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THE GREEN FUNCTIONS OF A CLASS OF BOUNDARY VALUE PROBLEMS AND THEIR TRACES

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ABSTRACT

Given $\alpha \in \mathbb{C} \setminus \{1\}$, for a fixed integer $n \geq 1$ the Green function of the two point boundary value problem $(-i)^n u^{(n)} = f$, $\alpha u^{(j)}(0) = u^{(j)}(1)$ $(0 \leq j \leq n-1)$ is constructed explicitly by means of the Eulerian polynomial $H_{n-1}(x|\alpha)$. If $|\alpha| = 1$, the eigenfunction expansion of the Green function is applied to obtain certain summation formulas in terms of the central factorial numbers.

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1. INTRODUCTION

Let $\alpha \in \mathbb{C} \setminus \{1\}$ be given. For a fixed integer $n \geq 1$, we consider the two point boundary value problem

$$\begin{aligned} &(-i)^n u^{(n)} = f \\ &\alpha\, u^{(j)}(0) = u^{(j)}(1) \quad (0 \leq j \leq n-1) \end{aligned} \tag{1}$$

on $[0,1]$. Obviously the homogeneous problem for (1) has only a trivial solution and hence the Green function $k_n(x,y|\alpha)$ for (1) exists (see, e.g., [8, p. 194]), so that for $f \in L^2[0,1]$

$$u(x) = K_\alpha^n f(x) \equiv \int_0^1 k_n(x,y|\alpha) f(y)\,dy \tag{2}$$

is the unique solution to the Problem (1), where K_α^n denotes the integral operator with the kernel $k_n(x,y|\alpha)$.

The object of this paper is twofold. We first construct explicitly in §2 the Green function $k_n(x,y|\alpha)$ by means of the Eulerian polynomial $H_{n-1}(x|\alpha)$ introduced by Euler in 1755 (see, e.g., [3]). We then assume $|\alpha| = 1$ in §3, so that Problem (1) becomes self-adjoint. Applying the eigenfunction expansion of the Green function, certain summation formulas are obtained in terms of the central factorial numbers (see, e.g., [2]), extending a familiar identity which is the case $n = 2$.

Recent interest in (1) has come in [4] in connection with some sharp inequalities for the eigenvalues of integral operators with smooth kernels. We refer to [5] for an explicit formula of the Green function for the Problem (1) when $\alpha = 1$.

2. THE GREEN FUNCTIONS

Given $\alpha \in \mathbb{C} \setminus \{1\}$, let $H_n(x|\alpha)$ $(n = 0, 1, 2, \ldots)$ be the polynomials in x defined by the generating function

$$\frac{1-\alpha}{e^t - \alpha} e^{xt} = \sum_{n=0}^{\infty} H_n(x|\alpha) \frac{t^n}{n!}$$

as in [3]. When $\alpha = -1$, $H_n(x|-1)$ are no other than the Euler polynomials $E_n(x)$ (see, e.g., [1, p. 804]). It is straightforward to verify that $H_n(x|\alpha)$ can be characterized by the properties

$$H_0(x|\alpha) \equiv 1, \tag{3}$$

$$\frac{d}{dx} H_{n+1}(x|\alpha) = (n+1) H_n(x|\alpha) \tag{4}$$

and the boundary condition

$$\alpha H_n(0|\alpha) = H_n(1|\alpha) \quad (n \geq 1) \tag{5}$$

when restricting to the interval [0, 1]. It follows either directly from the generating function or from the Properties (3–5) that

$$H_n(1-x|\alpha) = (-1)^n H_n\left(x|\frac{1}{\alpha}\right), \tag{6}$$

which may serve as definition of $H_n(x|1/\alpha)$ for the case $\alpha = 0$.

One of the main results is the following:

Theorem 1. *The Green function for the Problem* (*1*) *is*

$$k_n(x,y|\alpha) = \frac{i^n}{\alpha-1}\begin{cases} \alpha\dfrac{H_{n-1}(x-y|\alpha)}{(n-1)!} & \text{if } x > y \\ (-1)^{n-1}\dfrac{H_{n-1}(y-x|1/\alpha)}{(n-1)!} & \text{if } x < y. \end{cases}$$

Proof. We shall verify directly that for $f \in L^2[0,1]$, (2) gives indeed a solution u to (1). Now

$$u(x) = \frac{i^n}{\alpha-1}\left[\alpha\int_0^x \frac{H_{n-1}(x-y|\alpha)}{(n-1)!}f(y)\,dy + (-1)^{n-1} \times \int_x^1 \frac{H_{n-1}(y-x|1/\alpha)}{(n-1)!}f(y)\,dy\right].$$

Differentiating j times for $1 \le j \le n-1$ and using (4) we have

$$\begin{aligned}(-i)^n u^{(j)}(x) = {}& \frac{1}{\alpha-1}\left[\alpha\frac{H_{n-j}(0|\alpha)}{(n-j)!}f(x) + (-1)^{n-j-1}\frac{H_{n-j}(0|1/\alpha)}{(n-j)!}f(x)\right] \\ &+ \frac{1}{\alpha-1}\left[\alpha\int_0^x \frac{H_{n-j-1}(x-y|\alpha)}{(n-j-1)!}f(y)\,dy \right. \\ &\left. + (-1)^{n-j-1}\int_x^1 \frac{H_{n-j-1}(y-x|1/\alpha)}{(n-j-1)!}f(y)\,dy\right],\end{aligned}$$

where by (5) and (6) the term inside the first bracket vanishes. Moreover, it follows from (6) that

$$u^{(j)}(1) = \frac{i^n\alpha}{\alpha-1}\int_0^1 \frac{H_{n-j-1}(1-y|\alpha)}{(n-j-1)!}f(y)\,dy = \alpha\, u^{(j)}(0).$$

Finally we have

$$(-i)^n u^{(n)}(x) = \frac{1}{\alpha-1}[\alpha f(x) - f(x)] = f(x).$$

This completes the proof of the theorem.

By the corresponding property of the boundary value problems in (1), we have for integers $m, n \geq 1$

$$K_\alpha^{m+n} = K_\alpha^m K_\alpha^n \tag{7}$$

as integral operators. In terms of the Green functions we obtain the following simple consequence of Theorem 1 (cf. [1, p. 805]).

Corollary. *For integers $m, n \geq 0$,*

$$H_{m+n+1}(0|\alpha) = \frac{(m+n+1)!}{m!\,n!} \frac{(-1)^m}{\alpha - 1} \int_0^1 H_m\left(z|\frac{1}{\alpha}\right) H_n(z|\alpha)\,dz.$$

3. SUMMATION FORMULAS

We fix an integer $n \geq 1$ and assume $\alpha \in \mathbb{C} \setminus \{1\}$and $|\alpha| = 1$, so that the Problem (1) is selfadjoint. As is easily seen, the eigenvalues λ_m of (1) counting according to multiplicities, and the corresponding orthonormal eigenfunctions ϕ_m are given by

$$\lambda_m = (2\pi m + \beta)^n \quad \text{and} \quad \phi_m(x) = e^{i(2\pi m + \beta)x} \quad (m = 0, \pm 1, \pm 2, \ldots),$$

where

$$\alpha = e^{i\beta} = \cos\beta + i\sin\beta \quad (0 < \beta < 2\pi).$$

Since the eigenvalues of (1) are all nonzero, the eigenvalues of the integral operator K_α^n are precisely the reciprocals $1/\lambda_m$ of that of (1). Using the eigenfunction expansion of the kernel $k_n(x, y|\alpha)$ of K_α^n, we obtain the following series expansion.

Theorem 2. *For an integer $n \geq 2$*

$$\frac{\alpha}{\alpha - 1} \frac{H_{n-1}(x|\alpha)}{(n-1)!} = (-i)^n \sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m + \beta)^n} e^{i(2\pi m + \beta)x},$$

where the series converges absolutely and uniformly on $[0, 1]$.

Proof. We first consider the case $n = 2r$ ($r \geq 1$) is even. Then the integral operator K_α^{2r} is positive definite as seen from (7) with a continuous kernel. It follows from Mercer Theorem (see, e.g., [8, p. 376]) that the kernel $k_{2r}(x, y|\alpha)$ of K_α^{2r} has the eigenfunction expansion

$$k_{2r}(x, y|\alpha) = \sum_{m=-\infty}^{\infty} \frac{1}{\lambda_m} \phi_m(x)\overline{\phi_m(y)},$$

where the series converges absolutely and uniformly on the square $[0,1]^2$. Setting $y = 0$ we obtain

$$\frac{\alpha}{\alpha-1}\frac{H_{2r-1}(x|\alpha)}{(2r-1)!} = (-1)^r \sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m+\beta)^{2r}} e^{i(2\pi m+\beta)x}, \tag{8}$$

where the series converges absolutely and uniformly on $[0,1]$.

For the case $n = 2r+1$ $(r \geq 1)$ is odd, we replace r by $r+1$ in (8) and differentiate term by term. Then

$$\frac{\alpha}{\alpha-1}\frac{H_{2r}(x|\alpha)}{(2r)!} = (-i)^{2r+1} \sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m+\beta)^{2r+1}} e^{i(2\pi m+\beta)x}. \tag{9}$$

This completes the proof of the theorem.

The values of the left hand sides of (8) and (9) at $x = 0$ can be evaluated in terms of the central factorial numbers. For this purpose we consider

$$R_n(\alpha) = (\alpha-1)^n H_n(0|\alpha),$$

which is a polynomial of degree $n-1$ expressed by Euler in the powers of α with the integer coefficients known as the Eulerian numbers (see, e.g., [3]). Later Frobenius gave another expression for $R_n(\alpha)$ in the powers of $\alpha - 1$ with the coefficients related to the Stirling numbers of the second kind (see, e.g., [7, p. 244]). Recently a new representation for $R_n(\alpha)$ is obtained in terms of the central factorial numbers $T(2r, 2k)$ defined by

$$T(2r,2k) = \frac{2}{(2k)!}\sum_{j=1}^{k}(-1)^{k-j}\binom{2k}{k-j} j^{2r}.$$

It has been shown in [6] that for $r \geq 1$,

$$R_{2r-1}(\alpha) = \sum_{k=1}^{r}(2k-1)!\alpha^{k-1}(\alpha-1)^{2r-2k}T(2r,2k), \tag{10}$$

$$R_{2r}(\alpha) = (1+\alpha)\sum_{k=1}^{r} k(2k-1)!\alpha^{k-1}(\alpha-1)^{2r-2k}T(2r,2k). \tag{11}$$

We refer to [2] for an interesting exposition on central factorial numbers and a variety of their applications.

Finally we give the other main result in the following.

Theorem 3. *Let $r \geq 1$ be an integer. Then for $z \in \mathbb{C}$ not an integer*

$$\frac{1}{(2\pi)^{2r}} \sum_{m=-\infty}^{\infty} \frac{1}{(z+m)^{2r}} = \frac{1}{(2r-1)!} \times \sum_{k=1}^{r} (-1)^{r-k} \frac{(2k-1)!}{4^k \sin^{2k}(\pi z)} T(2r, 2k), \qquad (12)$$

$$\frac{1}{(2\pi)^{2r+1}} \sum_{m=-\infty}^{\infty} \frac{1}{(z+m)^{2r+1}} = \frac{\cot(\pi z)}{(2r)!} \times \sum_{k=1}^{r} (-1)^{r-k} \frac{k(2k-1)!}{4^k \sin^{2k}(\pi z)} T(2r, 2k). \qquad (13)$$

Proof. Since both sides of (12) and (13) are meromorphic functions in $z \in \mathbb{C}$, it suffices to prove them for $z = (\beta/2\pi)$ $(0 < \beta < 2\pi)$. Let $\alpha = e^{i\beta}$. Then

$$\alpha - 1 = 2ie^{i\beta/2} \sin(\beta/2) \quad \text{and} \quad \alpha + 1 = 2e^{i\beta/2} \cos(\beta/2).$$

Moreover, we have by (10)

$$\frac{\alpha}{\alpha - 1} H_{2r-1}(0|\alpha) = \sum_{k=1}^{r} (-1)^k (2k-1)! \frac{T(2r, 2k)}{4^k \sin^{2k}(\beta/2)}$$

and by (11)

$$\frac{\alpha}{\alpha - 1} H_{2r}(0|\alpha) = i \cot\left(\frac{\beta}{2}\right) \sum_{k=1}^{r} (-1)^{k-1} k(2k-1)! \frac{T(2r, 2k)}{4^k \sin^{2k}(\beta/2)}.$$

The equalities in (12) and (13) follow by setting $x = 0$ in (8) and (9).

For $r = 1$ (12) reduces to a familiar identity usually derived using the residue theory.

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