Australian Journal of Basic and Applied Sciences, 2(1): 114-124, 2008
ISSN 1991-8178

Approximate Solution for the System of Non-linear Volterra Integral Equations of the Second Kind by using Block-by-block Method

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Abstract: The aim of this paper is for finding the numerical solution (sometimes exact) for non-linear system of Volterra integral equations of the second kind (NSVIEK2) by using block-by-block method. Which avoid the need for special starting procedures, but uses numerical quadrature rule. Also some illustrative examples are presented, to elucidate the accuracy of this method.

Key words: block-by-block, system of non-linear Volterra integral equation

INTRODUCTION

A block method is essentially an extrapolation procedure which has advantage of being self-starting and produces a block of values at a time (Delves and Mohamed (1985); Delves and Walsh (1974)).

Linz (1969) describes two block (block-by-block) method and used this method to solve Volterra integral equations of the second kind. Also AL-Asdi (2002) used two and three blocks for solving Hammersetien Volterra integral equations of the second kind, while Saify (2005) used two, three and four blocks for solving a system of linear Volterra integral equation of the second kind.

In this paper, the approaches of two and three blocks are reformulated and applied to find the numerical solution for a system of non-linear VIEK2's, in which a block of two and three values are produced at each stage, and these values are obtained using the two three-point quadrature formula.

The resulting system of non-linear equations from this approaches are solved by modified Newton-Raphson method (mNRm) (Kincaid and Cheney (2002).

Some Integration Formulas: (Gerald and Wheatley (1984); Kincaid and Cheney (2002)

In this section we used some quadrature formulas and quadratic interpolation polynomials such as trapezoidal, Simpson's 1/3 and Newton-Gregory forward methods.

The form of Newton-Gregory forward is:

$$p_n(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \ldots + \binom{s}{n}\Delta^n f_0 = \sum_{j=0}^{n}\binom{s}{j}\Delta^j f_0 \qquad (1)$$

where $s = \dfrac{x - x_0}{h}$, $\Delta^{n+1} f_i = \Delta^n f_{i+1} - \Delta^n f_i \,, n \geq 0$

Here, to derive other integration formulas we used the emblematic method in terms of the stepping operator E.

$$f(x_s) = f_s = E^s f_0 \; where \; s = \frac{x - x_0}{h}$$

Thus

$$\int_{x_0}^{x_1} f(x)dx = h\int_0^1 E^s f_0 dp = \frac{hE^s}{\ln(E)} f_0 \Bigg|_0^1$$

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$$\int_{x_0}^{x_1} f(x)dx = \frac{h(E-1)}{\ln(E)} f_0 \quad (2)$$

Since,

$$E=1+\Delta \quad (3)$$

we expand ln(1+ Δ) as a power series, we get:

$$\ln(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots \quad (4)$$

Substituting equation (3) and (4) into the equation (2) to obtain:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h\Delta}{\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4} f_0$$

Using division we get:

$$\int_{x_0}^{x_1} f(x)\,dx = h\,(f_0 + \frac{1}{2}\Delta f_0 - \frac{1}{12}\Delta^2 f_0 + \frac{1}{24}\Delta^3 f_0 - \ldots) \quad (5)$$

When n terms are used, it represents a polynomial of degree n, fitting from x_0 to x_n but the integrated only from x_0 to x_1.
First, using three terms of equation (4.5) we obtain:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{12}\left[5f_0 + 8f_1 - f_2\right] + \frac{h^4}{24} f_0^3(\varepsilon)\ ,\ \varepsilon \in (x_0, x_1) \quad (6)$$

Second, using four terms of equation (4.5) we obtain:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{24}\left[9f_0 + 19f_1 - 5f_2 + f_3\right] + \frac{h5}{720} f_0^{(4)}(\varepsilon) \quad (7)$$

Moreover, the following formula is derived by using adaptive Simpson's 1/3 rule.

$$\int_{x_0}^{x_1} f(x)dx = S(x_0, x_1) = \frac{h}{6}\left[f_0 + 4f_{1/2} + f_1\right] \quad (8)$$

where $f_{1/2}$ can be found as follows:

In the equation (1), putting s=1/2 and n=2, we get:

$$p_2(1/2) = f_{1/2} = \frac{1}{8}\left[3f_0 + 6f_1 - f_2\right] \quad (9)$$

Block-by-block Method:

The basic interval [a, b] is divided into steps of width h, such as $x_j = jh$, $j = 0,1,.....n$ and $nh = b - a$. The approximate solution of $u_i(x)$ will be defined at mesh-points x_j and denoted by u_{ij}; $j = 0,1,....,n$ such as u_{ij} is an approximation to $u_i(x_j)$.

For solving system of non-linear Volterra integral equations

$$U(x) = F(x) + \int_0^x K(x,t,U(t))dt \tag{10}$$

where

$$U(x) = (u_1(x),\ \dots,\ u_m(t))^T, U(t) = (u_1(t),\ \dots,\ u_m(t)),$$
$$F(x) = (f_1(x),\ \dots,\ f_m(x))^T, K(x,t,U(t)) = (k_1(x,t,U(t),\ \dots,\ k_m(x,t,U(t)))^T,$$

Rewrite equation (10) as follows:

$$u_i(x_k) = f_i(x_k) + \int_a^{x_{pl}} k_i(x_k,t,U(t))dt + \int_{x_{pl}}^{x_k} k_i(x_k,t,U(t))dt\ , \tag{11}$$

where p is some integer and l is $\left[\left|\frac{k}{p}\right|\right]$. If the values u_{i0}, u_{i1}, ..., $u_{i,pl}$ are known, then the first integral can be approximated by standard quadrature methods. The second integral is estimated by a quadrature rule using values of the integrand at $t = x_{pl}, x_{pl+1}, \dots, x_{p(l+1)}$. Since the values of u_i at these points are unknown, we have a system of mp non-linear simultaneous equations

$$u_{ik} = f_i(x_k) + h\sum_{j=0}^{lp} w_{kj}k_i(x_k,t_j,u_{1j},\dots,u_{mj}) + h\sum_{j=0}^{p} w'_{kj}k_i(x_k,t_{lp+j},u_{1,lp+j},\dots,u_{m,lp+j}) \tag{12}$$

for $k = lp+1,\ lp+2,\ \dots,\ (l+1)p$, where w_{kj}, w'_{kj} depend on the quadrature rule used. For sufficiently small h the system we obtain from equation (12) has a unique solution which can be determined by iteration such as modified Newton-Raphson method. Thus, a 'block' of p values of u_i is obtained simultaneously.

Modified Method of Two Blocks:

For this method we take $p = 2$, the integration over $[a,\ x_{2l}]$ can be accomplished by Simpson's rule, and the integral over $[x_{2l}, x_k]$ by using a quadratic interpolation of the integrand at the point $x_{2l}, x_{2l+1}, x_{2l+2}$ Then equation (10) becomes:

$$u_{i,2l+1} = f_i(x_{2l+1}) + \int_a^{(2l+1)h} k_i(x_{2l+1},t,U(t))dt \tag{13}$$

and

$$u_{i,2l+2} = f_i(x_{2l+2}) + \int_a^{(2l+2)h} k_i(x_{2l+2},t,U(t))dt \tag{14}$$

where, $i = 1,2,\ldots,m. l = 0,1,\ldots$.

Or from equation (11), equation (13), (14) can be written as:

$$u_{i,2l+1} = f_i(x_{2l+1}) + \int_a^{2lh} k_i(x_{2l+1}, t, U(t))dt + \int_{2lh}^{(2l+1)h} k_i(x_{2l+1}, t, U(t))dt$$

and

$$u_{i,2l+2} = f_i(x_{2l+2}) + \int_a^{2lh} k_i(x_{2l+2}, t, U(t))dt + \int_{2lh}^{(2l+2)h} k_i(x_{2l+2}, t, U(t))dt$$

Therefore, by equation (6) the approximate solution is computed by:

$$u_{i,2l+1} = f_i(x_{2l+1}) + \frac{h}{3}\sum_{j=0}^{2l} w_j k_i(x_{2l+1}\ t_j, u_{1j}, \ldots, u_{mj})$$
$$+\frac{h}{12}\Big[5k_i(x_{2l+1}, t_{2l}, u_{1,2l}, \ldots, u_{m,2l}) + 8k_i(x_{2l+1}, t_{2l+1}, u_{1,2l+1}, \ldots, u_{m,2l+1})$$
$$-k_i(x_{2l+1}, t_{2l+2}, u_{1,2l+2}, \ldots, u_{m,2l+2})\Big] \quad (15)$$

$$u_{i,2l+2} = f_i(x_{2l+2}) + \frac{h}{3}\sum_{j=0}^{2l+2} \bar{w}_j k_i(x_{2l+2}, t_j, u_{1j}, \ldots, u_{mj}) \quad (16)$$

It should be noticed that in the equation (15) the kernel has to be evaluated at the point $\big[(2l+1),(2l+2), u_{i,2l+2}\big]$.

Thus, replace the second term in equation (15) by formula (8) and (9). Then the resulting equations are:

$$u_{i,2l+1} = f_i(x_{2l+1}) + \frac{h}{3}\sum_{j=0}^{2l} w_j k_i(x_{2l+1}, t_j, u_{1j}, \ldots, u_{mj})$$
$$+\frac{h}{6}\Big[k_i(x_{2l+1}, t_{2l}, u_{1,2l}, \ldots, u_{m,2l}) + 4k_i(x_{2l+1}, t_{2l+0.5}, (\frac{3}{8}u_{1,2l} + \frac{3}{4}u_{1,2l+1} - \frac{1}{8}u_{1,2l+2})$$
$$,\ldots,(\frac{3}{8}u_{m,2l} + \frac{3}{4}u_{m,2l+1} - \frac{1}{8}u_{m,2l+2})) + k_i(x_{2l+1}, t_{2l+1}, u_{1,2l+1}, \ldots, u_{m,2l+1})\Big] \quad (17)$$

$$u_{i,2l+2} = f_i(x_{2l+2}) + \frac{h}{3}\sum_{j=0}^{2l+2} \bar{w}_j k_i(x_{2l+2}, t_j, u_{1j}, \ldots, u_{mj}) \quad (18)$$

where

$$w_0 = w_{2l} = 1, w_j = 3-(-1)^j, j = 1, 2, \ldots, 2l-1$$

$$\bar{w}_0 = \bar{w}_{2l+2} = 1, \bar{w}_j = 3-(-1)^j, j = 1, 2, \ldots, 2l+1,$$
$$i = 1,2,\ldots,m, l = 0,1,\ldots$$

At each step we construct $2m$ non-linear simultaneous equations from (17) and (18) to find the unknowns u_{i2l+1} and u_{i2l+2}. Solve the resulting system of non-linear equations by using modified Newton-Raphson method.

Algorithm of MBLM2:

Step (1): Fix $u_{i0}(a) = f_i(a), i = 1, 2, \ldots, m$

Step (2): Letting $h = \frac{b-a}{n}, n \in N$.

Step (3): Calculate $f_i(x_{2l+1})\, and\, f_i(x_{2l+2})\, for\, i = 1, 2, \ldots, m$.

Step (4): Using equation (17) and (18) to find system of equation for the unknown's u_{i2l+1} and u_{i2l+2}.

Step (5): Find the value of u_{i2l+1} and u_{i2l+2} by using mNRm.

Step (6): Repeat steps (3)-(4) for l=1, 2, ...

Modified Method of Three Blocks:

The method of two blocks which presented in section 3.1 can be extended to produce $3m$ simultaneous non-linear equations at each step since in this method we have $p = 3$.

Then through the use of equation (11) equation (10) can be written as:

$$u_{i,3l+1} = f_i(x_{3l+1}) + \int_a^{3lh} k_i(x_{3l+1}, t, U(t))dt + \int_{3lh}^{(3l+1)h} k_i(x_{3l+1}, t, U(t))dt$$

$$u_{i,3l+2} = f_i(x_{3l+2}) + \int_a^{3lh} k_i(x_{3l+2}, t, U(t))dt + \int_{3lh}^{(3l+2)h} k_i(x_{3l+2}, t, U(t))dt$$

$$u_{i,3l+3} = f_i(x_{3l+3}) + \int_a^{3lh} k_i(x_{3l+3}, t, U(t))dt + \int_{3lh}^{(3l+3)h} k_i(x_{3l+3}, t, U(t))dt$$

In practice this method depends on the use of three quadrature formula, Simpson's 1/3 rule, trapezoidal rule and some quadrature interpolation formula.

Therefore, the approximate solution is computed as follows:

If l is even

$$u_{i,3l+1} = f_i(x_{3l+1}) + \frac{h}{3}\sum_{j=0}^{3l} w_j k_i(x_{3l+1}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{2}\sum_{j=3l}^{3l+1} k_i(x_{3l+1}, t_j, u_{1j}, \ldots, u_{mj}) \quad (19)$$

$$u_{i,3l+2} = f_i(x_{3l+2}) + \frac{h}{3}\sum_{j=0}^{3l} w_j k_i(x_{3l+2}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{2}\sum_{j=3l}^{3l+1} k_i(x_{3l+2}, t_j, u_{1j}, \ldots, u_{mj})$$
$$+\frac{h}{12}\Big[5k_i(x_{3l+2}, t_{3l+1}, u_{1,3l+1}, \ldots, u_{m,3l+1}) + 8k_i(x_{3l+2}, t_{3l+2}, u_{1,3l+2}, \ldots, u_{m,3l+2})$$
$$-k_i(x_{3l+2}, t_{3l+3}, u_{1,3l+3}, \ldots, u_{m3l+3})\Big] \quad (20)$$

$$u_{i,3l+3} = f_i(x_{3l+3}) + \frac{h}{3}\sum_{j=0}^{3l+2} \overline{w}_j k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{2}\sum_{j=3l+2}^{3l+3} k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) \quad (21)$$

If l is odd

$$u_{i,3l+1} = f_i(x_{3l+1}) + \frac{h}{3}\sum_{j=0}^{3l+1} Z_j k_i(x_{3l+1},\, t_j,\, u_{1j},\, \ldots,\, u_{mj}) \tag{22}$$

$$u_{i,3l+2} = f_i(x_{3l+2}) + \frac{h}{3}\sum_{j=0}^{3l+1} Z_j k_i(x_{3l+2},\, t_j,\, u_{1j},\, \ldots,\, u_{mj})$$
$$+\frac{h}{12}\Big[5k_i(x_{3l+2},\, t_{3l+1},\, u_{1,3l+1},\, \ldots,\, u_{m,3l+1}) + 8k_i(x_{3l+2},\, t_{3l+2},\, u_{1,3l+2},\, \ldots,\, u_{m,3l+2})$$
$$-k_i(x_{3l+2}, t_{3l+3}, u_{1,3l+3}, \ldots, u_{m3l+3})\Big] \tag{23}$$

$$u_{i,3l+3} = f_i(x_{3l+3}) + \frac{h}{3}\sum_{j=0}^{3l+3} \bar{Z}_j k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) \tag{24}$$

Setting s=1/2, n=3 and using equation (1) and (8) the last term in equations (19), (20), (22) and (23) becomes:

$$\int_{x_{3l+q}}^{x_{3l+q+1}} f(x)dx = \frac{h}{6}\Bigg[k_i(x_{3l+(q+1)}, t_{3l+q}, u_{1,3l+q}, \ldots, u_{m,3l+q}) + 4k_i\Bigg(x_{3l+(q+1)}, t_{3l+(q+0.5)}, \frac{1}{16}(5u_{1,3l}$$
$$+15u_{1,3l+1} - 5u_{1,3l+2} + u_{1,3l+3}), \ldots, \frac{1}{16}(5u_{m,3l} + 15u_{m,3l+1} - 5u_{m,3l+2} + u_{m,3l+3})\Bigg)$$
$$+k_i(x_{3l+(q+1)}, t_{3l+(q+1)}, u_{1,3l+(q+1)}, \ldots, u_{m,3l+(q+1)})\Bigg], \text{ where } q=0,\ 1 \tag{25}$$

The resulting equations (19)-(24) become:
If l is even

$$u_{i,3l+1} = f_i(x_{3l+1}) + \frac{h}{3}\sum_{j=0}^{3l} w_j k_i(x_{3l+1},\, t_j,\, u_{1j},\, \ldots,\, u_{mj}) + \frac{h}{6}\Big[k_i(x_{3l+1},\, t_{3l},\, u_{1,3l},\, \ldots,\, u_{m,3l})$$
$$+4k_i\Big(x_{3l+1}, t_{3l+0.5}, \frac{1}{16}(5u_{1,3l} + 15u_{1,3l+1} - 5u_{1,3l+2} + u_{1,3l+3}), \ldots, \frac{1}{16}(5u_{m,3l}$$
$$+15u_{m,3l+1} - 5u_{m,3l+2} + u_{m,3l+3})\Big) + k_i(x_{3l+1}, t_{3l+1}, u_{1,3l+1}, \ldots, u_{m,3l+1})\Big]. \tag{26}$$

$$u_{i,3l+2} = f_i(x_{3l+2}) + \frac{h}{3}\sum_{j=0}^{3l} w_j k_i(x_{3l+2}, t_j,\, u_{1j},\, \ldots,\, u_{mj}) + \frac{h}{2}\sum_{j=3l}^{3l+1} k_i(x_{3l+2},\, t_j,\, u_{1j},\, \ldots,\, u_{mj})$$
$$+\frac{h}{6}\Big[k_i(x_{3l+2},\, t_{3l+1},\, u_{1,3l+1},\, \ldots,\, u_{m,3l+1}) + 4k_i(x_{3l+2},\, t_{3l+1.5}, \frac{1}{16}(5u_{1,3l} + 15u_{1,3l+1}$$
$$-5u_{1,3l+2} + u_{1,3l+3}), \ldots, \frac{1}{16}(5u_{m,3l} + 15u_{m,3l+1} - 5u_{m,3l+2} + u_{m,3l+3}))$$
$$+k_i(x_{3l+2}, t_{3l+2}, u_{1,3l+2}, \ldots, u_{m,3l+2})\Big] \tag{27}$$

$$u_{i,3l+3} = f_i(x_{3l+3}) + \frac{h}{3}\sum_{j=0}^{3l+2} \overline{w}_j k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{2}\sum_{j=3l+2}^{3l+3} k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) \quad (28)$$

If l is odd

$$u_{i,3l+1} = f_i(x_{3l+1}) + \frac{h}{2}\sum_{j=0}^{3l} w_j k_i(x_{3l+1}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{6}\Big[k_i(x_{3l+1}, t_{3l}, u_{1,3l}, \ldots, u_{m,3l}) + 4k_i\Big(x_{3l+1}, t_{3l+0.5}, \frac{1}{16}(5u_{1,3l} + 15u_{1,3l+1} - 5u_{1,3l+2} + u_{1,3l+3}), \ldots, \frac{1}{16}(5u_{m,3l} + 15u_{m,3l+1} - 5u_{m,3l+2} + u_{m,3l+3})\Big) + k_i(x_{3l+1}, t_{3l+1}, u_{1,3l+1}, \ldots, u_{m,3l+1})\Big] \quad (29)$$

$$u_{i,3l+2} = f_i(x_{3l+2}) + \frac{h}{3}\sum_{j=0}^{3l+1} Z_j k_i(x_{3l+2}, t_j, u_{1j}, \ldots, u_{mj}) + \frac{h}{6}\Big[k_i(x_{3l+2}, t_{3l+1}, u_{1,3l+1}, \ldots, u_{m,3l+1}) + 4k_i\Big(x_{3l+2}, t_{3l+1.5}, \frac{1}{16}(5u_{1,3l} + 15u_{1,3l+1} - 5u_{1,3l+2} + u_{1,3l+3}), \ldots, \frac{1}{16}(5u_{m,3l} + 15u_{m,3l+1} - 5u_{m,3l+2} + u_{m,3l+3})\Big) + k_i(x_{3l+2}, t_{3l+2}, u_{1,3l+2}, \ldots, u_{m,3l+2})\Big] \quad (30)$$

$$u_{i,3l+3} = f_i(x_{3l+3}) + \frac{h}{3}\sum_{j=0}^{3l+3} \overline{Z}_j k_i(x_{3l+3}, t_j, u_{1j}, \ldots, u_{mj}) \quad (31)$$

where $w_0 = w_{3l} = 1, w_j = 3-(-1)^j, j = 1, 2, \ldots, 3l-1$

$\overline{w}_0 = \overline{w}_{3l+2} = 1, \overline{w}_0 = \overline{w}_{3l+2} = 1, j = 1, 2, \ldots, 3l+1$

$Z_0 = Z_{3l+1} = 1, Z_j = 3-(-1)^j, j = 1, 2, \ldots, 3l$

$\overline{Z}_0 = \overline{Z}_{3l+3} = 1, \overline{Z}_j = 3-(-1)^j, j = 1, 2, \ldots, 3l+2$

$i = 1, 2, \ldots, m, l = 0, 1, \ldots$

Therefore, at each step we construct $3m$ simultaneously non-linear equations from the equation (26)-(31) which can be solved for the unknown's $u_{i,3l+1}$, $u_{i,3l+2}$ and $u_{i,3l+3}$, $i = 1, 2, \ldots, m$ by using modified Newton-Raphson method.

Algorithm of MBLM3:

Step (1): Fix $u_{i0}(a) = f_i(a), i = 1, 2, \ldots, m$,.

Step (2): Letting $h = \frac{b-a}{n}, n \in N$

Step (3): If l is even then, calculate $f_i(x_{3l+1}), f_i(x_{3l+2}), and\ f_i(x_{2l+3}), for\ i = 1, 2, \ldots, m$.

Step (4): Using equation (26), (27) and (28) to find system of equations for the unknown's $u_{i,3l+1}$, $u_{i,3l+2}$ and $u_{i,3l+3}$.

Step (5): Find the value of $u_{i,3l+1}$, $u_{i,3l+2}$ and $u_{i,3l+3}$ by using mNRm.

Step (6): If l is odd then, calculate, $f_i(x_{3l+1}), f_i(x_{3l+2}), and\ f_i(x_{2l+3}), for\ i = 1,2,\ldots,m$.

Step (7): Using equation (29), (30) and (31) to find system of equations for the unknown's $u_{i,3l+1}$, $u_{i,3l+2}$ and $u_{i,3l+3}$.

Step (8): Find the value of $u_{i,3l+1}$, $u_{i,3l+2}$ and $u_{i,3l+3}$ by using mNRM.

Step (9): Repeat steps (3)-(8) for l=1, 2, ...

Illustrative Examples:

In this section, three examples are presented for demonstrating the methods and a comparison among the solutions obtained by these methods against the exact solution which has been made depending on the least square errors.

Example 1: (Babolian and Biazar (2000))
Solve a system of non-linear VIEK2's:

$$u_1(x) = x - x^2 + \int_0^x (u_1(t) + u_2(t))dt$$

$$u_2(x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \int_0^x (u_1^2(t) + u_2(t))dt$$

The exact solution of this system is: $u_1(x) = x$ and $u_2(x) = x$.

After solving this system using block-by-block methods with h=0.1 in equation (17), (18) and (26)-(31), we obtain the following numerical results.

Table 1: Comparison between the exact solution x and the numerical solution $u_1(x)$ and $u_2(x)$ of Example 1 taking h=0.1.

x	Exact solution	$u_1(x)$		$u_2(x)$	
		MBLM2	MBLM3	MBLM2	MBLM3
0	0	0	0	0	0
0.1	0.1	0.100000000	0.100833229	0.100000000	0.100357773
0.2	0.2	0.200000000	0.176892384	0.200000000	0.186656903
0.3	0.3	0.300000000	0.296875864	0.300000000	0.298216594
0.4	0.4	0.400000000	0.378635043	0.400000000	0.386304373
0.5	0.5	0.500000000	0.479765214	0.500000000	0.482703968
0.6	0.6	0.600000000	0.589047425	0.600000000	0.590387335
0.7	0.7	0.700000000	0.687421316	0.700000000	0.688497387
0.8	0.8	0.800000000	0.771205066	0.800000000	0.769194326
0.9	0.9	0.900000000	0.877902491	0.900000000	0.876460146
1	1	1	0.908963315	1	0.906343628
L.S.E.		0	1.1293×10^{-02}	0	1.1167×10^{-02}

Table 2: Shows the least square errors for $u_1(x)$ and $u_2(x)$ with different values of h for Example 1.

Numerical solution of	methods	least square errors		
		h=0.1	h=0.05	h=0.025
$u_1(x)$	MBLM2	0	0	0
	MBLM3	1.1293 X 10^{-02}	3.7858 X 10^{-04}	1.9642 X 10^{-05}
$u_2(x)$	MBLM2	0	0	0
	MBLM3	1.1167 X 10^{-02}	1.8703 X 10^{-04}	6.8744 X 10^{-0}

Example 2: (Jumaa (2005))

Solve a system of non-linear VIEK2's:

$$u_1(x) = \sec(x) - x + \int_0^x \left((u_1(t))^2 - (u_2(t))^2 \right) dt$$

$$u_2(x) = 3\tan(x) - x - \int_0^x \left((u_1(t))^2 + (u_2(t))^2 \right) dt$$

The exact solution of this system is:

$$u_1(x) = \sec(x) \, and \, u_2(x) = \tan(x)$$

After solving this system using block-by-block methods with h=0.1 in equation (17), (18) and (26)-(31), we obtain the following numerical solution.

Table 3: Comparison between the exact solution and the numerical solution $u_1(x)$ of Example 2 taking h=0.1.

x	$u_1(x)$		
	Exact solution	MBLM2	MBLM3
0	1	1	1
0.1	1.005020918	1.005017575	1.007523924
0.2	1.020338845	1.020337787	0.970798541
0.3	1.046751602	1.046739343	1.035951535
0.4	1.085704428	1.085698147	1.032529932
0.5	1.139493927	1.139461063	1.110647184
0.6	1.211628315	1.211603295	1.173484878
0.7	1.307459259	1.307365801	1.255720007
0.8	1.435324199	1.435227650	1.364968388
0.9	1.608725810	1.608392792	1.489661977
1	1.850815718	1.850368333	1.612411571
L.S.E.		3.3102×10^{-07}	8.6331×10^{-02}

Table 4: Comparison between the exact solution tan (x) and the numerical solution $u_2(x)$ of Example 2 taking h=0.1.

x	u_2 (x)		
	Exact solution	MBLM2	MBLM3
0	0	0	0
0.1	0.100334672	0.100339541	0.098044642
0.2	0.202710036	0.202707015	0.255045763
0.3	0.309336249	0.309354731	0.315452967
0.4	0.422793219	0.422785830	0.471151390
0.5	0.546302489	0.546353121	0.574948583
0.6	0.684136808	0.684117256	0.702163567
0.7	0.842288380	0.842435901	0.869183553
0.8	1.029638557	1.029572151	1.101025176
0.9	1.260158218	1.260694307	1.304581821
1	1.557407725	1.557098761	1.744432386
L.S.E.		4.1240×10^{-07}	4.9036×10^{-02}

Table 5: Shows the least square errors for u_1 (x) and u_2 (x) with different values of h for Example 2.

Numerical solution of	methods	least square errors		
		h=0.1	h=0.05	h=0.025
u_1 (x)	MBLM2	3.3102×10^{-07}	4.3687×10^{-12}	1.6184×10^{-15}
	MBLM3	8.6331×10^{-02}	1.9598×10^{-03}	3.4291×10^{-04}
u_2 (x)	MBLM2	4.1240×10^{-07}	1.0057×10^{-11}	4.4325×10^{-15}
	MBLM3	4.9036×10^{-02}	2.0168×10^{-03}	3.5022×10^{-04}

Example 3: (Jumaa (2005))
Solve a system of non-linear VIEK2's:

$$u_1(x) = \frac{1}{4} - \frac{1}{4}e^{2x} + \int_0^x (x-t)u_2^2(t)dt$$

$$u_2(x) = -xe^x + 2e^x - 1 + \int_0^x te^{-2u_1(t)}dt$$

The exact solution of this system is:

$$u_1(x) = -\frac{1}{2}x \; and \; u_2(x) = e^x$$

After solving this system by block-by-block methods with h=0.1 in equation (17), (18) and (26)-(31), we obtain the following numerical solution.

Table 6: Comparison between the exact solution -1/2x and the numerical solution $u_1(x)$ of Example 3 taking h=0.1.

x	u_1 (x) Exact solution	MBLM2	MBLM3
0	0	0	0
0.1	-0.05000000	-0.050000599	-0.049985827
0.2	-0.10000000	-0.100004137	-0.107222365
0.3	-0.15000000	-0.174591819	-0.150658747
0.4	-0.20000000	-0.199968611	-0.214820411
0.5	-0.25000000	-0.311173987	-0.259292136
0.6	-0.30000000	-0.299387391	-0.301059464
0.7	-0.35000000	-0.464940722	-0.362594003
0.8	-0.40000000	-0.395483188	-0.427750208
0.9	-0.45000000	-0.641146839	-0.453802023
1	-0.50000000	-0.477073658	-0.550934523
L.S.E.		$5.4641 \times 10_{-02}$	3.8971×10^{-03}

Table 7: Comparison between the exact solution e^x and the numerical solution u_2 (x) of Example 3 taking h=0.1.

x	u_2 (x) Exact solution	MBLM2	MBLM3
0	1	1	1
0.1	1.105170918	1.105170935	1.105155293
0.2	1.221402758	1.221403347	1.214963390
0.3	1.349858808	1.351004099	1.350422637
0.4	1.491824698	1.494546780	1.482562177
0.5	1.648721271	1.657759152	1.641555613
0.6	1.822118800	1.839100944	1.825764392
0.7	2.013752707	2.052469700	2.018653396
0.8	2.225540928	2.290521685	2.224455410
0.9	2.459603111	2.585934314	2.477989376
1	2.718281828	2.916104816	2.686692409
L.S.E.		$6.1193 \times 10_{-02}$	$1.5533 \times 10_{-03}$

Table 8: Shows the least square errors for u_1 (x) and u_2 (x) with different values of h for Example 3.

Numerical solution of	methods	least square errors h=0.1	h=0.05	h=0.025
u_1 (x)	MBLM2	$5.4641 \times 10_{-02}$	$1.5401 \times 10_{-03}$	$6.6302 \times 10_{-02}$
	MBLM3	$3.8971 \times 10_{-03}$	$5.2922 \times 10_{-05}$	$1.6481 \times 10_{-06}$
u_2 (x)	MBLM2	$6.1193 \times 10_{-02}$	$4.3482 \times 10_{-04}$	$8.0922 \times 10_{-08}$
	MBLM3	$1.5533 \times 10_{-03}$	$1.1318 \times 10_{-04}$	$5.1867 \times 10_{-06}$

Conclusions:

According to the numerical results which obtaining from the illustrative examples we conclude that the method of two blocks is the best but slower than the method of three blocks. If the function $f_i(x)$, i=1, 2, ..., m are polynomial for sufficiently small h we get a good accuracy (exact sometimes) hence by reducing step size length the least square error will be reduced.

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