

# Threshold Boolean Form for Joint Probabilistic Constraints with Random Technology Matrix

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## Abstract

We develop a new modeling and exact solution method for stochastic programming problems that include a joint probabilistic constraint in which the multirow random technology matrix is discretely distributed. We binarize the probability distribution of the random variables in such a way that we can extract a threshold partially defined Boolean function (pdBf) representing the probabilistic constraint. We then construct a tight threshold Boolean minorant for the pdBf. Any separating structure of the tight threshold Boolean minorant defines sufficient conditions for the satisfaction of the probabilistic constraint and takes the form of a system of linear constraints. We use the separating structure to derive three new deterministic formulations equivalent to the studied stochastic problem. We derive a set of strengthening valid inequalities for the reformulated problems. A crucial feature of the new integer formulations is that the number of integer variables does not depend on the number of scenarios used to represent uncertainty. The computational study, based on instances of the stochastic capital rationing problem, shows that the MIP reformulations are orders of magnitude faster to solve than the MINLP formulation. The method integrating the derived valid inequalities in a branch-and-bound algorithm has the best performance.

Keywords: Stochastic Programming, Boolean Function, Joint Probabilistic Constraint, Random Technology Matrix, Minorant, Threshold Function

## 1 Introduction

Consider the probabilistically constrained stochastic Problem with Multirow Random Technology Matrix (PMRTM)

$$\text{PMRTM :} \quad \max \quad q^T x \quad (1)$$

$$\text{subject to} \quad Ax \geq b \quad (2)$$

$$\mathbb{P}(Tx \leq d) \geq p \quad (3)$$

$$x \in \mathbb{R}_+^{|J|} \quad (4)$$

where  $T$  is an  $[r \times |J|]$ -matrix and its rows  $T_1^T, \dots, T_r^T$  are discretely distributed random vectors not necessarily independent. The notation  $|J|$  refers to the cardinality of the set  $J$ . Each component  $t_{ij}$  of  $T_i$  is given by  $t_{ij} = s_{ij} \xi_j$ , where  $s_{ij} \in \mathbb{R}_+^1$  is a fixed positive number and  $\xi$  is a  $|J|$ -dimensional random vector. We denote by  $x \in \mathbb{R}^{|J|}$  the vector of decision variables, by  $p$  a prescribed reliability level, by  $\mathbb{P}$  a probability measure, and by  $q^T x : \mathbb{R}^{|J|} \rightarrow \mathbb{R}$  the objective function. The system of inequalities (2), with  $A \in \mathbb{R}^{e \times |J|}$  and  $b \in \mathbb{R}^e$ , represents the set of deterministic constraints. The constraint (3) is a multirow

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probabilistic constraint with random technology matrix that ensures that the  $r$  inequalities  $T_i^\top x \leq d_i$  ( $i = 1, \dots, r$ ) hold jointly with a probability at least equal to  $p$ . Without loss of generality, we assume that the components of the  $r$ -dimensional vector  $d$  are constant. If  $d$  is random, we can simply introduce a new variable  $x_{|J|+1}$  set equal to 1 and look at  $\mathbb{P}(Tx - d x_{|J|+1} \leq 0)$ . For discretely distributed variables, the feasible set defined by **PMRTM** is non-convex even when all the decision variables are continuous [3].

**Remark 1** *The assumption that each coefficient  $s_{ij}$  is non-negative is non-restrictive and was made to ease the notations. Indeed, if one (or more) of the coefficients  $s_{ij}$  is negative, one can rewrite  $s_{ij} = s_{ij}^+ - s_{ij}^-$  with  $s_{ij}^+, s_{ij}^- \geq 0$  (and at least one of them equal to zero), introduce a random vector  $\tilde{\xi} = -\xi$ , and then rewrite the entries  $t_{ij}x_j = s_{ij}^+x_j\xi_j + s_{ij}^-x_j\tilde{\xi}_j$  in  $Tx$  in (3).*

Probabilistically constrained problems with random technology matrix have first been studied by Kataoka [15] and van de Panne and Popp [38]. In their pioneering work, these authors consider a one-row probabilistic constraint with random technology matrix

$$\mathbb{P}(\xi^\top x \leq d) \geq p, \quad (5)$$

where  $\xi$  has normal distribution with  $E[\xi] = \mu$  and variance-covariance matrix  $\Sigma$ . They show that (5) is equivalent to the constraint

$$\mu^\top x + \Phi^{-1}(p) \sqrt{x^\top \Sigma x} \leq d, \quad (6)$$

where  $\Phi^{-1}(p)$  is the  $p$ -quantile of the standard normal distribution. If  $p \geq 0.5$ , as it is usual in practice, then  $\Phi^{-1}(p) \geq 0$ , and, since  $\sqrt{x^\top \Sigma x}$  is a convex function, the feasible set determined by (6) is a second-order cone constraint defining a convex feasible region. In a more general setting, the probability distribution of  $\xi^\top x$  is often mixed or partially known and the value of the  $p$ -quantile of the distribution is unknown. The value of this latter can be approximated by using the well-known Cantelli, Chebychev [5] and Camp-Meidell [17] probability inequalities. Henrion [12] showed that the feasible set defined by the individual (i.e., one row) chance constraint  $\mathbb{P}(\xi^\top h(x) \leq d) \geq p$  is convex if all the components of  $h(x)$  are nonnegative and convex,  $p \geq 0.5$ ,  $\xi$  has an elliptical symmetric distribution, and the parameters of the distribution are nonnegative.

The complexity of stochastic problems with random technology matrix is further compounded as the number  $r$  of inequalities that must hold jointly with a large probability level increases. Very few results are known for constraints with multirow random technology matrix. Prékopa [30], Henrion and Strugarek [13], and Prékopa et al. [33] have studied the convexity properties of the feasible set  $G = \{x : \mathbb{P}(Tx \leq d) \geq p\}$  when each row  $T_i$  of  $T$  follows a continuous distribution. For instance, Prékopa [30] showed that  $G$  is a convex set if all covariance and cross-covariance matrices of the columns and rows of  $T$  are proportional to each other. Even if the rows  $T_i$  are independent, the convexity of the feasible set defined by (3) can only be guaranteed under restrictive assumptions and is due to the fact that the product of quasi-concave functions is not necessarily quasi-concave [33]. Henrion and Strugarek [13] showed

that the joint probabilistic constraint  $\mathbb{P}(h_i(x) \geq \zeta_i, i = 1, \dots, r) \geq p$  defines a convex feasible area if  $h_i(x)$  is  $(-e_i)$ -concave and  $\zeta_i, i = 1, \dots, r$  are independent random variables with  $(e_i + 1)$ -decreasing densities for some  $e_i > 0$  for large values of  $p$ . Considering a Gaussian technology matrix in (3), Van Ackooij et al. [37] designed an efficient method to compute the gradients and value of the multivariate Gaussian distribution functions with the code developed by Genz [9]. A possible approach is to approximate a multirow chance constraint with random technology matrix (3) with individual chance constraints (7). This can be done by using, for example, Boole's inequality and requiring that:

$$\mathbb{P}(T_i x \leq d_i) \geq p_i, \quad i = 1, \dots, r \quad (7)$$

$$\sum_{i=1}^r p_i - (r - 1) \geq p \quad (8)$$

$$0 \leq p_i \leq p. \quad (9)$$

However, this approach provides a very conservative solution.

In this paper, we consider the case in which the elements  $t_{ij}$  of  $T$  are discretely distributed. Discrete distributions are employed in multiple applications and are often created through sampling as approximations of the underlying distribution. Tanner and Ntairo [36] propose a large-scale mixed integer programming reformulation for **PMRTM**, derive the so-called irreducibly infeasible subsystem optimality cuts and solve the strengthened problem with a specialized branch-and-cut algorithm. Computational results are reported for instances of the optimal vaccine allocation problem in which the random technology matrix has one row. Ruszczyński [34] derives cutting planes based on precedence knapsack constraints which he uses within a branch-and-cut solution method. Beraldi and Bruni [3] consider the probabilistic set covering problem, in which random variables are present in the technology matrix and uncertainty is represented by a finite set of scenarios. The reformulated large-scale integer problem including knapsack constraints is solved with a specialized branch-and-bound algorithm. Using a Monte Carlo sample of the distribution of the random variables, sample approximation methods have been shown [23] to find good solutions for the original problem and provide statistical bounds on the solution quality.

The main objective of this paper is to develop a new modeling and exact solution method for a class of particularly complex stochastic programming problems. More precisely, we study stochastic problems that include a joint probabilistic constraint in which the elements of the multirow random technology matrix are discretely distributed. Our modeling approach involves the binarization of the probability distribution and is carried out by using a set of threshold values, called cut points. The binarization process generates binary images for the realizations of the random variables and the set of recombinations, each of which is a vector defining possibly sufficient conditions for the probabilistic constraint to hold. The binarization is done in such a way that we can derive a threshold partially defined Boolean function (pdBf) representing the probabilistic constraint (3). The pdBf is defined with respect to the binary projections of the recombinations and consistently separates the  $p$ -sufficient recombinations from the  $p$ -insufficient ones. A key and novel contribution of this approach is to derive a tight threshold Boolean minorant for the pdBf representing the probabilistic constraint. Any separating structure of the

tight threshold Boolean minorant defines sufficient conditions for the probability of  $Tx \leq d$  to be at least equal to  $p$  and takes the form of a system of linear constraints.

The separating structure of the tight threshold Boolean minorant is a compact and computationally effective way to represent the pdBf. We use it to derive three new deterministic formulations equivalent to the probabilistically constrained problem **PMRTM**. A crucial feature of these three new integer formulations is that the number of integer variables is equal to the number of cut points used in the binarization process and does not depend on the number of scenarios used to represent uncertainty. We also derive a set of strengthening valid inequalities for the reformulated problems. The computational experiments are based on multiple instances of the stochastic capital rationing problem and are used to analyze the computational efficiency of several algorithmic methods. The tests reveal that our mixed-integer reformulations make it possible to solve very fast and to optimality instances of the stochastic capital rationing problem in which the uncertainty is present in the rows of the technology matrix and is represented by a very large number of scenarios. The introduction of the proposed valid inequalities further accelerates the solution process.

In Section 2, we describe the binarization process and the derivation of the pdBf representing (3). The method extends the Boolean framework proposed in [18, 19] to handle probabilistic constraints with random technology matrix. Section 3 describes the properties of the pdBf and introduces the concepts of extension and minorant of a Boolean function. It also characterizes the separating structure of the tight threshold minorant for the pdBf representing the probabilistic constraint (3). In Section 4, we derive three new reformulations equivalent to the stochastic programming problem **PMRTM**. Section 5 describes the computational results, while Section 6 provides concluding remarks.

## 2 Combinatorial Modeling Framework

In this section, we present the binarization process / Boolean modeling framework used to reformulate probabilistic programming problems with multirow and random technology matrix. The modeling framework involves the following steps: (i) the construction of the set of recombinations, and (ii) the binarization of the probability distribution. A variant of the proposed binarization process is used in [18, 19] for joint probabilistic constraints with random right-hand sides and deterministic random technology matrix. In [18], a  $p$ -efficient point [31] is represented as a conjunction of literals, called  $e^p$ -pattern, and mathematical programming problems for the generation of such patterns are proposed. In [19], Lejeune introduces the concept of a  $p$ -pattern that defines sufficient conditions for a chance constraint with random right-hand side to hold. An MIP reformulation equivalent to the associated stochastic problem is proposed and several heuristics providing high-quality  $p$ -patterns are evaluated.

### 2.1 Construction of Set of Recombinations

We denote by  $\Omega \in \mathbb{R}_+^{|J|}$  the support set of  $\xi$ :  $\Omega$  contains the possible realizations of the  $|J|$ -random vector  $\xi = [\xi_i, \dots, \xi_{|J|}]$  with distribution function  $F$ . We refer to  $\xi_j$  as the  $j^{\text{th}}$  component of the vector  $\xi$ . A

realization  $k$  is represented by the  $|J|$ -deterministic numerical vector  $\omega^k$ . We distinguish  $p$ -sufficient and  $p$ -insufficient realizations.

**Definition 1** [18] *A realization  $k$  is called  $p$ -sufficient if and only if  $\mathbb{P}(\xi \leq \omega^k) = F(\omega^k) \geq p$  and is  $p$ -insufficient if  $F(\omega^k) < p$ .*

Let  $\text{diag}(\omega^k)$  be the  $[|J| \times |J|]$ -dimensional diagonal matrix whose entries are the  $|J|$  elements of the vector  $\omega^k$  and  $S \in \mathbb{R}_+^{r \times |J|}$  be the matrix containing the coefficients  $s_{ij}$  ( $t_{ij} = s_{ij} \xi_j$ ) in the random technology matrix  $T$ . As indicated by its name, a  $p$ -sufficient realization defines sufficient conditions for the probabilistic constraint (3) to hold. Indeed, if  $k$  is  $p$ -sufficient, we have  $F(\omega^k) = \mathbb{P}(Tx \leq S \text{diag}(\omega^k) x) \geq p$ , and, thus,

$$S \text{diag}(\omega^k) x \leq d \quad \Rightarrow \quad \mathbb{P}(Tx \leq d) \geq p. \quad (10)$$

We obtain a partition of the set  $\Omega$  of realizations into two disjoint sets of  $p$ -sufficient  $\Omega^+ = \{k \in \Omega : F(\omega^k) \geq p\}$  and  $p$ -insufficient  $\Omega^- = \Omega \setminus \Omega^+$  realizations. To obtain the optimal solution for problem **PMRTM**, we must not only take into account the realizations  $k \in \Omega$  of the random vector, but we should also consider all points that can be  $p$ -sufficient and that we shall call **recombinations**. We denote by  $F_j$  the marginal probability distribution of  $\xi_j$  and by  $F_j^{-1}(p)$  its  $p$ -quantile. The inequalities

$$\omega_j^k \geq F_j^{-1}(p), j = 1, \dots, |J| \quad (11)$$

are based on the univariate quantile rule and represent necessary, although not necessarily sufficient, conditions for the probabilistic constraint (3) to hold:  $\mathbb{P}(Tx \leq d) \geq p \Rightarrow \omega_j^k \geq F_j^{-1}(p), j = 1, \dots, |J|$ . The sets  $C_j$  are given by

$$C_j = \{\omega_j^k : F_j(\omega_j^k) \geq p, k \in \Omega\}, j \in J \quad (12)$$

and their direct product [18, 32]

$$\bar{\Omega} = C_1 \times \dots \times C_j \times \dots \times C_{|J|}, \quad (13)$$

provides the set  $\bar{\Omega}$  of recombinations that provides the exhaustive list of the points that can possibly be  $p$ -sufficient. Each  $k \in \bar{\Omega}$  is called a recombination and is represented by a numerical vector  $\omega^k$  satisfying the  $|J|$  conditions defined by (11). A recombination  $k \in \bar{\Omega} = \bar{\Omega}^+ \cup \bar{\Omega}^-$  can be  $p$ -sufficient or  $p$ -insufficient, and the disjoint sets  $\bar{\Omega}^+ = \{k \in \bar{\Omega}^+ : F(\omega^k) \geq p\}$  and  $\bar{\Omega}^- = \{k \in \bar{\Omega}^- : F(\omega^k) < p\}$  are respectively called sets of  $p$ -sufficient and  $p$ -insufficient recombinations. A recombination can be a realization of the random vector, but it is not necessarily the case. All  $p$ -sufficient realizations are included in the set of  $p$ -sufficient recombinations:  $\Omega^+ \subseteq \bar{\Omega}^+$ .

## 2.2 Binarization Process

Using **cut points** [6], we shall now proceed to the **binarization** of the probability distribution and the set  $\bar{\Omega}$ . We denote by  $n = \sum_{j \in J} n_j$  the sum of the number  $n_j$  of cut points for each component  $\xi_j$ . The notation  $\{0, 1\}^n$  refers to the  $n$ -dimensional binary vectors.

**Definition 2** [18] *The binarization process is the mapping  $\mathbb{R}^{|J|} \rightarrow \{0, 1\}^n$  of a numerical vector  $\omega^k$  into a binary one  $\beta^k = [\beta_{11}^k, \dots, \beta_{1n_1}^k, \dots, \beta_{jl}^k, \dots, \beta_{jn_j}^k, \dots]$ , such that the value of each Boolean component  $\beta_{jl}^k$  is defined with respect to a cut point  $c_{jl}$  as follows:*

$$\beta_{jl}^k = \begin{cases} 1 & \text{if } \omega_j^k \geq c_{jl} \\ 0 & \text{otherwise} \end{cases}, \quad (14)$$

where  $c_{jl}$  denotes the  $l^{\text{th}}$  cut point associated with component  $\xi_j$ ,

$$l' < l \Rightarrow c_{jl'} < c_{jl}, \quad j \in J, \quad l = 1, \dots, n_j. \quad (15)$$

**Definition 3** *A Boolean vector is called relevant if it is the binary mapping of a recombination.*

The binarization of  $\bar{\Omega}$  provides the set  $\bar{\Omega}_B \subseteq \{0, 1\}^n$  called set of relevant Boolean vectors. Note that the binarization process arranges the cut points in ascending order (15).

**Definition 4** *A set of Boolean vectors is called regularized with respect to a group (i.e., ordered subset)  $G_j = (\beta_{j1}, \dots, \beta_{jn_j})$  of Boolean variables if every vector  $\beta^k$  satisfies the following conditions:*

$$\beta_{jl}^k = 1 \Rightarrow \beta_{jl'}^k = 1, \quad l' < l \quad (16)$$

$$\beta_{jl}^k = 0 \Rightarrow \beta_{jl'}^k = 0, \quad l' > l. \quad (17)$$

Lemma 5 is a direct consequence of Definitions 2 and 4.

**Lemma 5** *The binarization process described in Definition 2 generates a regularized set of Boolean vectors, i.e., for every component  $\xi_j$ ,  $j \in J$ , if  $c_{jl'} < c_{jl}$ , then*

$$\beta_{jl}^k \leq \beta_{jl'}^k, \quad j \in J, \quad k \in \Omega. \quad (18)$$

**Definition 6** *Any vector  $\beta^k \in \{0, 1\}^n$  such that*

$$\beta_{jl}^k > \beta_{jl'}^k \quad \text{for any } l > l', \quad j \in J$$

*does not correspond to any realization in the original numerical space and is called not possible.*

We shall define a set of cut points, such that the binarization process (15) preserves the disjointedness between the binary projections  $\bar{\Omega}_B^+$  and  $\bar{\Omega}_B^-$  of  $\bar{\Omega}^+$  and  $\bar{\Omega}^-$  ( $\bar{\Omega}_B = \bar{\Omega}_B^+ \cup \bar{\Omega}_B^-$ ). We shall accomplish this by using the sufficient-equivalent [18] consistent set of cut points  $C^e$ .

**Definition 7** *The sufficient-equivalent consistent set of cut points is given by:*

$$C^e = \bigcup_{j=1}^{|J|} C_j, \quad (19)$$

where the sets  $C_j$  are defined by (12).

This ensures that no pair of  $p$ -sufficient and  $p$ -insufficient recombinations can have the same binary image. The set of relevant Boolean vectors is partitioned into the disjoint sets of  $p$ -sufficient  $\bar{\Omega}_B^+$  and  $p$ -insufficient  $\bar{\Omega}_B^-$  relevant Boolean vectors. Note that there is a one-to-one mapping between the numerical  $\omega^k \in \bar{\Omega}$  and the binary  $\beta^k$  vectors associated with a recombination  $k$  if the binarization process is carried out with the sufficient-equivalent set of cut points (19).

**Definition 8** [8] *A Boolean function  $f$  of  $n$  variables is a mapping  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ . A point  $X = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$  is a true point (resp., false point) of  $f$  if  $f(X) = 1$  (resp.,  $f(X) = 0$ ). We denote by  $\mathcal{T}(f)$  and  $\mathcal{F}(f)$  the disjoint sets of respectively true and false points of  $f$ .*

**Definition 9** *A partially defined Boolean function  $g$  defined by the pair of disjoint sets  $(\mathcal{T}, \mathcal{F}) \subseteq \{0, 1\}^n$  is a mapping  $g : (\mathcal{T} \cup \mathcal{F}) \rightarrow \{0, 1\}$  such that  $g(X) = 1$  (resp.,  $g(X) = 0$ ) if  $X \in \mathcal{T}$  (resp.,  $\mathcal{F}$ ).*

Note that a Boolean function is a special case of a partially defined Boolean function (pdBf): any pdBf  $g(\mathcal{T}, \mathcal{F})$  such that  $\mathcal{F} \cup \mathcal{T} = \{0, 1\}^n$  is a Boolean function. The binarization permits the derivation of a partially defined Boolean function representing the combination of the binary images  $\beta^k$  of the recombinations  $k$  with the prescribed probability level  $p$ .

**Definition 10** *Given two disjoint subsets  $\bar{\Omega}_B^+, \bar{\Omega}_B^-: \bar{\Omega}_B^+ \cup \bar{\Omega}_B^- = \bar{\Omega}_B \subseteq \{0, 1\}^n$ ,  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  is a partially defined Boolean function (pdBf) with the sets of true points  $\bar{\Omega}_B^+$  and false points  $\bar{\Omega}_B^-$ .*

In this pdBf, the set of  $p$ -sufficient recombinations is the set of true points and the set of  $p$ -insufficient recombinations is the set of false points. Observe that the binarization process (Definition 2) and the intra-group regularity of the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  (Definition 4) imply that some vectors of the  $n$ -dimensional cube  $\{0, 1\}^n$  do not correspond to any point in the original numerical space and are not possible (see Definition 6). For example, the binary vector  $(\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}) = (0, 1, 1, 0)$  is not possible, as (14)-(15) does not allow for  $\beta_{11}$  to be strictly smaller than  $\beta_{12}$ .

**Theorem 11** *Consider the binarization process obtained with  $C^e$ . The set  $\bar{\Omega}_B$  of relevant Boolean vectors is the binary projection of  $\bar{\Omega}$  and is given by*

$$\bar{\Omega}_B = \left\{ k : \beta^k \in \{0, 1\}^n, \beta_{j1}^k = 1, \beta_{jl}^k \leq \beta_{j,l-1}^k, j \in J, l = 2, \dots, n_j \right\}. \quad (20)$$

**Proof.** Consider an arbitrary point  $k$  and the associated numerical  $\omega^k$  and binary  $\beta^k$  vectors. It follows from the univariate quantile rule (11) that  $k$  is a recombination if and only if:

$$\omega_j^k \geq \min \{ \omega_j^k : k \in \Omega^+ \} = \min \{ \omega_j^k : \omega_j^k \in C_j \} = c_{j1} \text{ for any } j \in J \Leftrightarrow \beta_{j1}^k = 1 \text{ for any } j \in J. \quad (21)$$

The first equality follows from (19). The equivalence relationship is a consequence of the binarization process (14). The constraints  $\beta_{jl}^k \leq \beta_{j,l-1}^k$  in (20) follow from the regularization property (18).  $\square$

We shall use the stochastic programming problem (22) with multirow random technology matrix to illustrate our method:

$$\begin{aligned} & \max 3x_1 + x_2 + x_3 \\ & \text{subject to } \mathbb{P} \left\{ \begin{array}{l} 2\xi_1 x_1 + 3\xi_2 x_2 - 25 \leq 0 \\ 3\xi_1 x_1 + 6\xi_3 x_3 - 32 \leq 0 \end{array} \right\} \geq 0.7, \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \quad (22)$$

where  $s_{11} = 2$ ,  $s_{12} = s_{21} = 3$ ,  $s_{13} = s_{22} = 0$ ,  $s_{23} = 6$ , the random vectors are  $T_1 = [2\xi_1, 3\xi_2, 0]$  and  $T_2 = [3\xi_1, 0, 6\xi_3]$ . The cumulative probability distribution  $F$  of the random variable  $\xi = [\xi_1, \xi_2, \xi_3]$  and the marginal cumulative distributions  $F_j$ ,  $j = 1, 2, 3$  are given in Table 1. The realizations  $\omega^k = [\omega_1^k, \omega_2^k, \omega_3^k]$  of  $\xi$  are equally likely.

Table 1: Probability Distribution

$k$	1	2	3	4	5	6	7	8	9	10
$\omega_1^k$	7	1	1	3	2	3	7	7	7	7
$F_1(\omega_1^k)$	1	0.2	0.2	0.5	0.3	0.5	1	1	1	1
$\omega_2^k$	10	2	3	5	4	5	6	2	6	5
$F_2(\omega_2^k)$	1	0.2	0.3	0.7	0.4	0.7	0.9	0.2	0.9	0.7
$\omega_3^k$	2	1	3	2	1	1	1	4	3	5
$F_3(\omega_3^k)$	0.6	0.4	0.8	0.6	0.4	0.4	0.4	0.9	0.8	1
$F(\omega^k)$	0.6	0.1	0.2	0.4	0.2	0.3	0.4	0.2	0.7	0.7

The sufficient-equivalent set of cut points is:

$$C^e = \{c_{11} = 7; c_{21} = 5; c_{22} = 6; c_{23} = 10; c_{31} = 3; c_{32} = 4; c_{33} = 5\}.$$

To illustrate how the above cut points are selected, consider the second component  $\xi_2$  of the random vector  $\xi$ :  $\xi_2$  can take values 2, 3, 4, 5, 6, and 10 and the marginal cumulative probabilities  $F_2(2)$ ,  $F_2(3)$ ,  $F_2(4)$ ,  $F_2(5)$ ,  $F_2(6)$ , and  $F_2(10)$  are respectively equal to 0.2, 0.3, 0.4, 0.7, 0.9, and 1. The points included in the sufficient-equivalent set  $C^e$  are 5, 6, and 10 that are the only points with a cumulative marginal probability at least equal to the prescribed threshold  $p=0.7$ .

Table 2 displays the recombinations and their binary images (i.e., relevant Boolean vectors) obtained with  $C^e$ . We use the recombination  $k=11$  to illustrate how the values reported in Table 2 are obtained. Since  $\omega_1^{11} = 7$  is equal to  $c_{11} = 7$ , then the binarization process (14) implies that  $\beta_{11}^{11} = 1$ . As  $\omega_2^{11} = 5 = c_{21}$ ,  $\omega_2^{11} < c_{22} = 6$ , and  $\omega_2^{11} < c_{23} = 10$ , it follows that  $\beta_{21}^{11} = 1$  and  $\beta_{22}^{11} = \beta_{23}^{11} = 0$ . Similarly, as  $\omega_3^{11} = 3 = c_{31}$ ,  $\omega_3^{11} < c_{32} = 4$ , and  $\omega_3^{11} < c_{33} = 5$ , it follows that  $\beta_{31}^{11} = 1$  and  $\beta_{32}^{11} = \beta_{33}^{11} = 0$ . Since  $\beta^{11}$  is larger than or equal to the binary images  $\beta^2, \beta^3, \beta^4, \beta^5$ , and  $\beta^6$  of the realizations  $k = 2, 3, 4, 5, 6$ , and  $\beta^{11} \not\leq \beta^k, k = 1, 7, 8, 9, 10$ , the cumulative probability of  $k = 11$  is equal to  $0.5 < 0.7$ , which implies that  $k = 11 \in \bar{\Omega}_B^-$ .

It is easy to see that the numerical representation  $\Omega^+$  and  $\Omega^-$  of the sets of sufficient and insufficient numerical realizations are not linearly separable. Indeed, consider the two numerical insufficient realizations  $\omega^1, \omega^8 \in \Omega^-$ . The convex combination  $0.5\omega^1 + 0.5\omega^8$  is  $\omega^9 \in \Omega^+$ , which shows that the convex hulls of  $\Omega^+$  and  $\Omega^-$  are not disjoint and indicates the non-convexity of problem (22).



Table 2: Reformulations and Relevant Boolean Vectors

$k$	Numerical Vectors			Boolean Vectors							Relevant Boolean Sets
	$\omega_1^k$	$\omega_2^k$	$\omega_3^k$	$\beta_{11}^k$	$\beta_{21}^k$	$\beta_{22}^k$	$\beta_{23}^k$	$\beta_{31}^k$	$\beta_{32}^k$	$\beta_{33}^k$	
11	7	5	3	1	1	0	0	1	0	0	$\bar{\Omega}_B^-$
12	7	5	4	1	1	0	0	1	1	0	
9	7	6	3	1	1	1	0	1	0	0	$\bar{\Omega}_B^+$
10	7	5	5	1	1	0	0	1	1	1	
13	7	6	4	1	1	1	0	1	1	0	
14	7	6	5	1	1	1	0	1	1	1	
15	7	10	3	1	1	1	1	1	0	0	
16	7	10	4	1	1	1	1	1	1	0	
17	7	10	5	1	1	1	1	1	1	1	
				Cut Points							
				7	5	6	10	3	4	5	

### 3 Extension and Minorant of pdBf: Functional Forms and Properties

We shall now investigate how the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  representing  $(F, p)$  can be conveniently and efficiently used to solve problem **PMRTM**. With that objective in mind, we introduce the concepts of an extension and minorant of a pdBf and we continue with the characterization of the Boolean functional form under which  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  can be represented. This is instrumental for the derivation of a deterministic reformulation of problem **PMRTM**.

#### 3.1 Boolean Functional Form and Extension

**Definition 12** A Boolean function  $f$  is called an extension of a pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  if  $\bar{\Omega}_B^+ \subseteq \mathcal{T}(f)$  and  $\bar{\Omega}_B^- \subseteq \mathcal{F}(f)$ .

We shall now consider several Boolean functional forms that extensions of pdBf can take.

**Definition 13** [8] A Boolean function  $f$  is positive (resp., negative) monotone in the binary variable  $x_l, l \in \{1, \dots, n\}$  if  $f_{|x_l=0} \leq f_{|x_l=1}$  (resp.,  $f_{|x_l=1} \leq f_{|x_l=0}$ ) regardless of the values taken by the  $(n-1)$  other  $x_{l'} \in \{0, 1\}, l' \in \{1, \dots, n\} \setminus \{l\}$ . A Boolean function  $f$  is positive monotone (also called, increasing monotone or isotone) if it is positive in each of its variables.

The following result was derived in [18]:

**Theorem 14** Any consistent pdBf representing  $(F, p)$  can be extended as an isotone Boolean function.

**Definition 15** [8] A function  $f: \{0, 1\}^n \rightarrow \{0, 1\}$  is a threshold Boolean function if, for all  $(a_1, \dots, a_n) \in \{0, 1\}^n$ , there exists a vector of weights  $\lambda \in \mathbb{R}^n$  and a threshold  $\theta \in \mathbb{R}$ , such that

$$f(a_1, \dots, a_n) = 1 \quad \text{if and only if} \quad \sum_{l=1}^n \lambda_l a_l \geq \theta. \quad (23)$$

The hyperplane  $\left\{t \in \{0, 1\}^n : \sum_{l=1}^n \lambda_l t_l = \theta\right\}$  is a separator for  $f$  and the  $(n + 1)$ -tuple  $(\lambda, \theta)$  is the **separating structure** for the **threshold Boolean function**  $f$ .

The problem of proving that a Boolean function is a threshold one is known as the threshold recognition or synthesis problem [26]. The threshold nature of a Boolean function can be identified numerically, in polynomial time, by solving a linear programming problem [8].

**Theorem 16** *The pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  admits a threshold Boolean extension if and only if the system of inequalities*

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \geq \theta, \quad k \in \bar{\Omega}_B^+ \quad (24)$$

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq \theta - 1, \quad k \in \bar{\Omega}_B^- \quad (25)$$

is feasible. If this is the case, the solution  $(\lambda_{11}, \dots, \lambda_{|J|n_{|J|}}, \theta)$  defines the separating structure of  $g$ .

**Proof.** The **if part** directly follows from Definition 15 since the threshold Boolean function defined by the inequality in (24) is obviously an extension of the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . To prove the **only if part**, let us assume that the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  has a threshold extension defined by the separating structure  $(\lambda_{11}, \dots, \lambda_{|J|n_{|J|}}, \theta)$ . Then it follows from Definition 15 that all constraints (24) are satisfied. It also follows that all the following inequalities are satisfied:

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k < \theta, \quad k \in \bar{\Omega}_B^-. \quad (26)$$

Let us denote by  $\theta^* = \max\{\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \mid k \in \bar{\Omega}_B^-\}$ . Obviously,  $\theta^* < \theta$ . Let us define  $\theta' = \theta / (\theta - \theta^*)$ , and  $\lambda'_{jl} = \lambda_{jl} / (\theta - \theta^*)$ ,  $j \in J$ ,  $l = \{1, \dots, n\}$ . Then the so defined  $\theta'$  and  $\lambda'$  provide a feasible solution to the system of constraints (24) and (25).  $\square$

Theorem 16 illustrates the simplicity and convenience of the representation of a threshold function which is fully characterized by the  $(n + 1)$ -tuple  $(\lambda, \theta)$  defining its separating structure. This means that each  $p$ -sufficient (resp.,  $p$ -insufficient) realization is also labeled as  $p$ -sufficient (resp.,  $p$ -insufficient) by the linear rule defined by the separating structure  $(\lambda, \theta)$ . Sufficient conditions for satisfying the constraint (3) can thus be derived from the knowledge of the separating structure.

Consider problem (22) and the set  $\bar{\Omega}_B$  of relevant Boolean vectors displayed in Table 2. Note that, using (21), we dropped each realization  $k$  with Boolean vector  $\beta^k$  such that  $\beta_{j_1}^k = 0$  for at least one  $j$ , since (21) implies such a realization  $k$  cannot be  $p$ -sufficient. Minimizing the value of the threshold  $\theta$ ,

the threshold recognition problem for our example reads:

$$\begin{aligned}
& \min_{\lambda, \theta} \theta \\
\text{subject to} \quad & \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{31} \geq \theta & (k = 9) \\
& \lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{32} + \lambda_{33} \geq \theta & (k = 10) \\
& \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{31} + \lambda_{32} \geq \theta & (k = 13) \\
& \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{31} + \lambda_{32} + \lambda_{33} \geq \theta & (k = 14) \\
& \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{31} \geq \theta & (k = 15) \\
& \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{31} + \lambda_{32} \geq \theta & (k = 16) \\
& \lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{31} + \lambda_{32} + \lambda_{33} \geq \theta & (k = 17) \\
& \lambda_{11} + \lambda_{21} + \lambda_{31} \leq \theta - 1 & (k = 11) \\
& \lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{32} \leq \theta - 1 & (k = 12)
\end{aligned}$$

The optimal solution is  $(\lambda^*, \theta^*) = (0, 0, 1, 0, 0, 0, 1, 1)$ . Note that  $\beta_{j1}^k = 1, k \in \bar{\Omega}_B$ . Hence, the Boolean variables  $\beta_{j1}, j = 1, 2, 3$  do not play any role in the derivation of the separating structure and we do not need to associate a weight to them. Observe that the system of linear inequalities (24)-(25) is not always feasible.

We note that the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  may not be extendable as a threshold Boolean function. Let  $\xi$  be a four-dimensional vector with two cut points associated with each component. Consider

1. the sufficient Boolean vectors  $\beta^1 = (1, 1, 1, 1, 0, 1, 0)$  and  $\beta^2 = (1, 0, 1, 0, 1, 1, 1), k = 1, 2 \in \Omega^+$ , and
2. the insufficient Boolean vectors  $\beta^3 = (1, 1, 1, 0, 1, 0, 1)$  and  $\beta^4 = (1, 0, 1, 1, 1, 1, 0), k = 3, 4 \in \Omega^-$ .

Observe that the above set of Boolean vectors is valid for the problem at hand, since  $\beta_{j1}^k = 1, j = 1, \dots, 4$  and that there is no  $k \in \Omega^-$  such that  $\beta^k \geq \beta^{k'}$  for any  $k' \in \Omega^+$ .

Assume that  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  can be extended as a threshold Boolean function. In that case, there exists a tuple  $(\lambda, \theta)$  satisfying the system of linear inequalities:

$$\begin{cases}
\lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} + \lambda_{31} + \lambda_{41} \geq \theta & (k = 1) \\
\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{32} + \lambda_{41} + \lambda_{42} \geq \theta & (k = 2) \\
\lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{31} + \lambda_{41} + \lambda_{42} \leq \theta - 1 & (k = 3) \\
\lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{31} + \lambda_{32} + \lambda_{41} \leq \theta - 1 & (k = 4)
\end{cases}$$

The feasibility of the above system requires that the system

$$\begin{cases}
2\lambda_{11} + \lambda_{12} + 2\lambda_{21} + \lambda_{22} + 2\lambda_{31} + \lambda_{32} + 2\lambda_{41} + \lambda_{42} \geq 2\theta & (k = 1) + (k = 2) \\
2\lambda_{11} + \lambda_{12} + 2\lambda_{21} + \lambda_{22} + 2\lambda_{31} + \lambda_{32} + 2\lambda_{41} + \lambda_{42} \leq 2\theta - 2 & (k = 3) + (k = 4)
\end{cases}$$

obtained by summing up the first two inequalities and the last two be feasible. This obviously leads to a contradiction and shows that the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  is not always extendable as a threshold Boolean function.

However, we do not need to derive an extension of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . The knowledge of an extension  $f$  for  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  permits the derivation of linear inequalities representing sufficient conditions for the probabilistic constraint (3) to hold. To qualify as an extension for  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ , it can be seen from the aforementioned example that  $f$  must meet strict conditions. An extension  $f$  could very likely be such that we might not be able to extract the minimal conditions that must be met for (3) to be satisfied, thus impeding us in the to derivation of a deterministic formulation equivalent to **PMRTM**. Thus, in order to identify tighter, possibly minimal conditions for constraint (3) to hold, we shall enlarge the class of Boolean functions we consider beyond that of extensions and characterize several forms of minorant for Boolean functions.

### 3.2 Concept of Minorant

We shall now introduce the concepts of **minorant** and **tight minorant** of a Boolean function.

**Definition 17** [8] *A Boolean function  $f$  is a minorant of a pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  if  $\bar{\Omega}_B^- \subseteq \mathcal{F}(f)$ .*

A minorant is a very broad concept, as many functions whose structure is loosely tied to the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  could be a minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . For example, the constant 0 is a minorant for any pdBf. We shall now introduce the concept of tight minorant.

**Definition 18** *A Boolean function  $f$  is a tight minorant of a pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  if*

1.  $f$  is a minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ , and
2.  $\mathcal{T}(f) \cap \bar{\Omega}_B^+ \neq \emptyset$ .

Since a threshold Boolean function is characterized by a separating structure, Definition 18 leads to Lemma 19.

**Lemma 19** *A threshold Boolean function  $f$  defined by the separating structure  $(\lambda, \theta)$  is a tight minorant of a pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  if the system of inequalities*

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \geq \theta, \quad \text{for at least one } k \in \bar{\Omega}_B^+ \quad (27)$$

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq \theta - 1, \quad k \in \bar{\Omega}_B^- \quad (28)$$

*has a feasible solution.*

Observe that (27) is a disjunctive condition over the set  $\bar{\Omega}_B^+$ .

**Theorem 20** *Given a pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  that admits a positive monotone extension, there exists a tight minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  that is threshold.*

**Proof.** Since  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  admits a positive monotone extension (Theorem 14), we know that:

- For any  $k \in \bar{\Omega}_B^+$ ,

$$\text{there is no } k' \in \bar{\Omega}_B^- \text{ such that } \beta^{k'} \geq \beta^k. \quad (29)$$

- For any  $k \in \bar{\Omega}_B^-$ ,

$$\text{there is no } k' \in \bar{\Omega}_B^+ \text{ such that } \beta^{k'} \leq \beta^k. \quad (30)$$

Take an arbitrary  $k^* \in \bar{\Omega}_B^+$  and construct a threshold function  $f$  (Definition 15) whose separating structure has weights  $\lambda$  taking value 0 or 1 such that

$$\lambda_{jl} = \beta_{jl}^{k^*} \quad \text{and} \quad \theta = \sum_{j \in J} \sum_{l=1}^{n_j} \beta_{jl}^{k^*}. \quad (31)$$

By construction,

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k^*} = \sum_{j \in J} \sum_{l=1}^{n_j} (\beta_{jl}^{k^*})^2 = \theta, \quad (32)$$

and  $f(\beta^{k^*}) = 1$ . All that is left to show is that  $f(\beta^k) = 0$  for any  $k \in \bar{\Omega}_B^-$ .

Take an arbitrary point  $k' \in \bar{\Omega}_B^-$  and proceed to a componentwise comparison with  $\beta^{k^*}$ . Since  $\bar{\Omega}_B^+$  and  $\bar{\Omega}_B^-$  are disjoint, it follows that  $\beta^{k^*}$  and  $\beta^{k'}$  differ on at least one coordinate:

$$\sum_{j \in J} \sum_{l=1}^{n_j} |\beta_{jl}^{k^*} - \beta_{jl}^{k'}| \geq 1.$$

Since  $g$  is positive monotone (30) and  $\beta^{k^*} \neq \beta^{k'}$  since  $\bar{\Omega}_B^+$  and  $\bar{\Omega}_B^-$  are disjoint, we have that  $\beta_{jl}^{k'} < \beta_{jl}^{k^*}$  on at least one coordinate  $(j, l)$ . This implies that  $\beta^{k'}$  takes value 0 for at least one  $(j, l)$  on which  $\beta^{k^*}$  is equal to one. Therefore, by construction of  $\beta^{k^*}$  (31) and of the separating structure (32) of  $f$ , it follows that

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k'} = \sum_{jl: \beta_{jl}^{k^*} = 1} \beta_{jl}^{k'} \leq \theta - 1,$$

as we set out to prove. □

The inequalities (28) mean that each  $p$ -insufficient realization is also defined as  $p$ -insufficient by the separating structure  $(\lambda, \theta)$  characterizing the tight minorant  $f$ . Lemma 21 follows.

**Lemma 21** *The threshold tight minorant derived in the proof of Theorem 20 has an integral separating structure  $(\lambda_{11}, \dots, \lambda_{|J|n_j}, \theta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$  and is isotone by construction, i.e.,  $\lambda_{jl} \geq 0$ ,  $j \in J$ ,  $l = 1, \dots, n_j$ .*

A Boolean function  $f$  is said to depend on a Boolean variable  $\beta_{jl}$  if there are two Boolean vectors that differ only on  $\beta_{jl}$  and for which the values of the Boolean function are not the same. Note that a threshold function does not depend on  $\beta_{jl}$  if the corresponding weight  $\lambda_{jl}$  is zero in the separating structure of  $f$ .

We shall now introduce the concept of **irredundant tight minorant**.

**Definition 22** A threshold tight minorant is called *irredundant* if it has the minimum number of weights  $\lambda_{jl}$  with non-zero value.

Theorem 23 defines the number of non-zero weights in an irredundant threshold tight minorant  $f$  of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ , and a set of conditions allowing for their generation.

**Theorem 23** An irredundant threshold tight minorant  $f$  of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  contains  $|J|$  non-zero weights. Every feasible solution of the system of inequalities

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| - 1, \quad k \in \bar{\Omega}_B^- \quad (33)$$

$$\sum_{l=1}^{n_j} \lambda_{jl} = 1, \quad j \in J \quad (34)$$

$$\lambda_{jl} \in \{0, 1\}, \quad j \in J, l = 1, \dots, n_j \quad (35)$$

defines an irredundant threshold tight minorant  $f$  with integral separating structure  $(\lambda, |J|) \in \{0, 1\}^n \times \mathbb{Z}_+$ . Every point  $k$  such that  $\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k = |J|$  belongs to  $\bar{\Omega}_B^+$ .

**Proof.** For a joint chance constraint (3) to hold, at least one condition must be imposed on each component of  $\xi$  (see, e.g., [31, 34]). Thus, the minimal number of non-zero weights in a threshold tight minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  is  $|J|$ . This explains why, in the right-hand side of (33), we substitute  $(|J| - 1)$  in (33) for  $\theta$  as in (28).

Consider the extreme recombination  $k'$  with  $\beta_{jl}^{k'}$  being constant 1, i.e.,  $\beta_{jl}^{k'} = 1, l = 1, \dots, n_j, j \in J$ . Clearly,  $F(\omega^{k'}) = 1$  and  $k' \in \bar{\Omega}_B^+$ . For any feasible solution for (33)-(35), we have

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k'} = \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \geq \sum_{j \in J} 1 = |J|.$$

Thus, (27) is induced by (33)-(35) and is redundant.

Consider an arbitrary separating structure  $(\lambda, |J|)$  feasible for (33)-(35). All possible Boolean vectors (Definition 6) belong either to the set  $\bar{\Omega}_B^+$  of relevant  $p$ -sufficient Boolean vectors, or the set  $\bar{\Omega}_B^-$  of relevant  $p$ -insufficient Boolean vectors, or the set of  $(\Omega_B^- \setminus \bar{\Omega}_B^-)$  of non-relevant  $p$ -insufficient realizations. Constraint (33) prevents any  $k \in \bar{\Omega}_B^-$  from satisfying (36):

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \geq |J|. \quad (36)$$

Any  $k \in (\Omega_B^- \setminus \bar{\Omega}_B^-)$  is such that  $\beta_{jl}^k = 0$  for at least one  $j \in J$ . Combining this result with the regularization property (see (18) in Lemma 5), we have:

$$\beta_{jl}^k = 0, l = 1, \dots, n_j \text{ for at least one } j \in J, k \in (\Omega_B^- \setminus \bar{\Omega}_B^-).$$

Therefore,

$$\sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k = 0, \text{ for at least one } j \in J, k \in (\Omega_B^- \setminus \bar{\Omega}_B^-),$$

which implies that

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| - 1, \quad k \in (\Omega_B^- \setminus \bar{\Omega}_B^-),$$

and thus (36) does not hold for any  $k \in (\Omega_B^- \setminus \bar{\Omega}_B^-)$ . Hence, any  $k$  for which (36) holds belongs to  $\bar{\Omega}_B^+$ .  $\square$  Theorem 23 shows that we do not need to include the disjunction (27) to generate a tight minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ .

**Theorem 24** *Any feasible solution  $\lambda$  of the system of linear inequalities*

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| - 1, \quad k \in \bar{\Omega}_B^- \quad (37)$$

$$\sum_{l=1}^{n_j} \lambda_{jl} = 1, \quad j \in J \quad (38)$$

$$0 \leq \lambda_{jl} \leq 1, \quad j \in J, l = 1, \dots, n_j \quad (39)$$

can be used to obtain an integral separating structure  $(\bar{\lambda}, |J|) \in \{0, 1\}^n \times \mathbb{Z}_+$  for an irredundant threshold tight minorant  $f$  of  $g$  in the following way:

$$\bar{\lambda}_{jl_j^*} = 1 \quad \text{and} \quad \bar{\lambda}_{jl_j} = 0, \quad l_j \neq l_j^*, \quad \text{with} \quad l_j^* = \underset{l_j}{\operatorname{argmax}} \lambda_{jl_j} > 0, \quad j \in J, l_j = 1, \dots, n_j. \quad (40)$$

**Proof.** We have to show that constraints (33)-(35) hold for the separating structure  $(\bar{\lambda}, |J|)$ . It is straightforward to see that the rounding procedure (40) is such that (34)-(35) holds for  $\bar{\lambda}$ . What is left to prove is that (33) holds for  $\bar{\lambda}$ , which is equivalent to showing

$$\sum_{j \in J} \sum_{l=1}^{n_j} \bar{\lambda}_{jl} \beta_{jl}^k \leq \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k, \quad \forall k \in \bar{\Omega}_B^- \quad \Leftrightarrow \quad \sum_{j \in J} \sum_{l=1}^{l_j^*} \bar{\lambda}_{jl} \beta_{jl}^k \leq \sum_{j \in J} \sum_{l=1}^{l_j^*} \lambda_{jl} \beta_{jl}^k, \quad \forall k \in \bar{\Omega}_B^- \quad (41)$$

for  $\bar{\lambda}$  derived with (40) from  $\lambda$  feasible for (37)-(39).

Consider one component  $j \in J$  and let us show that:

$$\sum_{l=1}^{l_j^*} \bar{\lambda}_{jl} \beta_{jl}^k \leq \sum_{l=1}^{l_j^*} \lambda_{jl} \beta_{jl}^k, \quad \forall k \in \bar{\Omega}_B^-. \quad (42)$$

If  $\beta_{jl_j^*}^k = 1$ , then (40) implies that  $\sum_{l=1}^{l_j^*} \bar{\lambda}_{jl} \beta_{jl}^k = 1$ , and we have, from the regularization property (16), that:

$$\beta_{jl}^k = 1, \quad \forall l < l_j^* \quad \text{and thus that} \quad \sum_{l=1}^{l_j^*} \lambda_{jl} \beta_{jl}^k = \sum_{l=1}^{l_j^*} \lambda_{jl} = 1. \quad (43)$$

The last inequality follows from (40) and the fact that  $\lambda$  satisfies (38).

If  $\beta_{jl}^k = 0$ , then

$$\sum_{l=1}^{l_j^*} \bar{\lambda}_{jl} \beta_{jl}^k = 0 \leq \sum_{l=1}^{l_j^*} \lambda_{jl} \beta_{jl}^k. \quad (44)$$

Evidently, (43) and (44) imply (42). The same result is valid for any  $j \in J$ , which guarantees that (41) holds.  $\square$

In our example, the system of inequalities (33)-(35) reads:

$$\lambda_{11} + \lambda_{21} + \lambda_{31} \leq 2 \quad (45)$$

$$\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{32} \leq 2 \quad (46)$$

$$\lambda_{11} = 1 \quad (47)$$

$$\lambda_{21} + \lambda_{22} + \lambda_{23} = 1 \quad (48)$$

$$\lambda_{31} + \lambda_{32} + \lambda_{33} = 1 \quad (49)$$

$$\lambda_{jl} \in \{0, 1\}, \quad \forall j, l. \quad (50)$$

Any feasible solution for (45)-(49), such as  $\lambda = (1, 0, 1, 0, 1, 0, 0)$ , defines the separating structure of an irredundant threshold tight minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . The above results are used in the Section 4 to derive deterministic approximations and equivalents for the original stochastic problem **PMRTM**.

**Theorem 25** [8] *If the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  is a threshold function defined on  $\{0, 1\}^n$ , then the set of separating structures is a full-dimensional convex cone in  $\mathbb{R}^{n+1}$ .*

**Proof.** Let  $\alpha, \mu \geq 0$  and  $D = (\lambda, \theta)$  and  $D' = (\lambda', \theta')$  be two arbitrary separating structures of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . It is straightforward that  $\alpha D$  and  $\alpha D + \mu D'$  are also separating structures of  $g$ , which proves the above statement.  $\square$

## 4 Threshold Minorant Reformulation of PMRTM

In this section, we shall use the threshold structure of the tight minorant of the pdBf  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$  to reformulate the probabilistically constrained problem **PMRTM**. New inner approximation and deterministic equivalent reformulations for **PMRTM** are derived.

### 4.1 Inner Approximation Formulation

In this section, we derive a linear programming formulation that constitutes an inner approximation of the stochastic problem **PMRTM**. We recall that each component  $t_{ij} = s_{ij} \xi_j$  of the matrix  $T$  in (3) involves a stochastic component  $\xi_j$ .

**Theorem 26** *Let  $(w, |J|)$  be a feasible solution for (37)-(39) and define  $l_j^*$  with (40) for each  $j \in J$ . The linear deterministic constraints*

$$\sum_{j \in J} s_{ij} c_{jl}^* x_j \leq d_i, \quad i = 1, \dots, r \quad (51)$$

*define sufficient conditions for  $\mathbb{P}(Tx \leq d)$  to be at least  $p$ .*



**Proof.** Let us take an arbitrary point feasible for (37)-(39) and construct the irredundant tight minorant  $f^*$  with separating structure  $(\lambda, |J|)$  defined according to (40). It follows from Theorem 24 that the separating structure of  $f^*$  is such that

$$\sum_{j \in J} \beta_{jl_j^*}^k \leq |J| - 1, \quad k \in \bar{\Omega}_B^- \quad \text{and} \quad \sum_{j \in J} \beta_{jl_j^*}^k \geq |J| \quad \text{for at least one } k \in \bar{\Omega}_B^+. \quad (52)$$

This underlines that the conditions imposed by  $f^*$  (i.e., induced by its separating structure) are violated by each  $k \in \bar{\Omega}_B^-$  but hold for at least one  $k \in \bar{\Omega}_B^+$ . Using the definition of the binarization process (14)-(15), one can see that (52) is equivalent to:

$$\begin{cases} \omega_j^k < c_{jl_j^*} \beta_{jl_j^*}^k = c_{jl_j^*}, & \text{for at least one } j, \forall k \in \bar{\Omega}^- \\ \omega_j^k \geq c_{jl_j^*} \beta_{jl_j^*}^k = c_{jl_j^*}, & \forall j \in J, \text{ for at least one } k \in \bar{\Omega}^+ \end{cases} \quad (53)$$

Further, the construction of the sufficient-equivalent set of cut points (see (19) in Definition 7) implies

$$\omega_j^k = c_{jl_j^*}, \quad j \in J, \text{ for one } k \in \bar{\Omega}^+. \quad (54)$$

Let  $k' \in \bar{\Omega}^+$  be the recombination for which (54) holds:

$$\omega_j^{k'} = c_{jl_j^*}, \quad j \in J. \quad (55)$$

Every component  $t_{ij}$  of  $T$  is defined by:  $t_{ij} = s_{ij} \xi_j$ ,  $i = 1, \dots, r, j \in J$ . Thus, we have

$$\mathbb{P}(Tx \leq d) = \mathbb{P}\left(\sum_{j \in J} s_{ij} \xi_j x_j \leq d_i, \quad i = 1, \dots, r\right). \quad (56)$$

Since  $k' \in \bar{\Omega}^+$ , we have  $\mathbb{P}(\xi \leq \omega^{k'}) \geq p$  and

$$\mathbb{P}\left(\sum_{j \in J} s_{ij} \xi_j x_j \leq \sum_{j \in J} s_{ij} \omega_j^{k'} x_j, \quad i = 1, \dots, r\right) \geq p \quad (57)$$

given that  $x_j$  and  $s_{ij}$  are all non-negative. This, combined with (56), implies that:

$$\sum_{j \in J} s_{ij} \omega_j^{k'} x_j \leq d_i, \quad i = 1, \dots, r \quad \Rightarrow \quad \mathbb{P}\left(\sum_{j \in J} s_{ij} \xi_j x_j \leq d_i, \quad i = 1, \dots, r\right) \geq p. \quad (58)$$

Using (55), we can rewrite (58) as

$$\sum_{j \in J} s_{ij} c_{jl_j^*} x_j \leq d_i, \quad i = 1, \dots, r \quad \Rightarrow \quad \mathbb{P}(Tx \leq d) \geq p, \quad (59)$$

as we set out to prove. □

Corollary 27 follows immediately from Theorem 26.

**Corollary 27** *The linear program **LP***

$$\begin{aligned} \mathbf{LP} : \quad & \max \quad q^\top x \\ & \text{subject to} \quad (2); (4); (51) \end{aligned}$$

*is an inner approximation of the probabilistically constrained problem **PMRTM**. Its feasible set is included in the feasible set of **PMRTM**.*

Consider  $\lambda = (1, 0, 1, 0, 1, 0, 0)$  which is feasible for (45)-(49) and defines the separating structure of an irredundant threshold tight minorant of  $g(\bar{\Omega}_B^+, \bar{\Omega}_B^-)$ . The corresponding constraints (51) are:

$$2 \cdot 7 \cdot x_1 + 3 \cdot 6 \cdot x_2 \leq 25 \quad (60)$$

$$3 \cdot 7 \cdot x_1 + 6 \cdot 3 \cdot x_3 \leq 32. \quad (61)$$

The associated linear programming problem **LP** is

$$\begin{aligned} & \max \quad 3x_1 + x_2 + x_3 \\ & \text{subject to} \quad (60) - (61) \\ & \quad \quad \quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

## 4.2 Equivalent Formulations

In this section, we build on the results of Section 4 to derive three new deterministic formulations that are equivalent to the stochastic problem **PMRTM**. Instead of successively solving the system of linear inequalities (37)-(39) to derive a threshold tight minorant and to solve the resulting inner approximation linear programming problem **LP**, we propose in this section to execute the two tasks concurrently through the solution of a deterministic problem equivalent to **PMRTM**.

### 4.2.1 Bilinear Integer Programming Formulation

We first reformulate **PMRTM** as a bilinear integer programming problem.

**Theorem 28** *The quadratic integer optimization problem **QDE***

$$\begin{aligned} \mathbf{QDE} : \quad & \max \quad q^\top x \\ & \text{subject to} \quad (2); (4); (33); (34); (35) \\ & \quad \quad \quad \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} = y_j, \quad j \in J \quad (62) \\ & \quad \quad \quad \sum_{j \in J} s_{ij} y_j x_j \leq d_i, \quad i = 1, \dots, r \quad (63) \\ & \quad \quad \quad c_{j1} \leq y_j \leq c_{jn_j}, \quad j \in J \quad (64) \end{aligned}$$

*is equivalent to the probabilistically constrained problem **PMRTM**.*

**Proof.** (i) Any solution feasible for **QDE** is feasible for **PMRTM**. Since constraints (33), (34) and (35) are part of the constraint set, we can use Theorem 23 that indicates that any  $\hat{\lambda} \in \{\lambda : (33), (34), (35)\}$  defines the separating structure  $(\hat{\lambda}, |J|)$  of an irredundant threshold tight minorant  $f$ . In other words,  $G = \left\{ k \in \bar{\Omega}_B^- : \sum_{j \in J} \sum_{l=1}^{n_j} \hat{\lambda}_{jl} \beta_{jl}^k = |J| \right\} \subseteq \bar{\Omega}_B^+$ . Let  $L = \{(j, l) : \hat{\lambda}_{jl} = 1, j \in J, l = 1, \dots, n_j\}$ . The definition of the binarization process (14)-(15) implies that, for any  $k \in G$ , we have (see (53) in Theorem 26)

$$\omega_j^k \geq c_{jl}, (j, l) \in L, \quad (65)$$

and the construction of the sufficient-equivalent set of cut points (Definition 7) implies that there exists  $k' \in G$  such that

$$\omega_j^{k'} = c_{jl}, (j, l) \in L. \quad (66)$$

Constraints (34) and (35) ensure that exactly one term  $\hat{\lambda}_{jl} c_{jl}$  in the left-hand side of each constraint (62) is non-zero and equal to  $c_{jl}, (j, l) \in L$ , and thus  $y_j = c_{jl}, (j, l) \in L$  in (62). This allows the rewriting of (66) as

$$\omega_j^{k'} = c_{jl} = \sum_{i=1}^{n_j} \lambda_{il} c_{jl} = y_j, (j, l) \in L. \quad (67)$$

Since  $k' \in G \subseteq \bar{\Omega}_B^+$ , (57) and (58) hold, which, along with (67), implies that

$$\sum_{j \in J} s_{ij} y_j x_j \leq d_i, i = 1, \dots, r \quad \Rightarrow \quad \mathbb{P}(Tx \leq d) \geq p. \quad (68)$$

(ii) Any solution feasible for **PMRTM** is feasible for **QDE**. For any  $k'' \in \bar{\Omega}$  such that  $\mathbb{P}(\xi \leq \omega^{k''}) \geq p$ , the definition of the sufficient-equivalent set of cut points implies that there exists  $k' \in \bar{\Omega}_B^+$  such that  $\omega^{k'} \leq \omega^{k''}$  and  $\omega_j^{k'} = \bigvee_{l=1, \dots, n_j} c_{jl}, j \in J$ .

Let  $l_j^* = \operatorname{argmax}_l \{c_{jl} : c_{jl} = \omega_j^{k'}\}, j \in J$ . What is left to prove is that the vector  $\lambda'$

$$\lambda'_{jl} = \begin{cases} 1 & \text{if } l = l_j^* \\ 0 & \text{otherwise} \end{cases}, j \in J \quad (69)$$

is feasible for **QDE**, i.e., that  $\lambda' \in \{\lambda : (33), (34), (35), (62)\}$ . It is evident that  $\lambda'$  is feasible for (34), (35) and (62). By construction (69),  $\lambda'$  is feasible for (33) if

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda'_{jl} \beta_{jl}^k = \sum_{j \in J} \lambda'_{j l_j^*} \beta_{j l_j^*}^k = \sum_{j \in J} \beta_{j l_j^*}^k \leq |J| - 1, k \in \bar{\Omega}_B^-. \quad (70)$$

The feasibility of the above constraints is ensured if  $\beta_{j l_j^*}^k = 0$  for at least one  $j, \forall k \in \bar{\Omega}_B^-$ .

Note that, for any  $k' \in \bar{\Omega}_B^+$ , there is no  $k \in \bar{\Omega}_B^-$  such that  $\beta^k \geq \beta^{k'}$  (see Definition 19 of a sufficient-equivalent set of cut points). This implies that

$$(\beta_{j_1}^{k'}, \dots, \beta_{j_{n_j}}^{k'}) < (\beta_{j_1}^k, \dots, \beta_{j_{n_j}}^k) \text{ for at least one } j, \forall k \in \bar{\Omega}_B^-.$$

Let  $h \in J$  be the coordinate such that  $(\beta_{h_1}^k, \dots, \beta_{h_{n_h}}^k) < (\beta_{h_1}^{k'}, \dots, \beta_{h_{n_h}}^{k'})$  for an arbitrary  $k \in \bar{\Omega}_B^-$ . Since  $l_h^* = \operatorname{argmax}_l \beta_{hl}^{k'} = 1$ , we have for any  $l > l_h^*$  that  $\beta_{hl}^k = \beta_{hl}^{k'} = 0$ . Thus, the vectors  $\beta^{k'}$  and  $\beta^k$  differ only in terms of the first  $l_h^*$  components, and we have

$$(\beta_{h1}^k, \dots, \beta_{hl_h}^k) < (\beta_{h1}^{k'}, \dots, \beta_{hl_h}^{k'}).$$

The regularization property (16)-(17) indicates that this relationship can only be true if  $\beta_{hl_h}^k = 0 < \beta_{hl_h}^{k'} = 1$ . This shows that, for any  $k \in \bar{\Omega}_B^-$ ,  $\beta_{jl_j}^k = 0$  for at least one  $j$ , which results in  $\sum_{j \in J} \beta_{jl_j}^k$  to be bounded from above by  $(|J| - 1)$  for each  $k \in \bar{\Omega}_B^-$  and implies that  $\lambda'$  is feasible for (70) and thus (33).  $\square$

Note that (64) is a strengthening of  $y \in \mathbb{R}_+^{|J|}$  and follows immediately from (34), (35), and (62). Problem **QDE** is a mixed-integer nonlinear programming (MINLP) problem which is NP-hard. It includes  $n$  mixed-integer bilinear terms  $x_j \lambda_{jl}$  and its continuous relaxation is non-convex. In our example, the problem **QDE** reads:

$$\begin{aligned} & \max \quad 3x_1 + x_2 + x_3 \\ & \text{subject to} \quad (45) - (50) \\ & \quad 7\lambda_{11} = y_1 \\ & \quad 5\lambda_{21} + 6\lambda_{22} + 10\lambda_{23} = y_2 \\ & \quad 3\lambda_{31} + 4\lambda_{32} + 5\lambda_{33} = y_3 \\ & \quad 2y_1x_1 + 3y_2x_2 \leq 25 \\ & \quad 3y_1x_1 + 6y_3x_3 \leq 32 \\ & \quad 5 \leq y_2 \leq 10 \\ & \quad 3 \leq y_3 \leq 5 \\ & \quad x_1, x_2, x_3 \geq 0 \\ & \quad \lambda_{11}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{31}, \lambda_{32}, \lambda_{33} \in \{0, 1\} \end{aligned}$$

#### 4.2.2 Mixed-Integer Linear Programming Formulation

Instead of attempting to solve the **QDE** problem directly, we shall linearize the  $|J|$  products  $x_j (\sum_{l=1}^{n_j} \lambda_{jl} c_{jl})$  of continuous by binary variables (see, e.g., [1, 10]). Let  $u_j$  be an upper bound on the value taken by  $x_j$ .

**Theorem 29** *The bilinear set*

$$S_{B_i} = \left\{ (x, \lambda) \in \mathbb{R}_+^{|J|} \times \{0, 1\}^n : \sum_{j \in J} s_{ij} x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) \leq d_i ; (34) \right\} \quad (71)$$

*is equivalent to the linear set*

$$S_{L_i} = \left\{ (x, z, \lambda) \in \mathbb{R}_+^{|J|} \times \mathbb{R}^{|J|} \times \{0, 1\}^n : \sum_{j \in J} s_{ij} z_j \leq d_i ; z_j \geq c_{jl} x_j - (1 - \lambda_{jl}) (c_{jl} - c_{j1}) M_{ij}, j \in J, l = 1, \dots, n_j ; (34) \right\} \quad (72)$$

where

$$M_{ij} = \begin{cases} \min \left\{ u_j, \frac{d_i}{s_{ij} c_{j1}} \right\} & \text{if } s_{ij} c_{j1} > 0 \\ u_j & \text{otherwise} \end{cases}, j \in J. \quad (73)$$

**Proof.** First, we show that  $M_{ij}$  is an upper bound on the value that  $x_j$  can take if  $s_{ij}$  and  $c_{j1}$  have the same sign and are both different from 0. Consider the inequality

$$\sum_{j \in J} s_{ij} x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) \leq d_i \quad (74)$$

in  $S_{B_i}$  (71). The value that  $x_j$  can take is maximal if  $x_{j'} = 0, \forall j' \neq j$ . Since the cut points  $c_{jl}$  are arranged in increasing order and the set partitioning constraints (34) require that one  $\lambda_{jl}$  be equal to 1, while all the other  $\lambda_{jl}$  be equal to 0, the smallest value that the expression  $\left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right)$  can take is  $c_{j1}$  when  $\lambda_{j1} = 1$ . Therefore, if  $s_{ij}c_{j1} > 0$ , we have

$$x_j \leq \frac{d_i}{s_{ij} c_{j1}},$$

and the largest value that  $x_j$  can take in constraint (74) is:  $M_{ij} = \min \left\{ u_j, \frac{d_i}{s_{ij} c_{j1}} \right\}$ . On the other hand, if the product of  $s_{ij}$  by  $c_{j1}$  is not strictly positive, we set  $M_{ij} = u_j$ .

We shall now prove that the feasible sets defined by  $S_{B_i}$  and  $S_{L_i}$  are equivalent and that the substitution of  $z_j$  for  $x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right)$  in (74) is valid. In order to do so, we show that:

1. The linearized constraints

$$z_j \geq c_{jl} x_j - (1 - \lambda_{jl}) (c_{jl} - c_{j1}) M_{ij}, \quad j \in J, l = 1, \dots, n_j, \quad (75)$$

in  $S_{L_i}$  (72) imply that

$$z_j \geq x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right). \quad (76)$$

Due to (34), exactly one variable  $\lambda_{jl}, l = 1, \dots, n_j$  is equal to 1 for each  $j \in J$  while all the others are equal to 0. Let  $l'_j$  be such that  $\lambda_{jl'_j} = 1, \lambda_{jl} = 0, \forall l \neq l'_j$ . It follows that  $\sum_{l=1}^{n_j} \lambda_{jl} c_{jl} = c_{jl'_j}$  and (76) can be rewritten as:

$$z_j \geq x_j c_{jl'_j}. \quad (77)$$

For  $l = l'_j$ , the inequality (75) reads  $z_j \geq c_{jl'_j} x_j$  and is the same as (77). This shows that the constraints (75) imply (76). If we replace  $x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right)$  by  $z_j$  in (74), we have

$$\sum_{j \in J} s_{ij} z_j \leq d_i \quad \Rightarrow \quad \sum_{j \in J} s_{ij} x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) \leq d_i.$$

2. The linearized constraints do not cut any feasible solution of  $S_{B_i}$ .

For  $l = l'_j$ , the inequality (75) requires  $z_j$  to be at least equal to  $c_{jl'_j} x_j$  and is equivalent to (76).

Consider now the other indices  $l \neq l'_j$  for which  $\lambda_{jl} = 0$ . Along with the fact that the cut points are sorted in increasing order, the set partitioning constraints (34) imply that:

$$c_{jn_j} \geq \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} = c_{jl'_j} \geq c_{j1} \quad (78)$$

$$x_j \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \geq x_j c_{j1} \quad (79)$$

To prevent (75) from cutting any feasible solution, we must ensure that the right-hand sides of the constraints (75) ( $l \neq l'_j$ ) are each at most equal to  $x_j c_{j1}$ , which reads:

$$x_j c_{j1} \geq c_{jl} x_j - (c_{jl} - c_{j1}) M_{ij}, \quad \forall j \neq l'_j. \quad (80)$$

The inequalities (80) can be equivalently rewritten as:

$$M_{ij}(c_{jl} - c_{j1}) \geq x_j(c_{jl} - c_{j1}), \quad \forall j \neq l'_j,$$

which is always true since  $M_{ij}$  is an upper bound on the value that  $x_j$  can take. This provides the result that we set out to prove.  $\square$

A direct consequence of Theorem 29 is that the bilinear integer problem **QDE** can be equivalently reformulated as the MIP problem **MIP1**. Lemma 30 follows.

**Lemma 30** *The probabilistically constrained stochastic programming problem **PMRTM** can be equivalently reformulated as the MIP problem **MIP1**:*

$$\begin{aligned} \mathbf{MIP1} : \quad & \max && q^\top x \\ & \text{subject to} && (2); (4); (33); (34); (35) \end{aligned}$$

$$\sum_{j \in J} s_{ij} z_j \leq d_i, \quad i = 1, \dots, r \quad (81)$$

$$z_j \geq c_{jl} x_j - (1 - \lambda_{jl})(c_{jl} - c_{j1}) \bar{M}_j, \quad j \in J, l = 1, \dots, n_j, \quad (82)$$

where  $\bar{M}_j = \min_i M_{ij}$ ,  $j \in J$

In our example, **MIP1** is given by:

$$\begin{aligned} & \max && 3x_1 + x_2 + x_3 \\ & \text{subject to} && (45) - (50) \\ & && 2z_1 + 3z_2 \leq 25 \\ & && 3z_1 + 6z_3 \leq 32 \\ & && z_1 \geq 7x_1 \\ & && z_2 \geq 5x_2 && z_2 \geq 6x_2 - (1 - \lambda_{22}) \cdot \frac{25}{15}; && z_2 \geq 10x_2 - (1 - \lambda_{23}) \cdot 5 \cdot \frac{25}{15} \\ & && z_3 \geq 3x_3 && z_3 \geq 4x_3 - (1 - \lambda_{32}) \cdot \frac{32}{18}; && z_3 \geq 5x_3 - (1 - \lambda_{33}) \cdot 2 \cdot \frac{32}{18} \\ & && x_1, x_2, x_3 \geq 0 \\ & && \lambda_{11}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{31}, \lambda_{32}, \lambda_{33} \in \{0, 1\} \end{aligned}$$

### 4.2.3 Strengthened Mixed-Integer Linear Programming Formulation

We shall now strengthen the linearized formulation **MIP1** and derive valid inequalities. Consider the set

$$G_j = \left\{ (x_j, z_j, \lambda_{jl}) \in \mathbb{R}_+ \times \mathbb{R} \times \{0, 1\}^{n_j} : z_j \geq c_{jl} x_j - (1 - \lambda_{jl})(c_{jl} - c_{j1})\bar{M}_j, l = 1, \dots, n_j; \sum_{l=1}^{n_j} \lambda_{jl} = 1 \right\}. \quad (83)$$

**Theorem 31** *The inequalities*

$$z_j \geq c_{jl} x_j - (1 - v_{jl})(c_{jl} - c_{j1})\bar{M}_j, l = 1, \dots, n_j, \quad (84)$$

$$\text{with} \quad v_{jl} = \sum_{r=l}^{n_j} \lambda_{jr}, l = 1, \dots, n_j, \quad (85)$$

are valid for  $G_j$  and are tighter than

$$z_j \geq c_{jl} x_j - (1 - \lambda_{jl})(c_{jl} - c_{j1})\bar{M}_j, l = 1, \dots, n_j. \quad (86)$$

**Proof.** Let  $\lambda$  be feasible for the continuous relaxation of the set  $G_j$  and let  $l' = \operatorname{argmax}_l \lambda_{jl} > 0$ . For  $l \geq l'$ ,  $v_{jl} = \lambda_{jl}$  and the corresponding inequalities (84) and (86) in  $G_j$  are identical.

We show now that the inequalities (84) are tighter than (86) for  $l' > l$ . For  $l < l'$  ( $l' > 1$ ), we have  $v_{jl} \geq \lambda_{jl}$  from (85), which successively implies that

$$(1 - \lambda_{jl})\bar{M}_j \geq (1 - v_{jl})\bar{M}_j \quad \Rightarrow \quad c_{jl} x_j - (1 - v_{jl})(c_{jl} - c_{j1})\bar{M}_j \geq c_{jl} x_j - (1 - \lambda_{jl})(c_{jl} - c_{j1})\bar{M}_j. \quad (87)$$

Thus,  $z_j$  is limited from below by  $c_{jl} x_j - (1 - v_{jl})(c_{jl} - c_{j1})\bar{M}_j$  in (84) which is larger than or equal to the lower bound  $c_{jl} x_j - (1 - \lambda_{jl})(c_{jl} - c_{j1})\bar{M}_j$  on  $z_j$  defined by (86). Further, from (85) and the fact that  $l' = \operatorname{argmax}_l \lambda_{jl} > 0$ , it follows that  $v_{jl} > \lambda_{jl}, l = 1, \dots, l' - 1$ . Hence,  $(l' - 1)$  of the inequalities (87) are strict. This shows that the inequalities (84) ( $l < l'$ ) are tighter than (86).

We show now that the inequalities (84) do not cut any feasible solution of  $G_j$  and are thus valid.

Recall that  $l' = \operatorname{argmax}_l \lambda_{jl} > 0$ . From any fractional solution  $\lambda$ , we can easily derive a integral solution for (84) and (86) by setting:  $v_{jl} = 1, l = 1, \dots, l', v_{jl} = 0, l > l', \lambda_{j l'} = 1$ , and  $\lambda_{jl} = 0, l \neq l'$ . The inequalities (84) read

$$z_j \geq c_{jl} x_j, l = 1, \dots, l'. \quad (88)$$

Observe first that the inequality (86) corresponding to the component  $l'$  requires

$$z_j \geq c_{j l'} x_j. \quad (89)$$

Note further that, since the cut points are sorted in ascending order (15), we have that

$$c_{j l'} x_j \geq c_{jl} x_j, l = l', \dots, n_j. \quad (90)$$

The above relationships (89) and (90) imply that the constraints (88) do not cut any feasible solution for  $G_j$  and are thus valid.  $\square$

Observe that, by construction (see (85)), we have  $v_{j1} = 1$  and  $z_j \geq c_{j1}x_j$ . Lemma 32 is a direct consequence of Theorem (31).

**Lemma 32** *Problems MIP1 and MIP2 are equivalent and therefore problem PMRTM can be equivalently reformulated as the MIP problem MIP2:*

$$\begin{aligned} \text{MIP2 : } \max \quad & q^\top x \\ \text{subject to} \quad & (2); (4); (33); (34); (35); (81) \\ & z_j \geq c_{jl} x_j - (1 - v_{jl}) (c_{jl} - c_{j1}) \bar{M}_j, \quad j \in J, l = 1, \dots, n_j \quad (91) \\ & v_{jl} = \sum_{r=l}^{n_j} \lambda_{jr}, \quad j \in J, l = 1, \dots, n_j \quad (92) \end{aligned}$$

In our example, **MIP2** reads:

$$\begin{aligned} \max \quad & 3x_1 + x_2 + x_3 \\ \text{subject to} \quad & (45) - (50) \\ & 2z_1 + 3z_2 \leq 25 \\ & 3z_1 + 6z_3 \leq 32 \\ & z_1 \geq 7x_1 \\ & z_2 \geq 5x_2 \\ & z_2 \geq 6x_2 - (1 - v_{22}) \cdot \frac{25}{15} \\ & z_2 \geq 10x_2 - (1 - v_{23}) \cdot 5 \cdot \frac{25}{15} \\ & z_3 \geq 3x_3 \\ & z_3 \geq 4x_3 - (1 - v_{32}) \cdot \frac{32}{18} \\ & z_3 \geq 5x_3 - (1 - v_{33}) \cdot 2 \cdot \frac{32}{18} \\ & v_{11} = \lambda_{11} \\ & v_{21} = \lambda_{21} + \lambda_{22} + \lambda_{23}; \quad v_{22} = \lambda_{22} + \lambda_{23}; \quad v_{23} = \lambda_{23} \\ & v_{31} = \lambda_{31} + \lambda_{32} + \lambda_{33}; \quad v_{32} = \lambda_{32} + \lambda_{33}; \quad v_{33} = \lambda_{33} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The optimal solution  $(x^*, z^*, \lambda^*) = (\underbrace{1.52, 0.24, 0}_{x^*}; \underbrace{10.67, 1.22, 0}_{z^*}; \underbrace{1, 0, 0.15, 0.85, 1, 0, 0}_{\lambda^*})$  for the continuous relaxation of problem **MIP1** violates the constraint  $z_2 \geq 6x_2 - (1 - v_{22}) \cdot \frac{25}{15}$  in the continuous relaxation of problem **MIP2**.



## 5 Numerical Evaluation

### 5.1 Test Laboratory

In this section, we test the contribution of the solution framework for the numerical solution of optimization problems including multirow probabilistic constraints with random technology matrix. The tests are conducted on instances of a multi-period capital rationing problem in which uncertainty concerns the value of each possible project, the cash outflows associated with each of them, and the capital available at each period in the planning horizon. Random variables are in both the technology matrix and the right-hand sides of the probabilistic constraint.

The capital rationing process consists in the allocation of a limited budget to profitable projects. A survey considering Fortune 500 firms indicates that more than 60% of them place a limit on the internal capital available for investment plans [25]. We refer to [16] for a review of the mathematical programming approaches used for capital rationing purposes. The problem of rationing capital under uncertainty was first studied by Lorie and Savage [21] who describe the challenges of capital rationing problems involving multi-period projects that generate inflows and outflows dispersed across their lifetimes. The first chance-constrained formulation for capital rationing was proposed by Naslund [27] and maximizes the expected value of the adopted projects across a multi-period horizon, plus the amount of money lent, minus the amount of money borrowed. The model includes several individual chance constraints requiring that the probability of having a budget deficit in each period is kept low. Sarper [35] rejects the use of the normal distribution for characterizing the uncertain cash flows and supports the use of uniformly distributed variables. Gurgur and Luxhoj [11] assume that cash flows and available budgets are asymmetrically distributed random variables and follow a Weibull distribution. Most recently, Beraldi et al. [4] propose a formulation with joint probabilistic constraints in which the random variables are finitely distributed. They propose a specialized branch-and-bound algorithm and evaluate the impact of introducing risk measures.

Let  $r$  denote the number of periods in the horizon and  $J$  be the set of possible projects. The binary decision variables  $x_j$  define whether project  $j$  is selected (i.e.,  $x_j = 1$ ) or not. The random variable  $\xi_{ij}$ ,  $i = 1, \dots, r$ ,  $j \in J$  represents the cash outflow due to project  $j$  in period  $i$ , while  $\zeta_i$ ,  $i = 1, \dots, r$  denotes the random budget that will be available in period  $i$ . We denote by  $V_j$  the random value of project  $j$ . The initial cash outflows and the currently available budget are known and denoted by  $a_{0j}$ ,  $j \in J$  and  $b_0$ . The formulation reads:

$$\max \quad \sum_{j \in J} \mathbb{E}[V_j] \cdot x_j \quad (93)$$

$$\text{subject to} \quad \sum_{j \in J} a_{0j} \cdot x_j \leq b_0 \quad (94)$$

$$\mathbb{P} \left( \sum_{j \in J} \xi_{ij} \cdot x_j \leq \zeta_i, i = 1, \dots, r \right) \geq p \quad (95)$$

$$x \in \{0, 1\}^{|J|} \quad (96)$$

The objective function (93) maximizes the expected value of the selected projects. Constraint (94) ensures that the initial cash outflows do not exceed the current budget. Constraint (95) requires that the cash outflows of the selected projects do not create a budget shortfall at any of the periods of the horizon with a probability at least equal to  $p$ .

The ‘‘aggregated’’ random matrix  $T \in \mathbb{R}_+^{r \times (1+|J|r)}$  in **PMRTM** includes the random cash outflows  $\xi_{ij}$  and available budgets  $\zeta_i$ . Each row  $T_i$  includes up to  $(|J| + 1)$  non-zero components:

$$T_i = [0, \dots, 0, \xi_{i1}, \xi_{i2}, \dots, \xi_{i|J|}, 0, \dots, 0, -\zeta_i, 0, \dots, 0], \quad i = 1, \dots, r. \quad (97)$$

The first  $(r \cdot |J|)$  components of  $T_i$  are the cash flows incurred by each project  $j$  at the successive periods  $i$ , while the  $r$  last components of  $T_i$  are the budgets that will be available in the future periods  $i$ .

Let us omit the zero-components in (97) and rewrite the vector  $T_i$  of the random technology matrix  $T$  accordingly:  $T_i = [\xi_{i1}, \xi_{i2}, \dots, \xi_{i|J|}, -\zeta_i]$ . We denote by  $c_{j1}, \dots, c_{jn_j}$ ,  $j \in J$  the cut points associated with the random variable  $\xi_{ij}$ , and by  $c_{0l}$ ,  $l = 1, \dots, n_0$  the cut points associated with  $-\zeta_i$ .

Owing to the structure of the capital rationing problem, namely the stochasticity of the right-hand side, the set  $S_B$  (71) corresponding to the  $i^{\text{th}}$  inequality in (95) can be rewritten as

$$S_{B_i}^{CR} = \left\{ (x, \lambda) \in \{0, 1\}^{|J|+n+n_0} : \sum_{j \in J} s_{ij} x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) \leq \sum_{l=1}^{n_0} \lambda_{0l} c_{0l} ; (34) \right\}.$$

Note also that  $c_{jl} \geq 0$ ,  $l = 1, \dots, n_j$ ,  $j \in J$ , and  $s_{ij} = 1$  or  $= 0$ . It is straightforward to see that

$$S_{L_i}^{CR} = \left\{ (x, \lambda, z) \in \{0, 1\}^{|J|+n+n_0} \times \mathbb{R}^{|J|} : \sum_{j \in J} s_{ij} z_j \leq \sum_{l=1}^{n_0} \lambda_{0l} c_{0l}; z_j \geq c_{jl} x_j - (1 - \lambda_{jl}) M_{jl}, \forall j : s_{ij} > 0 ; (34) \right\},$$

where  $M_{jl}$  is a sufficiently large positive number, is equivalent to  $S_{B_i}^{CR}$ .

In order to find the tightest possible continuous relaxation of  $S_{B_i}^{CR}$ , the constants  $M_{jl}$ ,  $j \in J, l = 1, \dots, n_j$  should be set to the smallest values that yet ensure the validity of the formulation. Taking into account the specifics of the capital rationing problem, Proposition 33 provides the smallest values that can be assigned to the parameters  $M_{jl}$ .

**Proposition 33** For every inequality  $z_j \geq c_{jl} x_j - (1 - \lambda_{jl}) M_{jl}$ ,  $j \in J, l = 1, \dots, n_j$  in  $S_{L_i}^{CR}$ ,

$$M_{jl} = c_{jl} - c_{j1} \quad (98)$$

is the smallest value that  $M_{jl}$  can take to ensure the equivalence between  $S_{L_i}^{CR}$  and  $S_{B_i}^{CR}$ .

**Proof.** The equivalence between  $S_{L_i}^{CR}$  and  $S_{B_i}^{CR}$  is based on the substitution of  $z_j$  for  $\sum_{j \in J} x_j (\sum_{l=1}^{n_j} \lambda_{jl} c_{jl})$ , and the assignment of acceptable values to the constants  $M_{jl}$ . Finding the smallest, non-negative valid value for each  $M_{j'l'}$ ,  $l' = 1, \dots, n_j$  can be represented as the optimization problem:

$$\begin{aligned}
& \min M_{j l'} \\
& \text{subject to } x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) \geq c_{j l'} x_j - (1 - \lambda_{j l'}) M_{j l'} \\
& M_{j l'} \geq 0
\end{aligned} \tag{99}$$

The set partitioning constraints require that exactly one of the variables  $\lambda_{jl}$  takes value 1 while the others take value 0. Let  $\lambda_{j l'} = 1$ . The left-hand side in (99) becomes:  $x_j \left( \sum_{l=1}^{n_j} \lambda_{jl} c_{jl} \right) = x_j c_{j l'}$  and is lower-bounded by  $x_j c_{j1}$ . The following cases can occur:

1. If  $\lambda_{j l'} = \lambda_{j1} = 1$ , then  $M_{j l'}$  is unrestricted by (99) and can take value 0;
2. If  $\lambda_{j l'} \neq \lambda_{j1}$  ( $\lambda_{j1} = 0$ ) and  $c_{j l'} > c_{j1}$ , then  $M_{j l'}$  is unrestricted by (99) and can take value 0;
3. If  $\lambda_{j l'} \neq \lambda_{j1}$  ( $\lambda_{j1} = 0$ ),  $c_{j l'} < c_{j1}$ , and  $x_j = 0$ , then  $M_{j l'}$  is unrestricted by (99) and can take value 0;
4. If  $\lambda_{j l'} \neq \lambda_{j1}$  ( $\lambda_{j1} = 0$ ),  $c_{j l'} < c_{j1}$ , and  $x_j \neq 0$  (and thus  $x_j = 1$ ), then (99) is binding and restricts the value that can be assigned to  $M_{j l'}$ . In that case, (99) becomes  $c_{j l'} \geq c_{j1} - M_{j l'}$ , which requires:

$$M_{j l'} \geq c_{j1} - c_{j l'} .$$

The right-hand side of the above inequality is minimal when  $l' = 1$  and  $c_{j l'} = c_{j1}$ . Hence, we have  $M_{j l'} \geq c_{j1} - c_{j1}$ , as we set out to prove.  $\square$

To generate the problem instances, we use the approach described by Beraldi et al. [4] and decompose a planning horizon into a finite number of periods. The budget available is known for the first period, while it is stochastic for the next ones. We consider a number of potential projects ranging between 10 and 20. Such numbers of projects are typical for the capital rationing problem, as the project selection is typically preceded by a screening phase in which homogeneous groups of projects are defined and are later on subjected to a joint evaluation [7]. The project outflows  $\xi_{ij}$  and project values  $V_j$  have been randomly generated from a uniform distribution (see [4, 35]). We also generate the available budgets from a uniform distribution defined on  $[0.2C_i, C_i]$ , where  $C_i$  is the average (over all potential projects) cash outflow at period  $i$ .

We create 60 types of problem instances characterized by the tuple  $(|\Omega|, p, r, |J|, U)$ . The instances differ in terms of the number of realizations ( $|\Omega|=10000, 15000$ ), the number of periods ( $r=1, 2, 3$ ) in the planning horizon, the number of considered projects ( $|J|=10, 20$ ), and the enforced probability level ( $p=0.85, 0.90, 0.95$ ), and the interval  $U$  on which the uniformly distributed project outflows  $\xi_{ij}$  are defined. We consider two intervals for the project outflows:  $[300, 600]$  and  $[3000, 4000]$ . For each instance type, we generate ten problem instances and solve the 600 problem instances with four algorithms proposed in this paper and with three other formulations and methods (see Section 5.2). The binarization process employed for deriving the proposed MIP formulations is implemented in Matlab. The AMPL modeling language is used to formulate the mathematical programming problems. The mixed-integer programming

formulations are solved with the Cplex 12.5 solver and the mixed-integer nonlinear programming ones (with nonconvex continuous relaxations) are solved with the Couenne solver. Each problem instance is solved on a 64-bit Dell Precision T5400 Workstation with Quad Core Xeon Processor X5460 3.16GHz CPU, and 4X2GB of RAM. The next subsection analyzes the computational tractability of the several equivalent problem formulations proposed in Section 4.2.

## 5.2 Comparative Analysis of Equivalent Reformulations and Algorithmic Techniques

As mentioned above, we have created 600 problem instances that we solve with:

- The four methods and formulations presented in this study:
  1. the MINLP solver Couenne [2] to solve the bilinear integer problem **QDE** with nonconvex continuous relaxation;
  2. the Cplex 12.5 standard branch-and-bound algorithm to solve the MIP problem **MIP1**;
  3. the Cplex 12.5 standard branch-and-bound algorithm supplemented by the valid inequalities (84) to solve the MIP problem **MIP2**;
  4. a new branch-and-cut algorithm in which, at each node of the tree, the valid inequalities (84) violated by the optimal solution of the incumbent continuous relaxation are introduced in the formulation of the problem. At each node of the tree, all the valid inequalities violated by the solution of the current node's continuous relaxation are added at once.

The above four methods are thereafter referred to with the abbreviations **QDE**, **MIP1**, **MIP2-B&B**, and **MIP2-B&C**.

- The Cplex 12.5 solver applied to the basic MIP formulation that associates a binary variable to each realization. We refer to it with the abbreviation **BIP**;
- The two most recent proposed solution techniques for problems of form **PMRTM**:
  1. the specialized branch-and-bound algorithm proposed by Beraldi and Bruni [3];
  2. the algorithm based on the concepts of weakly and strongly feasible solutions proposed by Beraldi et al. [4].

The above two are referred to with the abbreviations **BB** and **BBV**.

The results of the experiments are presented in Tables 3 and 4. The first five columns in Table 3 describe the problem instances. Column 1 (resp., 2, 3, 4, and 5) indicates the number of scenarios (resp., reliability level, number of periods, number of projects, and the interval on which the random variables can take value). Columns 6 to 8 provide the average computational time with the algorithms **MIP1**, **MIP2-B&B**, and **MIP2-B&C** for each of the 30 types of problem instances in which 10000 scenarios

are considered. The last column provides the average number  $\bar{n}$  of cut points – equal to the number of binary variables necessary for the reformulation of the probabilistic constraint – in the mixed-integer reformulations. Table 3 has the same structure and contains the results for the 30 types of instances in which 15000 scenarios are considered.

Table 3: Average Computational Times for each Algorithm and Number of Cut Points when  $|\Omega| = 10000$

Families of Problem Instances ( $ \Omega , p, r,  J , U$ )					Average Computational Times			Cut Points
$ \Omega $	$p$	$r$	$ J $	$U$	<b>MIP1</b>	<b>MIP2-B&amp;B</b>	<b>MIP2-B&amp;C</b>	$\bar{n}$
10000	0.95	1	20	[300, 600]	0.02	0.02	0.02	30.1
10000	0.9	1	20	[300, 600]	6.42	5.2	5.98	60.9
10000	0.85	1	20	[300, 600]	10.04	8.15	7.62	88.4
10000	0.95	2	10	[300, 600]	0.03	0.02	0.02	32.1
10000	0.9	2	10	[300, 600]	5.24	5.15	5.33	62.1
10000	0.85	2	10	[300, 600]	11.56	7.81	8.42	94.5
10000	0.95	2	20	[300, 600]	0.09	0.08	0.09	60.1
10000	0.9	2	20	[300, 600]	50.08	31.12	44.87	115.2
10000	0.85	2	20	[300, 600]	107.13	87.29	84.21	176.7
10000	0.95	3	10	[300, 600]	0.02	0.02	0.02	31.8
10000	0.9	3	10	[300, 600]	6.79	6.57	6.69	64.7
10000	0.85	3	10	[300, 600]	9.4	10.02	11.16	96.1
10000	0.95	3	20	[300, 600]	0.16	0.19	0.19	92.1
10000	0.9	3	20	[300, 600]	121.32	84.23	97.09	179.7
10000	0.85	3	20	[300, 600]	143.31	115.14	112.71	265.1
10000	0.95	1	20	[3000, 4000]	4.46	2.96	3.3	102.2
10000	0.9	1	20	[3000, 4000]	52.21	40.12	37.89	200.1
10000	0.85	1	20	[3000, 4000]	98.04	64.89	70.28	296.2
10000	0.95	2	10	[3000, 4000]	8.81	8.24	7.98	107.6
10000	0.9	2	10	[3000, 4000]	87.29	71.23	74.52	208.5
10000	0.85	2	10	[3000, 4000]	103.57	90.33	88.74	313.9
10000	0.95	2	20	[3000, 4000]	9.36	7.85	10.82	207.1
10000	0.9	2	20	[3000, 4000]	153.39	96.98	102.36	400.7
10000	0.85	2	20	[3000, 4000]	201.23	142.92	163.68	608.8
10000	0.95	3	10	[3000, 4000]	4.05	4.23	4.52	161.4
10000	0.9	3	10	[3000, 4000]	102.36	74.63	72.06	312.3
10000	0.85	3	10	[3000, 4000]	157.94	98.36	103.14	471.8
10000	0.95	3	20	[3000, 4000]	13.72	12.26	14.63	309.9
10000	0.9	3	20	[3000, 4000]	160.45	107.18	104.97	621.3
10000	0.85	3	20	[3000, 4000]	229.49	163.04	205.74	894.2

The three algorithms **MIP1**, **MIP2-B&B**, and **MIP2-B&C** solve to optimality all the 600 problem instances. To distinguish among these three algorithms, we focus on the 24 problem instance types which require an average computing time of at least one minute. The **MIP2-B&B** (resp., **MIP2-B&C**) algorithm is the fastest for 17 instance types (resp., 7). Figure 1 displays the average computational

Table 4: Average Computational Times for each Algorithm and Number of Cut Points when  $|\Omega| = 15000$

Families of Problem Instances ( $ \Omega , p, r,  J , U$ )					Average Computational Times			Cut Points
$ \Omega $	$p$	$r$	$ J $	$U$	<b>MIP1</b>	<b>MIP2-B&amp;B</b>	<b>MIP2-B&amp;C</b>	$\bar{n}$
15000	0.95	1	20	[300, 600]	0.02	0.03	0.02	29.9
15000	0.9	1	20	[300, 600]	6.22	6.37	6.49	61.4
15000	0.85	1	20	[300, 600]	10.24	9.96	10.31	91.2
15000	0.95	2	10	[300, 600]	0.03	0.02	0.02	32.2
15000	0.9	2	10	[300, 600]	5.87	5.4	5.71	61.6
15000	0.85	2	10	[300, 600]	9.24	8.48	8.77	96.8
15000	0.95	2	20	[300, 600]	0.07	0.05	0.06	61
15000	0.9	2	20	[300, 600]	57.23	27.23	45.29	120.9
15000	0.85	2	20	[300, 600]	103.51	91.36	99.78	181.2
15000	0.95	3	10	[300, 600]	0.02	0.02	0.02	45.5
15000	0.9	3	10	[300, 600]	7.2	6.12	5.97	61.3
15000	0.85	3	10	[300, 600]	8.4	8.01	7.74	95.2
15000	0.95	3	20	[300, 600]	0.17	0.18	0.23	93.2
15000	0.9	3	20	[300, 600]	114.21	80.38	91.62	180.1
15000	0.85	3	20	[300, 600]	144.12	108.12	113.37	264.8
15000	0.95	1	20	[3000, 4000]	4.12	3.09	3.12	103.2
15000	0.9	1	20	[3000, 4000]	49.85	37.46	41.23	205.6
15000	0.85	1	20	[3000, 4000]	115.25	66.12	72.34	308.4
15000	0.95	2	10	[3000, 4000]	10.23	7.89	8.09	105.8
15000	0.9	2	10	[3000, 4000]	98.24	75.23	72.85	212.6
15000	0.85	2	10	[3000, 4000]	103.71	88.23	98.17	311.8
15000	0.95	2	20	[3000, 4000]	8.65	7.23	8.12	208
15000	0.9	2	20	[3000, 4000]	149.87	101.2	118.32	411.2
15000	0.85	2	20	[3000, 4000]	192.15	130.39	145.27	609.7
15000	0.95	3	10	[3000, 4000]	4.66	4.89	5.23	160.3
15000	0.9	3	10	[3000, 4000]	101.63	77.32	70.86	320.5
15000	0.85	3	10	[3000, 4000]	164.1	102.36	128.32	477.6
15000	0.95	3	20	[3000, 4000]	18.57	15.09	16.52	308.2
15000	0.9	3	20	[3000, 4000]	138.23	111.23	129.78	625.1
15000	0.85	3	20	[3000, 4000]	234.51	159.39	201.54	891.6

time for the 24 problem instances with computational time above 60 seconds and shows that the **MIP2-B&B** algorithm is on average more than 39.27% and 9.81% faster than the **MIP1** and **MIP2-B&C** methods. The **MIP2-B&C** method outpaces the **MIP1** method by 26.83%. The **MIP2-B&B** and **MIP2-B&C** methods which both use in some fashion the valid inequalities (84) are faster than the **MIP1** method, which points out the benefits of the valid inequalities (84). Among the two, the branch-and-bound algorithm **MIP2-B&B** outperforms the branch-and-cut algorithm **MIP2-B&C** for the instances considered in this study.

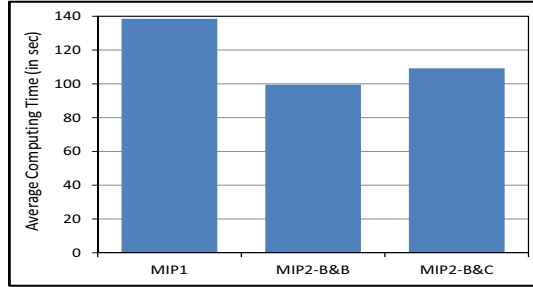


Figure 1: Average Computational Time for Instances Requiring more than 60 Seconds

The comparison of Tables 3 and 4 which respectively concern the instances with 10000 and 15000 scenarios shows clearly that the computing time does not increase with the number of scenarios used to represent uncertainty. This is highlighted by Figure 2 which displays the average computational time of the **MIP1**, **MIP2-B&B**, and **MIP2-B&C** algorithms for two of the challenging instance types  $((|\Omega|, p, r, |J|, U) = (10000, 0.85, 3, 20, U_2)$  and  $(15000, 0.85, 3, 20, U_2)$ ) that only differ in the considered number of scenarios (10000 vs. 15000). It can be seen that the average solution times with the **MIP2-B&B**, and **MIP2-B&C** algorithms are smaller for the instances including 15000 scenarios than for those including 10000 scenarios. This is due to the fact that the number of binary variables in the MIP reformulations is equal to the number of cut points and that the latter is not an increasing function of the number of scenarios. Table 5 focuses on the instances  $(10000, p, 3, 20, U_2)$  and  $(15000, p, 3, 20, U_2)$ . The average number of cut points in the instances of type  $(15000, 0.85, 3, 20, U_2)$  and  $(15000, 0.95, 3, 20, U_2)$  is marginally smaller than the average number of cut points in the instances of type  $(10000, 0.85, 3, 20, U_2)$  and  $(10000, 0.95, 3, 20, U_2)$ . When  $p$  is equal to 0.9, the average number of cut points is smaller when 10000 scenarios are considered.

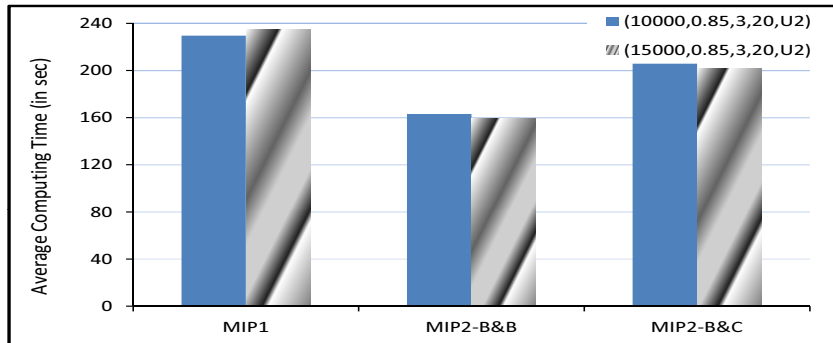


Figure 2: Average Computational Time as a Function of Number of Scenarios

Finally, we also notice that the computational time is a decreasing function of the enforced reliability level  $p$  and an increasing function of the interval on which the random variables can take value. As the value of  $p$  decreases and/or the range  $U$  increases, the number of cut points needed to binarize the probability distribution and therefore the number of binary variables in **MIP1** and **MIP2** increases. The computing time is also an increasing function of the size of the random vector. Not surprisingly, the

Table 5: Average Number of Cut Points and Binary Variables as a Function of Number of Scenarios

$\Omega$	Average Number of Cut Points and Binary Variables		
	$p=0.95$	$p=0.9$	$p=0.85$
10000	309.9	621.3	894.2
15000	308.2	625.1	891.6

MINLP formulation **QDE** takes much more time to solve than its three MIP counterparts. The **QDE** problems are solved with the MINLP solver Couenne and only two instance types can be solved to optimality within one hour of computing time. The average computational times for these four types of instances (10000, 0.95, 2, 10,  $U_1$ ) and (15000, 0.95, 2, 10,  $U_1$ ), are respectively equal to 2476.21 and 1508.2 seconds. This is why we have not included any computational time for **QDE** in Tables 3 and 4.

None of the instances could be solved to optimality by our implementation of the algorithms **BB** and **BBV** proposed in [3] and [4]. Note that the algorithms **BB** and **BBV** were respectively tested by their authors (see [3] and [4]) on problem instances comprising up to 500 and 200 realizations, while the instances considered in this paper have no less than 10000 realizations.

Similarly, the basic MIP formulation **BI** could not prove optimality, nor reach the optimal solution for any of the instances in one hour of CPU time. For example, for the problem instances of types (10000, 0.95, 2, 20,  $U_1$ ), (10000, 0.9, 2, 20,  $U_1$ ), and (10000, 0.65, 2, 20,  $U_1$ ) for which optimality is proven relatively easily with the proposed MIP formulations (see Table 3), the average relative MIP gap obtained after one hour of computing time with the **BI** formulation is respectively equal to 21.50%, 24.12%, and 26.89%.

## 6 Conclusion

We develop a new modeling and exact solution method for stochastic programming problems that include a joint probabilistic constraint in which the variables in the multirow random technology matrix are discretely distributed. We binarize the probability distribution of the random variables in such a way that we can extract a threshold partially defined Boolean function (pdBf) representing the probabilistic constraint. We then construct a tight threshold Boolean minorant for the pdBf. Any separating structure of the tight threshold Boolean minorant defines sufficient conditions for the satisfaction of the probabilistic constraint and takes the form of a system of linear constraints. We use the separating structure to derive three new deterministic formulations equivalent to the studied stochastic problem. We derive a set of strengthening valid inequalities for the reformulated problems. A crucial feature of the new integer formulations is that the number of integer variables does not depend on the number of scenarios used to represent uncertainty. The proposed reformulation method makes it possible to solve very efficiently probabilistically constrained stochastic programming problems with multirow random technology matrix. The computational study, based on instances of the stochastic capital rationing problem, shows that the MIP reformulations are much easier and orders of magnitude faster to solve than the MINLP



formulation. The method integrating the derived valid inequalities in a branch-and-bound algorithm performs best.

The proposed method and the one presented in [19] have similarities. Both are based on the binarization of the probability distribution of the random variables using a consistent set of cut points, model a chance constraint as a pdBf, and ultimately provide equivalent integer programming formulations in which the number of binary variables is typically orders of magnitude smaller than the number of scenarios. While germane to Lejeune's method [19] for joint probabilistic constraint with random right-hand sides, the present approach constitutes a major extension as it permits to solve efficiently more complex and more general probabilistic programming problems with random technology matrix. The method in [19] rests on the concept of  $p$ -pattern to model the satisfiability of a chance constraint with random right-hand sides and derive an MIP formulation equivalent to the stochastic programming problem. In this study, we define the concept of tight minorant of a threshold Boolean function to separate  $p$ -sufficient from  $p$ -insufficient recombinations and derive a set of mixed-integer quadratic inequalities to model the satisfiability of a chance constraint with random technology matrix. An additional step involving the linearization of the bilinear terms is needed to obtain several equivalent MIP reformulations.

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