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Research Article

Blowup Phenomena for a Modified Dullin-Gottwald-Holm Shallow Water System

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We discuss blowup phenomena for a modified two-component Dullin-Gottwald-Holm shallow water system. In this paper, some new blowup criteria of strong solutions involving the density and suitable integral form of the momentum are established.

1. Introduction

We consider the following two-component DGH type system:

$$\begin{gathered} y_t + 2\omega u_x + u y_x + 2 y u_x + \gamma u_{xxx} + g\rho\overline{\rho}_x = 0, \\ y = u - \alpha^2 u_{xx}, \\ \rho_t + (\rho u)_x = 0, \end{gathered} \tag{1}$$

where $u = u(x,t)$, $(x,t) \in (\mathbb{R}, \mathbb{R}^+)$ denotes the velocity field, g is the downward constant acceleration of gravity in applications to shallow water waves, and $\rho = (1-\partial_x^2)(\overline{\rho} - \overline{\rho}_0)$, where $\overline{\rho}_0$ is taken to be a constant. It is obvious that if $\rho \equiv 0$, then (1) reduces to the well-known Dullin-Gottwald-Holm equation [1] (DGH equation for short). There are some contributions to DGH equation concerning the well-posedness, scattering problem, blowup phenomenon, and so forth; see, for example, [2–5] and references therein. We find that (1) is expressed in terms of an averaged filtered density component $\overline{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho = (1-\partial_x^2)(\overline{\rho} - \overline{\rho}_0)$ and the velocity component u. The idea is actually from the recent work [6]. Our modification breaks the structure of DGH2 system derived by following Ivanov's approach [7] by the authors in [8]. The motivation of current research is stated as follows. From geometric point of view, (1) is the model for geodesic motion on the semidirect product Lie group of diffeomorphisms acting on densities, with respect to the H^1-norm of velocity u and the H^1-norm on filtered density. From a physical point of view, (1) admits wave breaking phenomena in finite time which attracts researchers' interest. We also find that the H^1-norm of $(u, \overline{\rho})$ is conserved with respect to time variable. This makes further different discussions on the singularities, unlike those for the DGH2 system or two-component Camassa-Holm system, possible. In the previous works [9–11] on the two-component Camassa-Holm equation and its modified version, blowup conditions were established in view of the negativity of initial velocity slope at some point; basically, the initial integral form of momentum is never involved. That is why we consider this kind of blowup condition in this paper. Precisely, we show the solutions blowup in finite time provided that the initial density and momentum satisfy certain sign conditions. To our knowledge, less results exist yet for the formation of singularities of (1) although the approaches we applied here are standard. The methods in previous works cannot be moved to this model parallelly. For convenience, let $v = \overline{\rho} - \overline{\rho}_0$ and $\Lambda = (1-\alpha^2\partial_x^2)^{-1}$; then the operator Λ can be expressed by its associated Green's function $G(x) = (1/2\alpha)e^{-|x/\alpha|}$ with

$$\Lambda f(x) = G * f(x) = \int_{\mathbb{R}} G(x-y) f(y)\, dy. \tag{2}$$

Using this identity, system (1) takes an equivalent form of a quasilinear evolution equation of hyperbolic type as follows:

$$
\begin{gathered}
u_t + u_x\left(u - \frac{\gamma}{\alpha^2}\right) \\
= -\partial_x G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + \left(2\omega + \frac{\gamma}{\alpha^2}\right)u + \frac{g}{2}v^2 - \frac{g}{2}v_x^2\right), \\
v_t + uv_x = -G * \left(\left(u_x v_x\right)_x + u_x v\right).
\end{gathered} \tag{3}
$$

The current paper is based on some results on the Camassa-Holm equation [12–19] and its two-component generalizations [20–27]. We investigate further formation of singularities of solutions to (3) with the case of $g = 1$ and $\alpha > 0$, just for simplicity mathematically. This paper is organized as follows. In Section 2, we recall some preliminary results on the well-posedness and blowup scenario. In Section 3, the detailed blowup conditions are presented.

2. Preliminaries

In this section, for completeness, we recall some elementary results and skip their proofs since they are not the main concern of this work. For convenience, in what follows, we let $\lambda = -\gamma/\alpha^2$ and $2\kappa = 2\omega + \gamma/\alpha^2$.

We can apply Kato's theory [28] to establish the following local well-posedness theorem for (3).

Theorem 1. *Assume an initial data $(u_0, v_0) \in H^s \times H^{s-1}$, $s \geq 5/2$. Then there exists a maximal $T = T(\|u_0, v_0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution*

$$
(u, v) \in C\left([0, T); H^s \times H^{s-1}\right) \cap C^1\left([0, T); H^{s-1} \times H^{s-2}\right) \tag{4}
$$

of system (3). Moreover, the solution (u, v) depends continuously on the initial value (u_0, v_0), and the maximal time of existence $T > 0$ is independent of s.

The proof of Theorem 1 is similar to the one in [11]. Moreover, using the techniques in [11], one can get the criterion for finite time wave breaking to (3) as follows.

Theorem 2. *Let $(u_0, v_0) \in H^s \times H^{s-1}$ with $s \geq 5/2$, and let $T > 0$ be the maximal time of existence of the solution (u, v) to (3) with initial data (u_0, v_0). Then the corresponding solution (u, v) blowsup in finite time if and only if*

$$
\lim_{t \uparrow T}\left\{\inf_{x \in \mathbb{R}} u_x(t, x)\right\} = -\infty. \tag{5}
$$

Lemma 3 (see [29]). *Assume that a differentiable function $y(t)$ satisfies*

$$
y'(t) \leq -Cy^2(t) + K \tag{6}
$$

with constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{K/C}$, then the solution to (9) goes to $-\infty$ before t tends to $1/(-Cy_0 + K/y_0)$.

Lemma 4 (see [19]). *Suppose that $\Psi(t)$ is twice continuously differential satisfying*

$$
\begin{gathered}
\Psi''(t) \geq C_0 \Psi'(t)\Psi(t), \quad t > 0,\ C_0 > 0, \\
\Psi(t) > 0, \qquad \Psi'(t) > 0.
\end{gathered} \tag{7}
$$

Then Ψ blowsup in finite time. Moreover the blowup time can be estimated in terms of the initial datum as

$$
T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}. \tag{8}
$$

We also need to introduce the standard particle trajectory method for later use. Consider now the following two initial value problems:

$$
\begin{gathered}
q_{1,t} = u(t, q_1) + \lambda, \quad t \in [0, T), \\
q_1(0, x) = x, \quad x \in \mathbb{R},
\end{gathered} \tag{9}
$$

$$
\begin{gathered}
q_{2,t} = u(t, q_2), \quad t \in [0, T), \\
q_2(0, x) = x, \quad x \in \mathbb{R},
\end{gathered} \tag{10}
$$

where $u \in C^1([0, T), H^{s-1})$ is the first component of the solution (u, v) to system (3) with initial data $(u_0, v_0) \in H^s \times H^{s-1}$ $(s \geq 5/2)$, and $T > 0$ is the maximal time of existence. By direct computation, we have

$$
q_{i,tx}(t, x) = u_x(t, q_i(t, x))\, q_{i,x}(t, x), \quad i = 1, 2. \tag{11}
$$

Then,

$$
q_{i,x}(t, x) = \exp\left(\int_0^t u_x(\tau, q_i(\tau, x))\, d\tau\right) > 0, \quad t > 0,\ x \in \mathbb{R}, \tag{12}
$$

which means that $q_i(t, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0, T)$. Consequently, the L^∞-norm of any function $v(t, \cdot)$ is preserved under the family of the diffeomorphisms $q_i(t, \cdot)$; that is,

$$
\begin{gathered}
\|v(t, \cdot)\|_{L^\infty} = \left\|v(t, q_1(t, \cdot))\right\|_{L^\infty} = \left\|v(t, q_2(t, \cdot))\right\|_{L^\infty}, \\
t \in [0, T).
\end{gathered} \tag{13}
$$

Similarly,

$$
\begin{gathered}
\inf_{x \in \mathbb{R}} v(t, x) = \inf_{x \in \mathbb{R}} v(t, q_1(t, x)) = \inf_{x \in \mathbb{R}} v(t, q_2(t, x)), \\
t \in [0, T), \\
\sup_{x \in \mathbb{R}} v(t, x) = \sup_{x \in \mathbb{R}} v(t, q_1(t, x)) = \sup_{x \in \mathbb{R}} v(t, q_2(t, x)), \\
t \in [0, T).
\end{gathered} \tag{14}
$$

3. Blowup Phenomenon

In this section, we show that blowup phenomenon is the only one way that singularity arises in smooth solutions. We start this section with the following useful lemma.

Lemma 5. *Let $X_0 = (u_0, v_0) \in H^s \times H^{s-1}$, $s \geq 2$. T is assumed to be the maximal existence time of the solution $X = (u, v)$ to system* (3) *corresponding to the initial data X_0. Then for all $t \in [0, T)$, one has the following conservation law:*

$$E(t) = \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2 + v^2 + v_x^2\right) dx. \tag{15}$$

Proof. We will prove that $E(t)$ is a conserved quantity with respect to time variable. Here we use the classical energy method. Multiplying the first equation in (3) by $u(x, t)$ and integrating by parts, we obtain

$$\int_{\mathbb{R}} u u_t dx + \int_{\mathbb{R}} \alpha^2 u_x u_{xt} dx = -\int_{\mathbb{R}} u v_x (v - v_{xx}) dx. \tag{16}$$

Similarly, we have the following inequality for the second equation (3):

$$\int_{\mathbb{R}} v v_t + v_x v_{xt} dx = \int_{\mathbb{R}} u v_x (v - v_{xx}) dx. \tag{17}$$

This implies that

$$\int_{\mathbb{R}} \left(u u_t + \alpha^2 u_x u_{xx} + v v_t + v_x v_{xx}\right) dx = 0. \tag{18}$$

Thus, we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2 + v^2 + v_x^2\right) dx \\ &\quad = 2 \int_{\mathbb{R}} \left(u u_t + \alpha^2 u_x u_{xx} + v v_t + v_x v_{xx}\right) dx = 0. \end{aligned} \tag{19}$$

This completes the proof. □

Using this conservation law, we obtain

$$\begin{aligned} &\|u(\cdot, t)\|_{L^\infty(\mathbb{R})}^2 + \|v(\cdot, t)\|_{L^\infty(\mathbb{R})}^2 \\ &\quad \leq \frac{1}{2\alpha} \|u(\cdot, t)\|_{H_\alpha^1(\mathbb{R})}^2 + \frac{1}{2} \|v(\cdot, t)\|_{H^1(\mathbb{R})}^2 \leq C_1 E(0), \end{aligned} \tag{20}$$

where

$$C_1 = \max\left\{\frac{1}{2\alpha}, \frac{1}{2}\right\}. \tag{21}$$

Theorem 6. *Suppose that $X_0 = (u_0, v_0) \in H^s \times H^{s-1}$, $s \geq 5/2$, $\rho_0(x_0) = y_0(x_0) + \kappa = 0$, and the initial data satisfies the following conditions:*

(i) $\rho_0(x) \geq 0$ *on* $(-\infty, x_0)$ *and* $\rho_0(x) \leq 0$ *on* (x_0, ∞),

(ii) $\int_{-\infty}^{x_0} e^{\xi/\alpha}(y_0(\xi) + \kappa) d\xi > 0$ *and* $\int_{-\infty}^{x_0} e^{-\xi/\alpha}(y_0(\xi) + \kappa) d\xi < 0$,

for some point $x_0 \in \mathbb{R}$. Then the solution to system (3) *with the initial value X_0 blowsup in finite time.*

Proof. Differentiating the first equation of (3) with respect to x, we obtain

$$\begin{aligned} &u_{xt} + u_x^2 + u u_{xx} + \lambda u_{xx} \\ &\quad + \partial_x^2 G * \left(u^2 + \frac{\alpha^2}{2} u_x^2 + 2\kappa u + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) = 0. \end{aligned} \tag{22}$$

Applying the relation $\partial_x^2 (G * f) = (1/\alpha^2)(G * f - f)$ yields

$$\begin{aligned} &u_{xt} + u_x^2 + u u_{xx} + \lambda u_{xx} \\ &\quad + \frac{1}{\alpha^2} G * \left(u^2 + \frac{\alpha^2}{2} u_x^2 + 2\kappa u + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) \\ &\quad - \frac{1}{\alpha^2} \left(u^2 + \frac{\alpha^2}{2} u_x^2 + 2\kappa u + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) = 0. \end{aligned} \tag{23}$$

From (23) we have

$$\begin{aligned} &\frac{d}{dt} u_x (q_1 (x_0, t), t) \\ &\quad = (u_{xt} + u u_{xx} + \lambda u_{xx}) (q_1 (x_0, t), t) \\ &\quad = -u_x^2 (q_1 (x_0, t), t) - \frac{1}{\alpha^2} G \\ &\qquad * \left(u^2 + \frac{\alpha^2}{2} u_x^2 + 2\kappa u + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) (q_1 (x_0, t), t) \\ &\qquad + \frac{1}{\alpha^2} \left(u^2 + \frac{\alpha^2}{2} u_x^2 + 2\kappa u + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) (q_1 (x_0, t), t) \\ &\quad \leq -\frac{1}{2} u_x^2 (q_1 (x_0, t), t) + \frac{1}{2\alpha^2} (u + \kappa)^2 (q_1 (x_0, t), t) \\ &\qquad + \frac{1}{2\alpha^2} \left(v^2 - v_x^2\right) (q_1 (x_0, t), t) \\ &\qquad - \frac{1}{2\alpha^2} \left(G * \left(v^2 - v_x^2\right)\right) (q_1 (x_0, t), t), \end{aligned} \tag{24}$$

where we used the fact proved in [30] that

$$G * \left((u + \kappa)^2 + \frac{\alpha^2}{2} u_x^2\right) \geq \frac{1}{2} (u + \kappa)^2. \tag{25}$$

In order to arrive at our result, we need the following three claims.

Claim 1. $y(q_1(x_0, t), t)) + \kappa = 0$ for all t in its lifespan; q_1 is defined in (9).

It is worth noting the equivalent form of the first equation in (3) in what follows:

$$y_t + u y_x + 2 y u_x + \lambda y_x + 2\kappa u_x + \rho v_x = 0. \tag{26}$$

From the previous equation, we can get

$$\begin{aligned}&\frac{d}{dt}\left(\left(y\left(q_1\left(x,t\right),t\right)+\kappa\right)q_{1,x}^2\left(x,t\right)\right)\\&\quad=\left(y_t+uy_x+2yu_x+\lambda y_x+2\kappa u_x\right)\left(q_1\left(x,t\right),t\right)q_{1,x}^2\left(x,t\right)\\&\quad=-\rho\left(q_1\left(x,t\right),t\right)v_x\left(q_1\left(x,t\right),t\right)q_{1,x}^2\left(x,t\right).\end{aligned}\tag{27}$$

Since $q_2(x,\cdot)$ defined by (10) is a diffeomorphism of the line for any $t\in[0,T)$, so there exists an $x_3(t)\in\mathbb{R}$ such that

$$q_2\left(t,x_3\left(t\right)\right)=q_1\left(t,x_0\right),\quad t\in[0,t).\tag{28}$$

When $t=0$, we have

$$x_3\left(0\right)=q_2\left(0,x_3\left(0\right)\right)=q_1\left(0,x_0\right)=x_0.\tag{29}$$

Now we prove that $\rho(t,q_1(t,x_0))=0$. It is easy to get

$$\frac{d}{dt}\rho\left(t,q_2\left(t,x_3\left(t\right)\right)\right)=-\left(\rho u_x\right)\left(t,q_2\left(t,x_3\left(t\right)\right)\right).\tag{30}$$

Since

$$\rho_0\left(x_0\right)=0,\tag{31}$$

integrating the previous equation, we can obtain

$$\begin{aligned}\rho\left(t,q_2\left(t,x_3\left(t\right)\right)\right)&=\rho\left(0,q_2\left(0,x_3\left(0\right)\right)\right)e^{-\int_0^t u_x(\tau,q_2(\tau,x_3(\tau)))d\tau}\\&=\rho_0\left(x_0\right)e^{-\int_0^t u_x(\tau,q_2(\tau,x_3(\tau)))d\tau}=0;\end{aligned}\tag{32}$$

thus we have

$$\rho\left(t,q_1\left(t,x_0\right)\right)=\rho\left(t,q_2\left(t,x_3\left(t\right)\right)\right)=0.\tag{33}$$

So we can get

$$\begin{aligned}&\frac{d}{dt}\left(\left(y\left(q_1\left(x_0,t\right),t\right)+\kappa\right)q_{1,x}^2\left(x_0,t\right)\right)\\&\quad=-\rho\left(q_1\left(x_0,t\right),t\right)v_x\left(q_1\left(x_0,t\right),t\right)q_{1,x}^2\left(x_0,t\right)=0;\end{aligned}\tag{34}$$

then we have

$$y\left(q_1\left(x_0,t\right),t\right)+\kappa=y_0\left(x_0\right)+\kappa=0.\tag{35}$$

Our claim is proved.

Claim 2. For any fixed t, $v_x^2(x,t)-v^2(x,t)\le v_x^2(q_1(x_0,t),t)-v^2(q_1(x_0,t),t)$ for all $x\in\mathbb{R}$. For any fixed t, if $x\le q_1(x_0,t)$, then

$$\begin{aligned}&v_x^2\left(x,t\right)-v^2\left(x,t\right)\\&\quad=-\left(\int_{-\infty}^{q_1(x_0,t)}e^{\xi}\rho\left(\xi,t\right)d\xi-\int_x^{q_1(x_0,t)}e^{\xi}\rho\left(\xi,t\right)d\xi\right)\\&\qquad\times\left(\int_{q_1(x_0,t)}^{\infty}e^{-\xi}\rho\left(\xi,t\right)d\xi+\int_x^{q_1(x_0,t)}e^{-\xi}\rho\left(\xi,t\right)d\xi\right)\\&\quad=v_x^2\left(q_1\left(x_0,t\right),t\right)-v^2\left(q_1\left(x_0,t\right),t\right)\\&\qquad-\int_{-\infty}^{x}e^{\xi}\rho\left(\xi,t\right)d\xi\int_x^{q_1(x_0,t)}e^{-\xi}\rho\left(\xi,t\right)d\xi\\&\qquad+\int_x^{q_1(x_0,t)}e^{\xi}\rho\left(\xi,t\right)d\xi\int_{q_1(x_0,t)}^{\infty}e^{-\xi}\rho\left(\xi,t\right)d\xi\\&\quad\le v_x^2\left(q_1\left(x_0,t\right),t\right)-v^2\left(q_1\left(x_0,t\right),t\right),\end{aligned}\tag{36}$$

where the condition (i) is used. Similarly, for $x\ge q_1(x_0,t)$, we also have

$$v_x^2\left(x,t\right)-v^2\left(x,t\right)\le v_x^2\left(q_1\left(x_0,t\right),t\right)-v^2\left(q_1\left(x_0,t\right),t\right).\tag{37}$$

So Claim 2 is proved. Consequently, we can obtain

$$\begin{aligned}&\left(G*\left(v^2-v_x^2\right)\right)\left(q_1\left(x_0,t\right),t\right)\\&\quad=\frac{1}{2\alpha}\int_{\mathbb{R}}e^{-|q_1(x_0,t)-\xi|/\alpha}\left(v^2-v_x^2\right)\left(\xi,t\right)d\xi\\&\quad\ge\frac{1}{2\alpha}\int_{\mathbb{R}}e^{-|q_1(x_0,t)-\xi|/\alpha}\left(v^2-v_x^2\right)\left(q_1\left(x_0,t\right),t\right)d\xi\\&\quad=\left(v^2-v_x^2\right)\left(q_1\left(x_0,t\right),t\right).\end{aligned}\tag{38}$$

Thus, one can get

$$\begin{aligned}\frac{d}{dt}u_x\left(q_1\left(x_0,t\right),t\right)\le&-\frac{1}{2}u_x^2\left(q_1\left(x_0,t\right),t\right)\\&+\frac{1}{2\alpha^2}\left(u+\kappa\right)^2\left(q_1\left(x_0,t\right),t\right).\end{aligned}\tag{39}$$

Claim 3. $(u+\kappa)^2(q_1(x_0,t),t)<\alpha^2u_x^2(q_1(x_0,t),t)$ for all $t\ge0$. Furthermore, $u_x(q_1(x_0,t),t)<0$ is strictly decreasing.

Suppose that there exists a t_0 such that $(u+\kappa)^2(q_1(x_0,t),t) < \alpha^2 u_x^2(q_1(x_0,t),t)$ on $[0,t_0)$ and $(u+\kappa)^2(q_1(x_0,t_0),t_0) = \alpha^2 u_x^2(q_1(x_0,t_0),t_0)$. From the expression of $u(x,t)$ in terms of $y(x,t)$, we can rewrite $u(x,t)+\kappa$ and $u_x(x,t)$ as follows:

$$\begin{aligned} u(x,t)+\kappa &= \frac{1}{2\alpha}e^{-x/\alpha}\int_{-\infty}^{x} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\ &\quad + \frac{1}{2\alpha}e^{x/\alpha}\int_{x}^{\infty} e^{-\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi, \\ u_x(x,t) &= -\frac{1}{2\alpha^2}e^{-x/\alpha}\int_{-\infty}^{x} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\ &\quad + \frac{1}{2\alpha^2}e^{x/\alpha}\int_{x}^{\infty} e^{-\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi. \end{aligned} \tag{40}$$

Letting

$$\begin{aligned} I(t) &= e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi, \\ II(t) &= e^{q_1(x_0,t)/\alpha}\int_{q_1(x_0,t)}^{\infty} e^{-\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi, \end{aligned} \tag{41}$$

then

$$\begin{aligned} \frac{dI(t)}{dt} &= -\frac{1}{\alpha}\left(u\left(q_1(x_0,t),t\right)+\lambda\right)e^{-q_1(x_0,t)/\alpha} \\ &\quad \times \int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\ &\quad + e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha} y_t(\xi,t)\,d\xi. \end{aligned} \tag{42}$$

Integrating by parts, the first term of (42) yields

$$\begin{aligned} &-\frac{1}{\alpha}\left(u\left(q_1(x_0,t),t\right)+\lambda\right)e^{-q_1(x_0,t)/\alpha} \\ &\quad \times \int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\ &= \left(\alpha u u_x - u^2 - \kappa u\right)\left(q_1(x_0,t),t\right) \\ &\quad - \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\lambda\left(y(\xi,t)+\kappa\right)d\xi. \end{aligned} \tag{43}$$

For the second term of (42), we have the following equation in the view of Claim 1:

$$\begin{aligned} &e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha} y_t(\xi,t)\,d\xi \\ &= -e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\Big(\left((y+\kappa)u\right)_x + \frac{1}{2}\left(u^2-\alpha^2u_x^2\right)_x \\ &\qquad + \lambda(y+\kappa)_x + \kappa u_x + \rho v_x\Big)d\xi \\ &= \left(\frac{\alpha^2}{2}u_x^2 - \frac{1}{2}u^2 - \kappa u\right)\left(q_1(x_0,t),t\right) \\ &\quad - e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\rho v_x\,d\xi \\ &\quad + \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\Big((y+\kappa)u + \frac{1}{2}\left(u^2-\alpha^2u_x^2\right) \\ &\qquad + \lambda(y+\kappa) + \kappa u\Big)d\xi \\ &= \left(\frac{\alpha^2}{2}u_x^2 - \frac{1}{2}u^2 - \kappa u + \frac{1}{2}\left(v_x^2 - v^2\right)\right)\left(q_1(x_0,t),t\right) \\ &\quad + \frac{1}{2\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(v^2 - v_x^2\right)d\xi \\ &\quad + \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\Big(\frac{3}{2}u^2 - \frac{\alpha^2}{2}u_x^2 - \alpha^2 u u_{xx} \\ &\qquad + \lambda(y+\kappa) + 2\kappa u\Big)d\xi \\ &= \left(\frac{\alpha^2}{2}u_x^2 - \alpha u u_x - \kappa u + \frac{1}{2}\left(v_x^2 - v^2\right)\right)\left(q_1(x_0,t),t\right) \\ &\quad + \frac{1}{2\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(v^2 - v_x^2\right)d\xi \\ &\quad + \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\Big(u^2 + \frac{\alpha^2}{2}u_x^2 \\ &\qquad + \lambda(y+\kappa) + 2\kappa u\Big)d\xi. \end{aligned} \tag{44}$$

Here we have used

$$\begin{aligned} &-\alpha e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(u u_{xx}\right)(\xi,t)\,d\xi \\ &= -\alpha\left(u u_x\right)\left(q_1(x_0,t),t\right) + \frac{1}{2}u^2\left(q_1(x_0,t),t\right) \\ &\quad + \alpha e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(u_x^2 - \frac{1}{2\alpha^2}u^2\right)(\xi,t)\,d\xi. \end{aligned} \tag{45}$$

Combining the previous equations together, and with the help of (38), (42) reads as

$$\begin{aligned} \frac{dI(t)}{dt} &= \left(\frac{\alpha^2}{2}u_x^2 - u^2 - 2\kappa u\right)\left(q_1(x_0,t),t\right) \\ &\quad + \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha}\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + 2\kappa u\right)d\xi \\ &\quad + \frac{1}{2}\left(v_x^2 - v^2\right)\left(q_1(x_0,t),t\right) + \frac{1}{2\alpha}e^{-q_1(x_0,t)/\alpha} \\ &\quad \times \int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(v^2 - v_x^2\right)d\xi \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha^2}{2}u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t) + \frac{1}{\alpha}e^{-q_1(x_0,t)/\alpha} \\
&\quad \times \int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(\frac{\alpha^2}{2}u_x^2 + (u+\kappa)^2\right)d\xi \\
&\quad + \frac{1}{2}\left(v_x^2 - v^2\right)(q_1(x_0,t),t) + \frac{1}{2\alpha}e^{-q_1(x_0,t)/\alpha} \\
&\quad \times \int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(v^2 - v_x^2\right)d\xi \\
&\geq \frac{1}{2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t) > 0, \quad \text{on } [0,t_0),
\end{aligned}
\tag{46}
$$

where Claim 2 and the inequality [30]

$$
\int_{-\infty}^{x} e^{\xi/\alpha}\left((u+\kappa)^2 + \frac{\alpha^2}{2}u_\xi^2\right)(\xi,t)\,d\xi \geq \frac{\alpha}{2}e^{x/\alpha}(u+\kappa)^2 \tag{47}
$$

have been used. From the continuity property, we have

$$
\begin{aligned}
&e^{-q_1(x_0,t_0)/\alpha}\int_{-\infty}^{q_1(x_0,t_0)} e^{\xi/\alpha}\left(y(\xi,t_0)+\kappa\right)d\xi \\
&\qquad > e^{-x_0/\alpha}\int_{-\infty}^{x_0} e^{\xi/\alpha}\left(y_0(\xi)+\kappa\right)d\xi > 0.
\end{aligned}
\tag{48}
$$

Similarly,

$$
\frac{dII(t)}{dt} \leq \frac{1}{2}\left((u+\kappa)^2 - \alpha^2 u_x^2\right)(q_1(x_0,t),t) < 0, \quad \text{on } [0,t_0). \tag{49}
$$

Thus, by continuity property,

$$
\begin{aligned}
&e^{q_1(x_0,t_0)/\alpha}\int_{-\infty}^{q_1(x_0,t_0)} e^{-\xi/\alpha}\left(y(\xi,t_0)+\kappa\right)d\xi \\
&\qquad < e^{x_0/\alpha}\int_{-\infty}^{x_0} e^{-\xi/\alpha}\left(y_0(\xi)+\kappa\right)d\xi < 0.
\end{aligned}
\tag{50}
$$

Summarizing (48) and (50), we obtain

$$
\begin{aligned}
&\alpha^2 u_x^2(q_1(x_0,t_0),t_0) - (u+\kappa)^2(q_1(x_0,t_0),t_0) \\
&= -\frac{1}{\alpha^2}\int_{-\infty}^{q_1(x_0,t_0)} e^{\xi/\alpha}\left(y(\xi,t_0)+\kappa\right)d\xi \\
&\quad \times \int_{q_1(x_0,t_0)}^{\infty} e^{-\xi/\alpha}\left(y(\xi,t_0)+\kappa\right)d\xi \\
&> -\frac{1}{\alpha^2}\int_{-\infty}^{x_0} e^{\xi/\alpha}\left(y_0(\xi)+\kappa\right)d\xi\int_{x_0}^{\infty} e^{-\xi/\alpha}\left(y_0(\xi)+\kappa\right)d\xi \\
&= \alpha^2 u_{0x}^2(x_0) - (u_0+\kappa)^2(x_0) > 0.
\end{aligned}
\tag{51}
$$

That is a contradiction. On the other hand, from the expression of $u_x(x,t)$ in terms of $y(x,t)$, we can easily get that $u_x(q_1(x_0,t),t) < 0$. So we complete the proof of Claim 3.

Furthermore, due to (46) and (49), we can obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t) \\
&= -\frac{1}{\alpha^2}\frac{d}{dt}\left(\int_{-\infty}^{q_1(x_0,t)} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi\right. \\
&\qquad\qquad \left.\times\int_{q_1(x_0,t)}^{\infty} e^{-\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi\right) \\
&\geq -\frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t)\,e^{q_1(x_0,t)/\alpha} \\
&\quad \times \int_{-\infty}^{x} e^{-\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\
&\quad + \frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t)\,e^{-q_1(x_0,t)/\alpha} \\
&\quad \times \int_{x}^{\infty} e^{\xi/\alpha}\left(y(\xi,t)+\kappa\right)d\xi \\
&= -u_x(q_1(x_0,t),t)\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t).
\end{aligned}
\tag{52}
$$

Integrating (39) and then substituting it into the previous inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t) \\
&\quad \geq \frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,t),t) \\
&\quad\quad \times\left(\int_0^t\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q_1(x_0,\tau),\tau)\,d\tau\right. \\
&\quad\quad\quad \left. -2\alpha^2 u_{0x}(x_0)\right).
\end{aligned}
\tag{53}
$$

Let $\Psi(t) = \int_0^t(\alpha^2u_x^2 - (u+\kappa)^2)(q_1(x_0,\tau),\tau)d\tau - 2\alpha^2u_{0x}(x_0)$; then we can complete the proof with the help of Lemma 4. □

Remark 7. We note that if the condition (i) is replaced by the following one:

(i′) $\rho_0(x) \leq 0$ on $(-\infty, x_0)$ and $\rho_0(x) \geq 0$ on (x_0,∞),

then Claim 2 also holds; that is, the theorem always holds with anyone of (i) and (i′).

As a corollary of Theorem 6, we have the following.

Theorem 8. *Suppose that $X_0 = (u_0, v_0) \in H^s\times H^{s-1}$, $s \geq 5/2$, and the initial data satisfies the following conditions:*

(i) $\rho_0(x) \geq 0$ *on* $(-\infty, x_0)$ *and* $\rho_0(x) \leq 0$ *on* (x_0,∞) *(or* $\rho_0(x) \leq 0$ *on* $(-\infty,x_0)$ *and* $\rho_0(x) \geq 0$ *on* (x_0,∞)*),*

(ii) $u_0'(x_0) \leq -(\sqrt{C_1E_0} + |\kappa|)/\alpha$, *for some point $x_0 \in \mathbb{R}$. Then the solution to system* (3) *with the initial value X_0 blowsup in finite time.*

Proof. As shown in Theorem 6, condition (i) guarantees that $v_x^2(x,t) + v^2(x,t) \le (v_x^2 - v^2)(q_1(x_0,t),t)$ for all $x \in \mathbb{R}$. Then,

$$\begin{aligned} \frac{d}{dt} u_x\left(q_1\left(x_0,t\right),t\right) &\le -\frac{1}{2} u_x^2\left(q_1\left(x_0,t\right),t\right) \\ &\quad + \frac{1}{2\alpha^2}(u+\kappa)^2\left(q_1\left(x_0,t\right),t\right) \\ &\le -\frac{1}{2} u_x^2\left(q_1\left(x_0,t\right),t\right) \\ &\quad + \frac{1}{2\alpha^2}\left(\sqrt{C_1 E_0} + |\kappa|\right)^2 \\ &:= -\frac{1}{2} u_x^2\left(q_1\left(x_0,t\right),t\right) + K^2, \end{aligned} \tag{54}$$

where $K > 0$ is a constant. By setting $\varphi(t) = u_x(q_1(x_0,t),t)$, we obtain

$$\frac{d\varphi}{dt} = -\frac{1}{2}\varphi^2 + K^2. \tag{55}$$

Applying Lemma 3, we have

$$\lim_{t \uparrow T} \varphi(t) = -\infty \quad \text{with } T = \frac{1}{-(1/2)\varphi_0 - \left(K^2/\varphi_0\right)}, \tag{56}$$

when

$$\varphi_0 < -\sqrt{2}K = -\frac{\sqrt{C_1 E_0} + |\kappa|}{\alpha}. \tag{57}$$

This completes the proof. □

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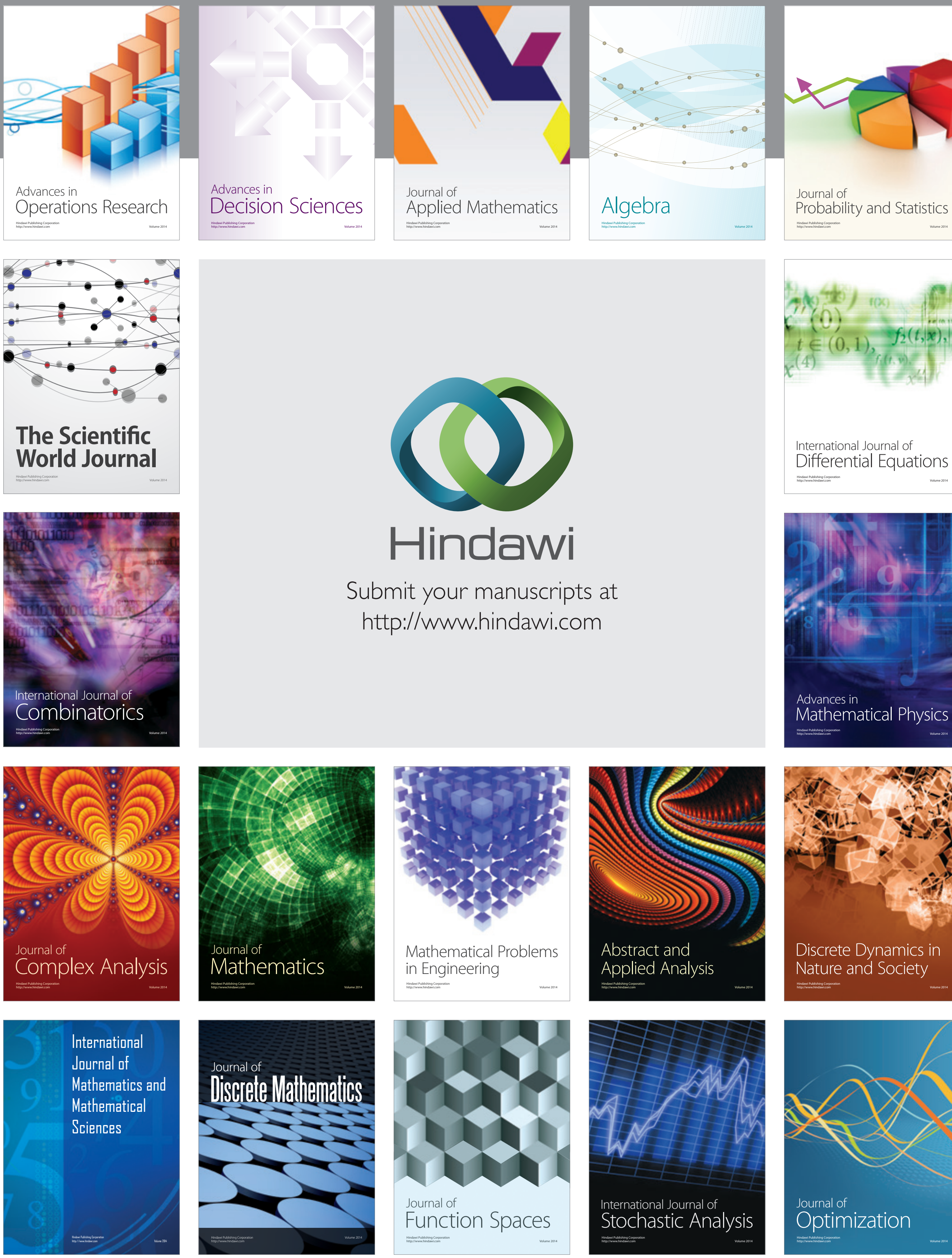
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