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Effects of Currents on Super- and Sub-Harmonic Waves in a Two-Fluid System

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Abstract

Effects of currents on wave motions in a two-fluid system are theoretically studied in this paper. Since super- and sub-harmonic waves are naturally generated by the interaction of arbitrary linear waves, it is of theoretical and practical interest to investigate the influence of pure currents not only on linear waves, but also on second-order waves. Solutions are derived using the perturbation technique. Internal Stokes waves recovered from present solutions are firstly discussed. Analyses of the super- and sub-harmonic interactions are carried out for the deep- and shallow-water configurations, respectively. Conditions resulting in the elimination of second-order waves are determined. Present solutions include and unify most existing theories for two- and single-fluid systems, and are of great importance to the analysis of random internal waves.

Keywords

Currents, harmonic waves, two-fluid system

1. Introduction

Interactions between waves and currents in a two-fluid system are of great importance to geophysical and engineering research. For pure wave motions, second-order bound waves are generated from the interaction between two arbitrary linear waves. There are two components of second-order waves, the sum-frequency and the difference-frequency components, which refer to the super- and sub-harmonic components, respectively. Stokes [1847] derived the second-order solutions describing the self-interaction of the first-order waves for a single-fluid system. This self-interaction is a special case of the superharmonic interactions. The Stokes theory was extended by Hunt [1961] and Thorpe [1968] to examine the interfacial wave motion between two fluids of different densities. For random second-order waves generated by arbitrary linear waves, Liu [2006] derived solutions of super- and sub-harmonic waves in a two-fluid system in which the current effects and surface tension are not included. Most existing theories either for a single-fluid system or for a two-fluid system can be recovered from Liu's solutions.

In the presence of stream effects, most studies have focused upon the interactions between Stokes waves and currents. These studies assume that the speeds of upper and lower currents are uniform but different in each layer. For example, Miles [1986] derived the solutions of wave-current interaction and analyzed the associated Kelvin-Helmholtz instability. Though current effects on Stokes waves have been well investigated, the theory of coupled effects between currents and super- and sub-harmonic waves is not well developed. Hence, the objective of the present study is to theoretically establish the second-order solutions of super- and sub-harmonic waves arising from the interaction between arbitrary linear waves and currents. Internal Stokes waves are examined. Solutions for the deep- and shallow-water configurations are investigated and critical conditions which result in the elimination of second-order waves are consequently determined.

2. Derivation

A two-fluid system bounded by two rigid plates, as shown in Fig. 1, is used for analysis. The flow in each layer is assumed to be irrotational and inviscid. The symbols Φ_i, U_i, ρ_i and h_i represent the velocity potential for pure wave motions, the current speed, the density, and the undisturbed layer thickness for the upper (i=1) and lower (i=2) fluids, respectively. The horizontal coordinate is denoted by x, while z indicates the vertical coordinate starting at the undisturbed interface and pointing upward. The displacement of the interface is represented by η. The current in each layer is assumed to be uniformly flowing along the x direction. The governing equations and boundary conditions for the two-dimensional wave motions are as follows

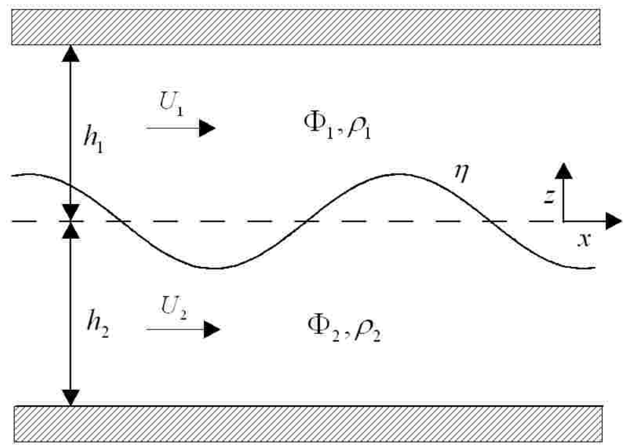

Fig.1 Definition Sketch

$$\frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial z^2} = 0, \qquad i = 1,2, \tag{1}$$

$$\frac{\partial \Phi_i}{\partial z} = \frac{\partial \eta}{\partial t} + \left(\frac{\partial \Phi_i}{\partial x} + U_i\right)\frac{\partial \eta}{\partial x} \qquad \text{at} \quad z = \eta, \qquad i = 1,2, \tag{2}$$

$$\left\langle \rho_i \left[g\eta + \frac{\partial \Phi_i}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi_i}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi_i}{\partial z}\right)^2 + U_i \frac{\partial \Phi_i}{\partial x} \right] \right\rangle + \sigma \frac{\partial^2 \eta}{\partial x^2}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-\frac{3}{2}} = 0 \qquad \text{at} \quad z = \eta \tag{3}$$

$$\frac{\partial \Phi_1}{\partial z} = 0 \quad \text{at} \quad z = h_1 \text{ and } \frac{\partial \Phi_2}{\partial z} = 0 \quad \text{at} \quad z = -h_2, \tag{4}$$

where g and σ are the gravity and the surface tension coefficient, respectively. The symbol $\langle F_i \rangle$ is the difference in a quantity across the interface, namely $\langle F_i \rangle = F_1 - F_2$ where the subscripts 1 and 2 indicate the upper and lower layers, respectively. The velocity potentials and the displacement of interface are then expanded as

$$[\Phi_1, \Phi_2, \eta] = \sum_{n=1}^{2} \left[\Phi_1^{(n)}, \Phi_2^{(n)}, \eta^{(n)}\right], \tag{5}$$

where the superscripts indicate the order of magnitude in powers of the wave slope ε, i.e. $\Phi_1^{(1)} = O(\varepsilon)$,

$\Phi_1^{(2)} = O(\varepsilon^2)$, etc. Inserting Eq.(5) into Eqs.(1) to (4) and expanding boundary conditions about $z = 0$ using the Taylor-series expansion generate the wave system order by order. The first-order components are represented in the following complex forms

$$\eta^{(1)} = \frac{D^{(1)}}{2} e^{i(kx-\omega t)} + \text{c.c.}, \tag{6}$$

$$\Phi_1^{(1)} = -i\frac{A^{(1)}}{2}\cosh k(z-h_1)\cdot e^{i(kx-\omega t)} + \text{c.c.}, \tag{7}$$

$$\Phi_2^{(1)} = -i\frac{B^{(1)}}{2}\cosh k(z+h_2)\cdot e^{i(kx-\omega t)} + \text{c.c.}, \tag{8}$$

where c.c. denotes the complex conjugate of the preceding term, and. k andω are the wavenumber and frequency of linear waves. The relations between $A^{(1)}$, $B^{(1)}$ and $D^{(1)}$ are readily obtained

$$A^{(1)} = \frac{-\omega + U_1 k}{k\sinh kh_1}\cdot D^{(1)}, \tag{9}$$

$$B^{(1)} = \frac{\omega - U_2 k}{k\sinh kh_2}\cdot D^{(1)}. \tag{10}$$

The linear dispersion relation associated with the effects of currents and surface tension is

$$\rho_1\frac{(\omega - U_1 k)^2}{\tanh kh_1} + \rho_2\frac{(\omega - U_2 k)^2}{\tanh kh_2} + (\rho_1 - \rho_2)gk - \sigma k^3 = 0. \tag{11}$$

From Eq.(11), the critical condition for the well-known Kelvin-Helmholtz instability (for detailed descriptions, see Drazin and Reid [1981]) can be recovered and shown as

$$(U_1 - U_2)^2 = \frac{\tanh kh_1\cdot\tanh kh_2}{\rho_1\rho_2 k}\left(\frac{\rho_1}{\tanh kh_1} + \frac{\rho_2}{\tanh kh_2}\right)\left((\rho_2 - \rho_1)g - \sigma k^2\right). \tag{12}$$

This critical condition indicates that when the difference of U_1 and U_2 exceeds the critical value, the initial disturbance will grow with time. It is also obvious that the existence of surface tension is advantageous to suppress the unstable phenomenon.

Based on the above first-order solutions, the corresponding governing equations and boundary conditions for second-order components are

$$\frac{\partial^2\Phi_i^{(2)}}{\partial x^2} + \frac{\partial^2\Phi_i^{(2)}}{\partial z^2} = 0, \quad i = 1,2, \tag{13}$$

$$\frac{\partial\Phi_i^{(2)}}{\partial z} - \frac{\partial\eta^{(2)}}{\partial t} - U_i\frac{\partial\eta^{(2)}}{\partial x} = -\frac{\partial^2\Phi_i^{(1)}}{\partial z^2}\eta^{(1)} + \frac{\partial\Phi_i^{(1)}}{\partial x}\frac{\partial\eta^{(1)}}{\partial x}, \text{ at } \quad z = 0, \quad i = 1,2 \tag{14}$$

$$\left\langle\rho_i\left[g\eta^{(2)} + \frac{\partial\Phi_i^{(2)}}{\partial t} + U_i\frac{\partial\Phi_i^{(2)}}{\partial x}\right]\right\rangle = -\left\langle\rho_i\left[\frac{\partial^2\Phi_i^{(1)}}{\partial t\partial z}\eta^{(1)} + \frac{1}{2}\left(\frac{\partial\Phi_i^{(1)}}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\Phi_i^{(1)}}{\partial z}\right)^2 + U_i\frac{\partial^2\Phi_i^{(1)}}{\partial x\partial z}\eta^{(1)}\right]\right\rangle,$$

at $\quad z = 0,$ (15)

$$\frac{\partial\Phi_1^{(2)}}{\partial z} = 0 \quad \text{at } z = h_1 \text{ and } \quad \frac{\partial\Phi_2^{(2)}}{\partial z} = 0 \quad \text{at } z = -h_2. \tag{16}$$

In Eqs.(13) to (16), all second-order components appear in the left-hand sides, and the terms on the right denote the nonlinear effects arising from interactions of first-order components. The second-order components, which contain the super- and sub-harmonic parts generated by the interaction of the linear waves (m-wave and n-wave), are assumed to be:

$$\eta_{mn}^{(2)\pm} = \frac{D_{mn}^{(2)\pm}}{2}\exp\left(ik_{mn}^{\pm}x - i\omega_{mn}^{\pm}t\right) + \text{c.c.}, \tag{17}$$

$$\Phi_{1mn}^{(2)\pm} = -i\frac{A_{mn}^{(2)\pm}}{2}\cosh k_{mn}^{\pm}(z-h_1)\exp\left(ik_{mn}^{\pm}x - i\omega_{mn}^{\pm}t\right) + \text{c.c.}, \tag{18}$$

$$\Phi_{2mn}^{(2)} = -i\frac{B_{mn}^{(2)\pm}}{2}\cosh k_{mn}^{\pm}(z+h_2)\exp\left(ik_{mn}^{\pm}x - i\omega_{mn}^{\pm}t\right) + \text{c.c.}, \tag{19}$$

where $k_{mn}^{\pm} = k_m \pm k_n$, $\omega_{mn}^{\pm} = \omega_m \pm \omega_n$ and the superscripts plus and minus denote the super- and sub-harmonic interactions, respectively. After substituting Eqs.(17) to (19) into Eqs.(13) to (16), the second-order amplitude $D_{mn}^{(2)\pm}$ is solved

$$D_{mn}^{(2)\pm} = \frac{D_m^{(1)}D_n^{(1)}}{2}\frac{\Delta_n^{\pm}}{\Delta_d^{\pm}}, \tag{20}$$

where

$$\Delta_d^{\pm} = \rho_1\frac{\left(V_{mn1}^{\pm}\right)^2}{k_{mn}^{\pm}T_{mn1}^{\pm}} + \rho_2\frac{\left(V_{mn2}^{\pm}\right)^2}{k_{mn}^{\pm}T_{mn2}^{\pm}} + (\rho_1-\rho_2)g - \sigma\left(k_{mn}^{\pm}\right)^3, \tag{21}$$

$$\Delta_n^{\pm} = \left\langle \rho_i\left[V_{mi}^2 + V_{ni}^2 - V_{mi}V_{ni}\left(\frac{1}{T_{mi}T_{ni}} \mp 1\right) - \frac{V_{mni}^{\pm}}{T_{mni}^{\pm}}\left(\frac{V_{mi}}{T_{mi}} + \frac{V_{ni}}{T_{ni}}\right)\right]\right\rangle, \tag{22}$$

$$\begin{cases} V_{mi} = k_m U_i - \omega_m \\ V_{ni} = k_n U_i - \omega_n \\ V_{mni}^{\pm} = k_{mn}^{\pm} U_i - \omega_{mn}^{\pm} \end{cases} \quad \text{and} \quad \begin{cases} T_{mi} = \tanh k_m h_i \\ T_{ni} = \tanh k_n h_i \\ T_{mni}^{\pm} = \tanh k_{mn}^{\pm} h_i \end{cases}, \quad i = 1,2. \tag{23}$$

Note that the magnitude of defined in Eq.(23) suggests the magnitude of current speed in comparison to the individual wave phase velocity. The coefficients of velocity potentials are solved and given below

$$A_{mn}^{(2)\pm} = \frac{A_m^{(1)}A_n^{(1)}}{2}\cdot\frac{k_m k_n T_{m1}T_{n1}}{V_{m1}V_{n1}\left(T_{m1}+T_{n1}\right)}\left[\frac{V_{m1}}{T_{m1}} + \frac{V_{n1}}{T_{n1}} + \frac{V_{mn1}^{\pm}}{k_{mn}^{\pm}}\frac{\Delta_n^{\pm}}{\Delta_d^{\pm}}\right], \tag{24}$$

$$B_{mn}^{(2)\pm} = \frac{B_m^{(1)}B_n^{(1)}}{2}\cdot\frac{k_m k_n T_{m2}T_{n2}}{V_{m2}V_{n2}\left(T_{m2}+T_{n2}\right)}\left[\frac{V_{m2}}{T_{m2}} + \frac{V_{n2}}{T_{n2}} - \frac{V_{mn2}^{\pm}}{k_{mn}^{\pm}}\frac{\Delta_n^{\pm}}{\Delta_d^{\pm}}\right]. \tag{25}$$

3. Internal Stokes waves

Internal Stokes waves are of great significance and are examined based on above solutions in this section. The Stokes theory is apparently a special case of superharmonic interactions and always attracts a great deal of attention. The interface displacement for the second-order Stokes theory with current effects is recovered from the superharmonic component of Eq.(20)

$$\eta_S = \frac{D_S^{(1)}}{2}e^{i(kx-\omega t)} + D_S^{(1)}\frac{kD_S^{(1)}}{8}\left[\frac{\rho_1 V_1^2\left(1-3T_1^{-2}\right) + \rho_2 V_2^2\left(3T_2^{-2}-1\right)}{\rho_1 V_1^2 T_1 + \rho_2 V_2^2 T_2 - 3\sigma k^3}\right]\cdot e^{2i(kx-\omega t)} + \text{c.c.}, \tag{26}$$

where S represents the quantity for Stokes waves and

$$\begin{cases} V_i = kU_i - \omega \\ T_i = \tanh kh_i \end{cases}, \quad i = 1,2. \tag{27}$$

Eq.(26) is equivalent to Mile's result [1986]. If one neglects the current effects and surface tension, Eq.(26) can be further reduced to

$$\eta_S = \frac{D_S^{(1)}}{2} e^{i(kx-\omega t)} + D_S^{(1)} \frac{k D_S^{(1)}}{8} \left[\frac{\rho_1 \left(1 - 3T_1^{-2}\right) + \rho_2 \left(3T_2^{-2} - 1\right)}{\rho_1 T_1 + \rho_2 T_2} \right] \cdot e^{2i(kx-\omega t)} + \text{c.c.} \tag{28}$$

Eq.(28) is identical to that given by Hunt [1961] and Thorpe [1968]. The well-known phenomenon of Stokes waves is that the shape of wave crests and troughs will be slightly modified due to the second-order effects (*cf*. Liu [2006] and Liu and Hwung [2006]). It is noted that only the superharmonic component is considered in the classical Stokes theory. As for the role of subharmonic components, Chen [2006] analyzed the set-down phenomenon for second-order Stokes theory for a single-fluid system by comparing the Stokes theory and the bichromatic wave theory. It is observed that the subharmonic wave components play an important role in changing the mean water level. Therefore, in present study, if one assumes frequencies of two first-order waves approach to each other, the subharmonic solution is ($\omega_n \to \omega_m = \omega$, $k_n \to k_m = k$)

$$D_{mn}^{(2)-} = \frac{D_m^{(1)} D_n^{(1)}}{2} \frac{\Delta_n^-}{\Delta_d^-}, \tag{29}$$

where

$$\Delta_n^- = \left\langle \rho_i k^2 \left[C^2 \left(1 - \frac{1}{\tanh^2 kh_i} \right) - \frac{2CC_g}{kh_i \tanh kh_i} \right] \right\rangle, \tag{30}$$

$$\Delta_d^- = \left(\frac{\rho_1}{h_1} + \frac{\rho_2}{h_2} \right) C_g^2 + (\rho_1 - \rho_2) g, \tag{31}$$

and $C \equiv \omega/k$, $C_g \equiv \partial\omega/\partial k$ in which C and C_g can be readily obtained from the linear dispersion relation. With the assumption $D_m^{(1)} = D_n^{(1)} = D_S^{(1)}/2$, above limiting subharmonic solution which is non-oscillatory and influences only the mean water level should be incorporated into Eq.(28) in order to represent Stokes theory more completely. Since general second-order wave theory (cf. Liu[2006]) contains the subharmonic effects, one could achieve the coincidence between the Stokes theory and general theory by adjusting the Bernoulli constant appearing in the dynamic boundary condition.

4. Analyses of the deep- and shallow-water approximations

In this section, the super- and sub-harmonic interactions are examined for the deep-water and shallow-water configurations, respectively. It should be noted that the assumption that $k_m > k_n > 0$ is made throughout this section. For the deep-water configuration, wavelengths of all waves are assumed to be much smaller than thicknesses of both layers. Hence, values of all hyperbolic tangent functions appearing in Eqs.(20) to (23) will approach unity. The resulting second-order amplitude is

$$D_{mn}^{(2)\pm} = \frac{D_m^{(1)} D_n^{(1)}}{2} \cdot \left(\pm k_{mn}^{\pm} \right) \cdot \frac{R^{\pm} - 1}{R^{\pm} + 1}, \tag{32}$$

where

$$\begin{cases} R^+ = \dfrac{\rho_2 V_{m2} V_{n2}}{\rho_1 V_{m1} V_{n1}} \\ R^- = \dfrac{\rho_2 (V_{m2} - V_{n2}) V_{n2}}{\rho_1 (V_{m1} - V_{n1}) V_{n1}} \end{cases}. \tag{33}$$

Assuming $\rho_1 \cong \rho_2$, the condition for $D_{mn}^{(2)+} = 0$ is determined by

$$U_1 + U_2 = \frac{\omega_m}{k_m} + \frac{\omega_n}{k_n}. \tag{34}$$

Eq.(34) indicates that, when the summation of current speeds is equivalent to the summation of phase velocities of two basic waves, the second-order effects on wave amplitude will be zero. As for the subharmonic case, the condition becomes

$$U_1 + U_2 = \frac{\omega_n}{k_n} + \frac{\omega_m - \omega_n}{k_m - k_n}. \tag{35}$$

which results in $D_{mn}^{(2)-} = 0$. Eq.(35) shows that no second-order amplitude occurs if the summation of current speeds is equivalent to the summation of the group velocity and the phase velocity of the longer basic wave.

For the shallow-water configurations, wavelengths of all waves are assumed to be much longer than thicknesses of both layers. This leads to

$$D_{mn}^{(2)\pm} = \frac{D_m^{(1)} D_n^{(1)}}{2} \cdot \left(\frac{\pm k_{mn}^{\pm}}{h_1} \right) \cdot \frac{\Delta_n^{\pm}}{\Delta_d^{\pm}}, \tag{36}$$

where

$$\Delta_n^{\pm} = \rho_1 \left[\frac{V_{m1}^2}{k_{mn}^{\pm} k_m} \pm \frac{V_{n1}^2}{k_{mn}^{\pm} k_n} + \frac{2 V_{m1} V_{n1}}{k_m k_n} \right] - \rho_2 \left(\frac{h_1}{h_2} \right)^2 \left[\frac{V_{m2}^2}{k_{mn}^{\pm} k_m} \pm \frac{V_{n2}^2}{k_{mn}^{\pm} k_n} + \frac{2 V_{m2} V_{n2}}{k_m k_n} \right], \tag{37}$$

$$\Delta_d^{\pm} = \rho_1 \left[V_{m1} \sqrt{\frac{k_n}{k_{mn}^{\pm} k_m}} - V_{n1} \sqrt{\frac{k_m}{k_{mn}^{\pm} k_n}} \right]^2 + \rho_2 \left(\frac{h_1}{h_2} \right) \left[V_{m2} \sqrt{\frac{k_n}{k_{mn}^{\pm} k_m}} - V_{n2} \sqrt{\frac{k_m}{k_{mn}^{\pm} k_n}} \right]^2. \tag{38}$$

It is found that the coupled effects between waves and the upper (lower) current dominate the magnitude of second-order amplitude while $h_1/h_2 << 1$ ($h_1/h_2 >> 1$). If one further imposes the restriction of narrow-band frequencies into above results, Eq.(36) can be further simplified to be

$$D_{mn}^{(2)+} = \frac{D_m^{(1)} D_n^{(1)}}{2} \cdot \left(\frac{2}{h_1} \right) \cdot \frac{\rho_1 \left(V_{m1}^2 + 4 V_{m1} V_{n1} + V_{n1}^2 \right) - \rho_1 \left(\frac{h_1}{h_2} \right)^2 \left(V_{m2}^2 + 4 V_{m2} V_{n2} + V_{n2}^2 \right)}{\rho_1 \left(V_{m1} - V_{n1} \right)^2 + \rho_2 \left(\frac{h_1}{h_2} \right) \left(V_{m2} - V_{n2} \right)^2} + O\left(\frac{\Delta k}{\bar{k}} \right), \tag{39}$$

$$D_{mn}^{(2)-} = \frac{D_m^{(1)} D_n^{(1)}}{2} \cdot \left(\frac{-1}{h_1} \right) \left(\frac{\Delta k}{\bar{k}} \right) \cdot \frac{\rho_1 \left(V_{m1}^2 - V_{n1}^2 \right) - \rho_1 \left(\frac{h_1}{h_2} \right)^2 \left(V_{m2}^2 - V_{n2}^2 \right)}{\rho_1 \left(V_{m1} - V_{n1} \right)^2 + \rho_2 \left(\frac{h_1}{h_2} \right) \left(V_{m2} - V_{n2} \right)^2} + O\left(\frac{\Delta k}{\bar{k}} \right)^2, \tag{40}$$

where $\bar{k} = \frac{1}{2}(k_m + k_n)$, $\Delta k = k_m - k_n$ and $\Delta k << \bar{k}$. On the basis of Eqs.(39) and (40), one can determine the conditions resulting in the elimination of second-order amplitude under specific assumptions. For example, if the speed of upper current is equivalent to the group velocity while $h_1 << h_2$, the second-order amplitude for the subharmonic case would be zero.

5. Conclusions

The wave-current interaction in a two-fluid system is investigated in present study. Super- and sub-harmonic components are included in second-order solutions. Based on present solutions, subharmonic component missing in the classical Stokes theory is discussed. For the deep-water and shallow-water configurations, the conditions resulting in the elimination of second-order amplitude are determined. Moreover, most existing theories for two- and single-fluid systems are included and unified in present solutions.

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References

Chen, X. B. (2006), The set-down in the second-order Stokes' waves, *ICHD'2006*, Ischia, Italy.

Drazin, P. G. and W. H. Reid (1981), *Hydrodynamic stability*, Cambridge Univ. Press, New York.

Hunt, J. N. (1961), Interfacial waves of finite amplitude, *La Houille Blanche*, 4, 515-531.

Liu, C. M. (2006), Second-order random internal and surface waves in a two-fluid system, *Geophys. Res. Lett.*, 33, L06610.

Liu, C. M. and H. H. Hwung (2006), Second-order effects on wave properties in a two-fluid system, in submission.

Miles, J. W. (1986), Weakly nonlinear Kelvin-Helmholtz waves, *J. Fluid Mech.*, 172, 513-529.

Stokes, G. G. (1847), On the theory of oscillatory waves, *Trans. Cambridge Philos. Soc.*, 8, 441-455.

Thorpe, S. A. (1968), On the shape of progressive internal waves, *Phil. Tran. R. Soc. Lond. A.*, 263, 563-614.