{-1} + 2\alpha^2 a_1^3 b_1 b_0^2\right) k,$$

$$B_5 = -15\Big(-16\mu\gamma a_1^3 b_1^2 b_0 - 304\gamma^2 k^2 a_1^3 b_1^2 b_0 + 304\gamma^2 k^2 a_0 b_1^3 a_1^2 - \alpha^2 a_0 b_1^3 a_1^2 + 76\alpha\gamma k a_1^3 b_1^2 b_0$$
$$-76\alpha\gamma k a_0 b_1^3 a_1^2 + \alpha^2 a_1^3 b_1^2 b_0 + 16\mu\gamma a_0 b_1^3 a_1^2\Big)k. \tag{3.6}$$

Substituting (3.4) into (1.1) and equating the coefficients of all powers of $e^{n\eta}$ to zero yields a system of algebraic equations for a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k and ω. Solving the system with the help of Maple, we obtain the following cases.

Case 1. Consider

$$\left\{a_1 = 0, a_0 = \frac{a_{-1}b_1}{b_0}, a_{-1} = a_{-1}, b_1 = b_1, b_0 = b_0, b_{-1} = 0, k = k, \omega = \omega\right\}. \tag{3.7}$$

Case 2. Consider

$$\left\{a_1 = \frac{a_0 b_0}{b_{-1}}, a_0 = a_0, a_{-1} = 0, b_1 = 0, b_0 = b_0,\right.$$
$$\left. b_{-1} = b_{-1}, k = k, \omega = \frac{k^2\left(15\alpha^2 - 76\alpha k\gamma - 76\gamma^2 k^2 - 316\mu\gamma\right)}{76\gamma}\right\}. \tag{3.8}$$

Case 3. Consider

$$\left\{a_1 = \frac{a_0 b_0}{b_{-1}}, a_0 = a_0, a_{-1} = 0, b_1 = 0, b_0 = b_0,\right.$$
$$\left. b_{-1} = b_{-1}, k = \pm\frac{\sqrt{-\mu\gamma}}{\gamma}, \omega = \frac{\mu\left(240\mu\gamma \pm 76\alpha\sqrt{-\mu\gamma} - 15\alpha^2\right)}{76\gamma^2}\right\}. \tag{3.9}$$

Case 4. Consider

$$\left\{a_1 = \frac{a_0 b_0}{b_{-1}}, a_0 = a_0, a_{-1} = 0, b_1 = 0, b_0 = b_0,\right.$$
$$b_{-1} = b_{-1}, k = \frac{1}{4}\,\frac{-19\alpha\gamma \pm \sqrt{437\alpha^2\gamma^2 - 1216\gamma^3\mu}}{38\gamma^2},$$
$$\omega = -\frac{20848\mu\alpha^2\gamma + 39936\mu^2\gamma^2 + 707\alpha^4}{184832\gamma^3}$$
$$\left. -\frac{1168\gamma\alpha\mu\sqrt{19}\sqrt{23\alpha^2 - 64\mu\gamma} - 41\alpha^3\sqrt{19}\sqrt{23\alpha^2 - 64\mu\gamma}}{184832\gamma^3}\right\}. \tag{3.10}$$

Substituting these cases into (3.4) we obtain the following new solitary wave solutions to (1.1):

$$u(x,t) = \frac{15k}{76\gamma\nu}\frac{c_3e^{6\eta}+c_2e^{5\eta}+c_1e^{4\eta}+c_0e^{3\eta}+c_{-1}e^{2\eta}+c_{-2}e^{\eta}+c_{-3}}{(b_0+b_1e^{\eta})^6},$$
$$c_{-3}=\left(\alpha^2-16\mu\gamma\right)b_1^6,\qquad c_{-2}=\left(6\alpha^2-96\mu\gamma\right)b_1^5b_0,\qquad c_{-1}=\left(15\alpha^2-240\mu\gamma\right)b_0^2b_1^4,$$
$$c_0=\left(20\alpha^2-320\mu\gamma\right)b_0^3b_1^3,\qquad c_1=\left(15\alpha^2-240\mu\gamma\right)b_1^2b_0^4,$$
$$c_2=\left(6\alpha^2-96\mu\gamma\right)b_0^5b_1,\qquad c_3=\left(\alpha^2-16\mu\gamma\right)b_0^6, \tag{3.11}$$

where $\eta = kx+\omega t$, k and ω are determined in Case 1, b_0 and b_1 are arbitrary constants. Consider

$$u(x,t) = -\frac{15k}{76\gamma\nu}\frac{c_3e^{6\eta}+c_2e^{5\eta}+c_1e^{4\eta}+c_0e^{3\eta}+c_{-1}e^{2\eta}+c_{-2}e^{\eta}+c_{-3}}{(b_0e^{\eta}+b_{-1})^6},$$
$$c_{-3}=\left(\alpha^2-16\mu\gamma\right)b_0^6,\qquad c_{-2}=\left(6\alpha^2-96\mu\gamma\right)b_0^5b_{-1},\qquad c_{-1}=\left(15\alpha^2-240\mu\gamma\right)b_{-1}^2b_0^4,$$
$$c_0=\left(20\alpha^2-320\mu\gamma\right)b_0^3b_{-1}^3,\qquad c_1=\left(15\alpha^2-240\mu\gamma\right)b_{-1}^4b_0^2,$$
$$c_2=\left(6\alpha^2-96\mu\gamma\right)b_0b_{-1}^5,\qquad c_3=\left(\alpha^2-16\mu\gamma\right)b_{-1}^6, \tag{3.12}$$

where $\eta = kx+\omega t$, k and ω are determined in Case 2, b_0 and b_{-1} are arbitrary constants. Consider

$$u(x,t) = -\frac{15k}{76\gamma\nu}\frac{c_3e^{6\eta}+c_2e^{5\eta}+c_1e^{4\eta}+c_0e^{3\eta}+c_{-1}e^{2\eta}+c_{-2}e^{\eta}+c_{-3}}{(b_0e^{\eta}+b_{-1})^6},$$
$$c_{-3}=\left(\alpha^2-16\gamma\mu\right)b_0^6,\qquad c_{-2}=\left(6\alpha^2-96\gamma\mu\right)b_0^5b_{-1},\qquad c_{-1}=\left(15\alpha^2-240\mu\gamma\right)b_0^4b_{-1}^2,$$
$$c_0=\left(20\alpha^2-320\mu\gamma\right)b_0^3b_{-1}^3,\qquad c_1=\left(15\alpha^2-240\mu\gamma\right)b_{-1}^4b_0^2,$$
$$c_2=\left(6\alpha^2-96\gamma\mu\right)b_{-1}^5b_0,\qquad c_3=\left(\alpha^2-16\gamma\mu\right)b_{-1}^6, \tag{3.13}$$

where $\eta = kx + \omega t$, k and ω are determined in Case 3, b_0 and b_{-1} are arbitrary constants. Consider

$$u(x,t) = -\frac{15k}{2888\gamma\nu}\frac{c_3e^{6\eta} + c_2e^{5\eta} + c_1e^{4\eta} + c_0e^{3\eta} + c_{-1}e^{2\eta} + c_{-2}e^{\eta} + c_{-3}}{(b_0e^{\eta} + b_{-1})^6},$$

$$c_{-3} = \left(\alpha^2 - 16\mu\gamma\right)b_0^6, \qquad c_{-2} = \left(6\alpha^2 - 96\mu\gamma\right)b_0^5b_{-1}, \qquad c_{-1} = \left(15\alpha^2 - 240\mu\gamma\right)b_0^4b_{-1}^2,$$

$$c_0 = \left(20\alpha^2 - 320\mu\gamma\right)b_0^3b_{-1}^3, \qquad c_1 = \left(15\alpha^2 - 240\mu\gamma\right)b_0^2b_{-1}^4,$$

$$c_2 = \left(6\alpha^2 - 96\mu\gamma\right)b_0b_{-1}^5, \qquad c_3 = \left(\alpha^2 - 16\mu\gamma\right)b_{-1}^6, \tag{3.14}$$

where $\eta = kx + \omega t$, k and ω are determined in Case 4, b_0 and b_{-1} are arbitrary constants.

The correctness of solutions (3.11)–(3.14) is ensured by testing them on computer with the aid of symbolic computation software *Maple*. It should be noted that these solutions have not been found in the literature. We also applied the Exp-function method to the KdV-Burgers-Kuramoto equation (1.1) to obtain its solitary wave solutions. As a result, we found only trivial solutions for the cases (i) $p = c = q = d = 1$, (ii) $p = c = 2, q = d = 1$, (iii) $p = c = 1$, $q = d = 2$, and one nontrivial solution for the case $p = c = 2$, $q = d = 2$, and we expect that more nontrivial solutions will be found for other cases. This indicates that the solutions obtained in this work cannot be directly constructed by the Exp-function method particularly in the case where $p = c = q = d = 1$.

4. Conclusion

In this paper, we have successfully implemented the Exp-function method based on the truncated Painlevé expansion method and obtained new generalized solitary wave solutions of the KdV-Burgers-Kuramoto equation. Our approach yields new travelling wave solutions with some free parameters. The approach considered in this paper and the Exp-function method complement each other in that our approach may provide nontrivial solutions in the cases that the Exp-function method may provide only trivial solutions. The result reveals that the Exp-function method is a promising tool because it can provide a variety of new soliton solutions with distinct physical structures.

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0025877).

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