Value of Accuracy in Linear Systems

Miguel Bagajewicz
School of Chemical Engineering and Materials Science, University of Oklahoma, Norman, OK 73019

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A connection between precision and downside expected lost revenue for linear systems was developed in previous work. Having the value of precision and accuracy in economic terms helps justify the upgrade of instrumentation and/or the increase in corrective maintenance repair rate. The connection between accuracy and economic value is extended in this article to the case where biases are present and in the context of existing corrective maintenance capabilities.
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Introduction

Better monitoring, that is, more precise or accurate measurements or estimators of key process variables, is a desirable goal. In particular, we need these good data for several activities: control, on-line optimization, parameter estimation, and production accounting, among others. One way in which this is done is through data reconciliation.[1] One problem that is associated with good monitoring is the efficient location of sensors, for which there are a variety of methods.[2]

One of the major obstacles in industry for the justification of instrumentation upgrade projects or the purchase/installation of software that will improve monitoring is that the cost is known, but the benefit (in economic terms) is unclear in some cases. In previous articles,[3,4] a statistical analysis was made to determine the economic value of precision. A formula was developed for such value based on the downside expected loss that occurs when an operator adjusts the throughput of a plant when the measurements or estimators obtained through data reconciliation suggest that the targeted production is met or surpassed. However, there is a finite probability that the measurement or estimator is above the target when, in fact, the real flow is below it; hence, the expected financial loss calculation. The associated probability (25%) is viewed as the confidence with which these expected losses is known. For the case of low process variability (steady state), the expected financial loss is proportional to the precision (standard deviation) of the estimator, a remarkably simple formula.

While precision is important, most instruments present biases and, therefore, the theory of economic value of precision needs to be extended to include them. To understand how biases corrupt the estimators, Bagajewicz[5] has defined the concept of software accuracy, which is based on the notion that data reconciliation with some test statistics is used to detect biases. This definition is based on an extension of the definition of accuracy for individual measurement. Indeed, accuracy is defined as the sum of precision and bias,[5] which in the case of individual measurements cannot be assessed from the measurement itself. In turn, if measurements are redundant through a model, one can perform statistical tests and detect these biases, but only after they have reached a certain threshold value. Below this threshold, the bias goes undetected and smears all the estimators, including those of the variables for which the corresponding instrument has no bias, called induced bias. Thus, software accuracy of a variable is defined as the sum of the precision (standard deviation) of the estimator plus the maximum possible undetected induced bias in that variable due to a sensor bias anywhere in the system, including the instrument measuring the variable itself. We use this concept here to guide the analysis.

This article is organized as follows: Previous work is reviewed first. Then, the probability of financial losses in the presence of biases is discussed and an expression is derived. The cases of one, two, and many biases present in the system are treated and then illustrated for selected special cases. Next, an expression for the downside financial loss is presented. Finally, the way to analyze the trade-off between value and cost through the use of net present value is briefly discussed. Finally, an example is shown.

M. Bagajewicz's e-mail address is: bagajewicz@ou.edu.

Figure 1. Material balance in a refinery.

Economic Value of Accuracy in Mass Flow Rates

Preliminaries

Bagajewicz et al.[3] argued that a typical refinery consists of several tank units that receive the crude, several processing units, and several tanks where products are stored, all this summarized in three blocks as in Figure 1. In this Figure, the last block represents the tanks, and the inventory associated is represented by $H_S(T)$, where T is the window of time under consideration. The system has feed streams m_r, intermediate streams m_I, and product stream m_p

They argued that the probability of not meeting the targeted production is $P\{H_S(T) \le H_S^*\}$, where H_S^* is a target value. This probability can be rewritten as $P\{m_p(t) \le m_p^*\}$, where m_p^* is a target flowrate. This is the probability of the true value of m_p being smaller than the target. Let $\hat{m}_p$ be the estimate one has of the true value of m_p obtained through direct measurement or through data reconciliation. Consider that production is adjusted to meet the targeted value, based on the estimate. In other words, if $\hat{m}_p < m_p^*$, production is increased; and vice versa, if $\hat{m}_p > m_p^*$, production is decreased. Bagajewicz et al.[3] assumed, however, that when $\hat{m}_p > m_p^*$, that is, when the measurement indicates that the target has been met or exceeded, the operator would not do any correction to the set points. They argue that the probability of this being incorrect, that is, that reality is that the true flowrate is lower than the target, is given by the following conditional probability $P\{m_p \le m_p^* | \hat{m}_p \ge m_p^*\}$, for which they derive the following expression:

$$P\{\hat{m}_p \ge m_p^* | m_p \le m_p^*\} = \int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi;\, m_p,\, \hat{\sigma}_p) d\xi \right\} g_P(m_p;\, m_p^*,\, \sigma_p) dm_p \quad (1)$$

where $g_M(\xi; m_p, \hat{\sigma}_p)$ is the probability distribution of measurements around the real value, and $g_P(m_p; m_p^*, \sigma_p)$ is the probability distribution of the real value around the targeted mean. The integral is taken over all possible values of m_p below the target because of the underlying assumption that m_p is lower than the target.

Effect of biases on the probability distribution

When there is a bias, induced or not, it could go undetected, which means it has an absolute value size smaller than $\hat{\delta}_{p,\max}^{i1,\ldots,in_b}$, which is the maximum induced bias that goes undetected by the maximum power measurement test when there are n_b gross errors. This value is a function of the existing instrumentation precision and the method being used to detect gross errors. When there is no redundancy, this value is, theoretically, infinite, but in practical terms, when the bias reaches a certain value $\delta_p^{\#}$, it becomes truly apparent to the operator that there is a bias and, hopefully, the instrument is calibrated. When there is redundancy, the value is finite and depends on the method used. We therefore concentrate on redefining $g_M(\xi; m_p, \hat{\sigma}_p)$ to include the possibility of biases.

Assuming one gross error in variable i and none in the others (that is, $n_b = 1$), we have

$$g_M = g_M^{R,i}(\xi;\, m_p,\, \hat{\sigma}_{p,i}^R) \quad |\hat{\delta}_p^i| > \hat{\delta}_{p,\max}^i \quad (2)$$

where $\hat{\sigma}_{p,i}^R$ is the residual precision left after the measurement of variable i has been eliminated, $\hat{\delta}_p^i$ is the induced bias in the estimator of m_p, which is a function of the original bias, δ_i, that is, $\hat{\delta}_p^i = \hat{\delta}_p^i(\delta_i)$. We note that under the condition of one and only one bias present in the system, certain tests like the maximum power and GLR tests are consistent, that is, if one bias is present, then when these tests flag positive, they point to the correct location of the bias. This assumption is important because in the absence of consistency, the assumption implicit in Eq. 2, that the right measurement will be eliminated, is no longer valid. From now on, we will assume consistency in gross error detection. In turn, when the induced gross error is smaller than the threshold of detection, then it is undetected and therefore:

$$g_M = g_M(\xi;\, m_p + \hat{\delta}_p^i,\, \hat{\sigma}_p) \quad |\hat{\delta}_p^i| \le \hat{\delta}_{p,\max}^i \quad (3)$$

Thus, the probability of the estimate to be higher than the true value, given a bias in measurement i, is now given by:

$$P\{\hat{m}_p \ge m_p^* | \delta_i\} = \begin{cases} \displaystyle\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi;\, m_p,\, \hat{\sigma}_{p,i}^R) d\xi \right\} g_P(m_p;\, m_p^*,\, \sigma_p) dm_p & |\hat{\delta}_p^i| > \hat{\delta}_{p,\max}^i \\ \displaystyle\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi;\, m_p + \hat{\delta}_p^i,\, \hat{\sigma}_p) d\xi \right\} g_P(m_p;\, m_p^*,\, \sigma_p) dm_p & |\hat{\delta}_p^i| \le \hat{\delta}_{p,\max}^i \end{cases} \quad (4)$$

which is a direct extension of Eq. 1 to the presence of one gross error.

Let us assume that when an instrument fails, which happens according to a certain probability $f_i(t)$ (a function of time), the size of the bias follows a certain distribution $h_i(\theta; \bar{\delta}_i, \rho_i)$ with mean $\bar{\delta}_i$ and variance ρ_i^2. Note that, depending on the value of the measurement in the range of the instrument, the mean is very likely to be nonzero.[3,6] We are also assuming here that the gross error size distribution is independent of time. Thus, we now need to integrate over all possible values of the gross error and multiply by the probability of such bias to develop. Therefore, if we assume that one instrument fails at a time, then, the probability of instrument i failing and the others not is given by: $\Phi_i^1 = f_i(t) \prod_{s \ne i} [1 - f_s(t)]$. Thus,

$$P\{\hat{m}_p \geq m_p^*|i\} = \Phi_i^1 \int_{-\infty}^{\infty} P\{\hat{m}_p \geq m_p^*|\theta\} h_i(\theta; \bar{\delta}_i, \rho_i)d\theta = \Phi_i^1 \int_{-\infty}^{-\bar{\delta}_i^p} \left[\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi; m_p, \hat{\sigma}_{p,i}^R)d\xi \right\} g_P(m_p; m_p^*, \sigma_p)dm_p \right]$$
$$\times h_i(\theta; \bar{\delta}_i, \rho_i)d\theta + \Phi_i^1 \int_{-\bar{\delta}_i^p}^{\bar{\delta}_i^p} \left[\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi; m_p + \hat{\delta}_p^i(\theta), \hat{\sigma}_p)d\xi \right\} g_P(m_p; m_p^*, \sigma_p)dm_p \right] h_i(\theta; \bar{\delta}_i, \rho_i)d\theta$$
$$+ \Phi_i^1 \int_{\bar{\delta}_i^p}^{\infty} \left[\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi; m_p, \hat{\sigma}_{p,i}^R)d\xi \right\} g_P(m_p; m_p^*, \sigma_p)dm_p \right] h_i(\theta; \bar{\delta}_i, \rho_i)d\theta\} \quad (5)$$

where $P\{\hat{m}_p \geq m_p^*|i\}$ indicates that the probability is conditional to the presence of one gross error in stream *i.* In turn, $\tilde{\delta}_i^p$ is the absolute value of the error in stream i that corresponds to the maximum undetectable (or minimum detectable) induced bias in stream p ($\hat{\delta}_{p,\max}^i$). Eq. 5 has three terms on the right hand side, one integral from $-\infty$ to $-\tilde{\delta}_i^p$, another term from $-\tilde{\delta}_i^p$ to $\tilde{\delta}_i^p$, and a third one from $\tilde{\delta}_i^p$ to ∞, representing the three different regions where the integrand changes. Operating:

$$\begin{aligned} &P\{\hat{m}_p \geq m_p^*|i\} \\ &= \Phi_i^1 \left[\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi; m_p, \hat{\sigma}_{p,i}^R)d\xi \right\} g_P(m_p; m_p^*, \sigma_p)dm_p \right] \\ &\quad \left[\int_{-\infty}^{\tilde{\delta}_i^p} h_i(\theta; \bar{\delta}_i, \rho_i)d\theta + \int_{\tilde{\delta}_i^p}^{\infty} h_i(\theta; \bar{\delta}_i, \rho_i)d\theta \right] \\ &\quad + \Phi_i^1 \int_{-\tilde{\delta}_i^p}^{\tilde{\delta}_i^p} \left[\int_{-\infty}^{m_p^*} \left\{ \int_{m_p^*}^{\infty} g_M(\xi; m_p + \hat{\delta}_p^i(\theta), \hat{\sigma}_p)d\xi \right\} \right. \\ &\quad \left. \times g_P(m_p; m_p^*, \sigma_p)dm_p \right] h_i(\theta; \bar{\delta}_i, \rho_i)d\theta \end{aligned} \quad (6)$$

Thus, the probability of having the measurement above the target in the presence of one and only one gross error in the system is given by the probability of one gross error in variable $i1$, OR one gross error in variable $i2$, and so on. Therefore, since all the events are assumed independent, the probability of having one measurement to be above the target when one gross error is present in the system is:

$$P\{\hat{m}_p \geq m_p^*|n_b = 1\} = \sum_{\forall i} P\{\hat{m}_p \geq m_p^*|i\} \quad (7)$$

Effect of biases on the probability distribution for more than one gross error

Consider now that two instruments at a time can fail. Then we write: $\Phi_{i1,i2}^2 = f_{i1}(t) f_{i2}(t) \prod_{s \neq i1, s \neq i2} [1 - f_s(t)]$. Thus, we write:

$$P\{\hat{m}_p \geq m_p^*|i1, i2\} = \Phi_{i1,i2}^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$$
$$\times \; P\{\hat{m}_p \geq m_p^*|\theta_1, \theta_2\} h_{i1}(\theta_1; \bar{\delta}_{i1}, \rho_{i1}) h_{i2}(\theta_2; \bar{\delta}_{i2}, \rho_{i2}) d\theta_1 d\theta_2 \quad (8)$$

To evaluate the above integral, one needs to recognize that one cannot use any of the fixed limits obtained for the maximum undetectable (or minimum detectable) induced bias in stream p calculated for single gross errors, $\tilde{\delta}_{i1}^p$ and $\tilde{\delta}_{i2}^p$ (corresponding to $\hat{\delta}_{p,\max}^{i1}$ and $\hat{\delta}_{p,\max}^{i2}$), as limits. These limits of detectability come in pairs, and there are infinite pairs that constitute thresholds. In other words, $\hat{\delta}_{p,\max}^{i1,i2} = \hat{\delta}_{p,\max}^{i1,i2}(\tilde{\delta}_{i1}^p, \tilde{\delta}_{i1}^p)$, does not have a unique inverse. This means that one has to have a test to choose which distribution function needs to be used for each value of the integrand. We therefore define the following function:

$$G(\delta_{i1}, \ldots \delta_{ip}|\delta_{i1}, \ldots \delta_{in_b}) = \begin{cases} 1 & \textit{If the MP test will flag positive for } \delta_{i1}, \ldots \delta_{ip} \\ 0 & \end{cases} \quad (9)$$

where the test would flag for all streams $\delta_{i1}, \ldots \delta_{ip}$, in the presence of the n_b errors located in $\delta_{i1}, \ldots \delta_{in_b}$. Consistency in this context means that the set $\{\delta_{i1}, \ldots \delta_{ip}\}$ is a subset of $\{\delta_{i1}, \ldots \delta_{in_b}\}$, that is, some errors are too small to be detected, but those detected are in fact true biases. This detection can take place consecutively in a serial elimination fashion or by means of some other collective tests like UBET[7] or SICC.[8] Therefore:

$$P\{\hat{m}_p \geq m_p^*|i1, i2\} = \Phi_{i1,i2}^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P\{\hat{m}_p$$
$$\geq m_p^*|\theta_1, \theta_2\} h_{i1}(\theta_1; \bar{\delta}_{i1}, \rho_{i1}) h_{i2}(\theta_2; \bar{\delta}_{i2}, \rho_{i2}) d\theta_1 d\theta_2 \quad (10)$$

where

$$P\{\hat{m}_p \geq m_p^* | \theta_1,\ \theta_2\} = \int_{-\infty}^{m_p^*} \left\{ \begin{array}{l} G(\theta_1,\ \theta_2|\theta_1,\ \theta_2) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p,\ \hat{\sigma}_{p,i1,i2}^R) d\xi\ + \\ [1 - G(\theta_1,\ \theta_2|\theta_1,\ \theta_2)] \\ \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1,i2}(\theta_1,\ \theta_2),\ \hat{\sigma}_{p,m}) d\xi \\ \quad + G(\theta_1|\theta_1,\ \theta_2) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_{p,m}^{i2}(\theta_2),\ \hat{\sigma}_{p,i1}^R) d\xi \\ \quad + G(\theta_2|\theta_1,\ \theta_2) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1}(\theta_1),\ \hat{\sigma}_{p,i2}^R) d\xi \end{array} \right\} g_P(m_p;\ m_p^*,\ \sigma_p) dm_p \tag{11}$$

where the condition $G(\theta_1,\ \theta_2|\theta_1,\ \theta_2) + G(\theta_1|\theta_1,\ \theta_2) + G(\theta_2|\theta_1,\ \theta_2) \leq 1$ holds naturally, that is, either the two gross errors flag, or only one of them flags, or none flags. For example, if $G(\theta_1,\ \theta_2|\theta_1,\ \theta_2) = 0$, no bias has been identified and the proper distribution is the one around the true value plus the induced bias $\hat{\delta}_p^{i1,i2}(\theta_1,\ \theta_2)$ with the estimator's variance. Conversely, if $G(\theta_1,\ \theta_2|\theta_1,\ \theta_2) = 1$, the two gross errors θ_1 and θ_2 are of sufficient size to be identified, and the proper distribution is one around the true value with the residual variance $\hat{\sigma}_{p,i1,i2}^R$ obtained after the measurements on *i1* and *i2* have been eliminated. The rest of the terms for the cases where only one bias is identified and the other not, are similar. Note again that we are assuming consistency. Thus, we have:

$$P\{\hat{m}_p \geq m_p^* | n_b = 2\} = \sum_{\forall i,k} P\{\hat{m}_p \geq m_p^* | i,\ k\} \tag{12}$$

for three gross errors, $\Phi_{i1,i2,i3}^3 = f_{i1}(t) f_{i2}(t) f_{i3}(t) \prod_{s \neq i1,i2,i3} [1 - f_s(t)]$, and we write

$$P\{\hat{m}_p \geq m_p^* | i1,\ i2,\ i3\} = \Phi_{i1,i2,i3}^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\{\hat{m}_p \geq m_p^* | \theta_1,\ \theta_2,\ \theta_3\} h_{i1}(\theta_1;\ \bar{\delta}_{i1},\ \rho_{i1}) h_{i2}(\theta_2;\ \bar{\delta}_{i2},\ \rho_{i2}) h_{i3}(\theta_3;\ \bar{\delta}_{i3},\ \rho_{i3}) d\theta_1 d\theta_2 d\theta_3 \tag{13}$$

where

$$P\{\hat{m}_p \geq m_p^* | \theta_1,\ \theta_2,\ \theta_3\} = \int_{-\infty}^{m_p^*} \left\{ \begin{array}{l} G(\theta_1,\ \theta_2,\ \theta_3|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p,\ \hat{\sigma}_{p,i1,i2,i3}^R) d\xi\ + \\ [1 - G(\theta_1,\ \theta_2,\ \theta_3|\theta_1,\ \theta_2,\ \theta_3)] \\ \quad \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1,i2,i3}(\theta_1,\ \theta_2,\ \theta_3),\ \hat{\sigma}_p) d\xi \\ \quad + G(\theta_1,\ \theta_2|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i3}(\theta_3),\ \hat{\sigma}_{p,i1,i2}^R) d\xi \\ \quad + G(\theta_1,\ \theta_3|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i2}(\theta_2),\ \hat{\sigma}_{p,i1,i3}^R) d\xi \\ \quad + G(\theta_2,\ \theta_3|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1}(\theta_1),\ \hat{\sigma}_{p,i2,i3}^R) d\xi \\ \quad + G(\theta_1|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i2,i3}(\theta_2,\ \theta_3),\ \hat{\sigma}_{p,i1}^R) d\xi \\ \quad + G(\theta_2|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1,i3}(\theta_1,\ \theta_3),\ \hat{\sigma}_{p,i2}^R) d\xi \\ \quad + G(\theta_3|\theta_1,\ \theta_2,\ \theta_3) \displaystyle\int_{m_p^*}^{\infty} g_M(\xi;\ m_p + \hat{\delta}_p^{i1,i2}(\theta_1,\ \theta_2),\ \hat{\sigma}_{p,i3}^R) d\xi \end{array} \right\} g_P(m_p;\ m_p^*,\ \sigma_p) dm_p \tag{14}$$

Thus,

$$P\{\hat{m}_p \geq m_p^* | n_b = 3\} = \sum_{\forall i,k,s} P\{\hat{m}_p \geq m_p^* | i, k, s\} \quad (15)$$

Generalizing $\Phi^{n_T}_{i1,i2,\ldots,in_b} = f_{i1}(t) \ldots f_{in_b}(t) \prod_{s \neq i1,\ldots,s \neq in_b} [1 - f_s(t)]$, we write:

$$P\{\hat{m}_p \geq m_p^* | i1, \ldots, in_b\} = \Phi^{n_T}_{i1,\ldots in_b} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} P\{\hat{m}_p \geq m_p^* | \theta_1, \ldots, \theta_{in_b}\} h_{i1}(\theta_1; \bar{\delta}_{i1}, \rho_{i1}) \ldots h_{in_b}(\theta_{in_b}; \bar{\delta}_{in_b}, \rho_{in_b}) d\theta_1 \ldots d\theta_{in_b} \quad (16)$$

where

$$P\{\hat{m}_p \geq m_p^* | \theta_1, \ldots, \theta_{in_b}\} = \int_{-\infty}^{m_p^*} \left\{ \begin{array}{l} G(\theta_1, \ldots \theta_{n_b} | \theta_1, \ldots, \theta_{n_b}) \int_{m_p^*}^{\infty} g_M(\xi; m_p, \hat{\sigma}^R_{p,i1,\ldots,in_b}) d\xi \\ [1 - G_{n_b}(\theta_1, \ldots \theta_{n_b} | \theta_1, \ldots, \theta_{n_b})] \\ \int_{m_p^*}^{\infty} g_M(\xi; m_p + \hat{\delta}_p^{i1,\ldots,in_b}(\theta_1, \ldots, \theta_{in_b}), \hat{\sigma}_{p,m}) d\xi \\ + \sum_k H_{n_b-1}(\theta_k) \\ \int_{m_p^*}^{\infty} g_M(\xi; m_p + \bar{\delta}_p^k, \hat{\sigma}^R_{p,k}) d\xi \\ + \sum_{k,s} H_{n_b-2}(\theta_k, \theta_s) \int_{m_p^*}^{\infty} \\ g_M(\xi; m_p + \bar{\delta}_p^{k,s}, \hat{\sigma}^R_{p,k,s}) d\xi \\ + \ldots + \sum_k G_1(\theta_k | \theta_1, \ldots, \theta_{n_b}) \\ \int_{m_p^*}^{\infty} g_M(\xi; m_p + \hat{\delta}^{i2}_{p,m}(\theta_2), \hat{\sigma}^R_{p,\overline{\{k\}}}(\theta_1)) d\xi \end{array} \right\} g_P(m_p; m_p^*, \sigma_p) dm_p \quad (17)$$

In this expression we have used $H_s(\theta_{k_1}, \theta_{k_2} \cdots \theta_{k_s}) = G(\theta_1, \ldots \theta_{k_1-1}, \theta_{k_1+1}, \ldots \theta_{k_2-1}, \theta_{k_2+1}, \ldots \theta_{k_3-1}, \theta_{k_3+1}, \ldots, \theta_{n_b} | \theta_1, \ldots, \theta_{n_b})$. In addition, we define $\hat{\sigma}^R_{p,\overline{\{k\}}}$ as the residual precision when all but the gross error in stream k is eliminated, that is, $\hat{\sigma}^R_{p,\overline{\{k\}}} = \hat{\sigma}^R_{p,i1,\ldots k-1,k+1,\ldots in_b}$. Finally, $\bar{\delta}_p^{k1,k2,\ldots ks} = \hat{\delta}_p^{i1,\ldots ik1-1,ik1+1,\ldots,ik2-1,ik2+1,\ldots,iks-1,iks+1,\ldots,in_b}$. Thus,

$$P\{\hat{m}_p \geq m_p^* | n_T\} = \sum_{\forall i1,i2,\ldots,in_T} P\{\hat{m}_p \geq m_p^* | i1, i2, \ldots in_b\} \quad (18)$$

Since all events are mutually exclusive, the probability is given by:

$$P\{\hat{m}_p \geq m_p^*\} = \sum_{r=0}^{n} P\{\hat{m}_p \geq m_p^* | r\} \quad (19)$$

Note that the summation includes the case $r = 0$, which is the case where no gross errors are present. Thus, using a result from Bagajewicz et al.,[1] we have

$$P\{\hat{m}_p \geq m_p^*\} = \Phi^0 \left[\frac{1}{4} + \frac{1}{2\sqrt{\pi}} \int_0^{\infty} erfc(z\sigma_p/\hat{\sigma}_{p,m}) e^{-z^2} dz \right] + \sum_{r=1}^{n} P\{\hat{m}_p \geq m_p^* | r\} \quad (20)$$

where Φ^0 is the probability of no gross error being present.

Examples for special cases

We now explore some special cases. For example, for normal distributions and $\sigma_p/\hat{\sigma}_p \to 0$, $P\{\hat{m}_p \geq m_p^* | i\}$ becomes:

$$P\{\hat{m}_p \geq m_p^* | i\} = \frac{\Phi_i^1}{4} \left[\int_{-\infty}^{-\bar{\tilde{\delta}}_i^p} h_i(\theta; \delta_i, \rho_i) d\theta + \int_{\tilde{\delta}_i^p}^{\infty} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \right] + \frac{\Phi_i^1}{2} \int_{-\tilde{\delta}_i^p}^{\tilde{\delta}_i^p} \left\{ \begin{array}{ll} \frac{1}{2} erfc\left\{ -\frac{\hat{\delta}_p^i(\theta)}{\sqrt{2}\,\hat{\sigma}_p} \right\} & \hat{\delta}_p^i(\theta) \leq 0 \\ 1 - \frac{1}{2} erfc\left\{ \frac{\hat{\delta}_p^i(\theta)}{\sqrt{2}\,\hat{\sigma}_p} \right\} & \hat{\delta}_p^i(\theta) > 0 \end{array} \right\} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \quad (21)$$

We now assume that h_i is also normal, and also note that[5] $\hat{\delta}_p^i(\theta) = \alpha_p^i \theta$. In such a case,

$$P\{\hat{m}_p \geq m_p^* | i\} = \frac{\Phi_i^1}{4}\left\{\begin{array}{ll} \frac{1}{2}\,erfc\left\{\frac{(\tilde{\delta}_i^p + \bar{\delta}_i)}{\sqrt{2}\,\rho_i}\right\} + 1 - \frac{1}{2}\,erfc\left\{-\frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i}\right\} & \tilde{\delta}_i^p \leq \bar{\delta}_i < \infty \\ \frac{1}{2}\,erfc\left\{\frac{(\tilde{\delta}_i^p + \bar{\delta}_i)}{\sqrt{2}\,\rho_i}\right\} + \frac{1}{2}\,erfc\left\{\frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i}\right\} & -\tilde{\delta}_i^p \leq \bar{\delta}_i < \tilde{\delta}_i^p \\ 1 - \frac{1}{2}\,erfc\left\{-\frac{(\bar{\delta}_i + \tilde{\delta}_i^p)}{\sqrt{2}\,\rho_i}\right\} + \frac{1}{2}\,erfc\left\{\frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i}\right\} & \bar{\delta}_i \leq -\tilde{\delta}_i^p \end{array}\right\} + \frac{\Phi_i^1}{4}$$

$$+ \frac{\Phi_i^1}{4\sqrt{\pi}}\left\{\begin{array}{ll} \int_{[(-\tilde{\delta}_i^p - \bar{\delta}_i)/\sqrt{2}\,\rho_i]}^{(-\bar{\delta}_i/\sqrt{2}\,\rho_i)} erfc\left\{-\frac{\alpha_p^i(\bar{\delta}_i + t\sqrt{2}\,\rho_i)}{\sqrt{2}\,\hat{\sigma}_p}\right\} e^{-t^2} dt - \int_{(-\bar{\delta}_i/\sqrt{2}\,\rho_i)}^{[(\tilde{\delta}_i^p - \bar{\delta}_i)/\sqrt{2}\,\rho_i]} erfc\left\{\frac{\alpha_p^i(\bar{\delta}_i + t\sqrt{2}\,\rho_i)}{\sqrt{2}\,\hat{\sigma}_p}\right\} e^{-t^2} dt & \alpha_p^i \geq 0 \\ -\int_{[(-\tilde{\delta}_i^p - \bar{\delta}_i)/\sqrt{2}\,\rho_i]}^{(-\bar{\delta}_i/\sqrt{2}\,\rho_i)} erfc\left\{\frac{\alpha_p^i(\bar{\delta}_i + t\sqrt{2}\,\rho_i)}{\sqrt{2}\,\hat{\sigma}_p}\right\} e^{-t^2} dt + \int_{(-\bar{\delta}_i/\sqrt{2}\,\rho_i)}^{[(\tilde{\delta}_i^p - \bar{\delta}_i)/\sqrt{2}\,\rho_i]} erfc\left\{-\frac{\alpha_p^i(\bar{\delta}_i + t\sqrt{2}\,\rho_i)}{\sqrt{2}\,\hat{\sigma}_p}\right\} e^{-t^2} dt & \alpha_p^i < 0 \end{array}\right\} \quad (22)$$

which for the case of zero mean ($\bar{\delta}_i = 0$) reduces to:

$$P\{\hat{m}_p \geq m_p^* | i\} = \frac{\Phi_i^1}{4} \quad (23)$$

Then, for a normal distribution of biases with zero mean and negligible process variation, we have

$$\lim_{\sigma_p/\hat{\sigma}_p \to 0} P\{\hat{m}_2 \geq m_2^*\} = \frac{\Phi^0}{4} + \frac{\Phi_i^1}{4} + \sum_{r=2}^{n} P\{\hat{m}_2 \geq m_2^* | r\} \quad (24)$$

In turn, for two gross errors and under the same assumptions of normality and negligible process variations, we have:

$$P\{\hat{m}_p \geq m_p^* | \theta_1, \theta_2\} = \frac{1}{2}\left\{\begin{array}{l} G(\theta_1, \theta_2 | \theta_1, \theta_2)\,\frac{1}{2} + \\ [1 - G(\theta_1, \theta_2 | \theta_1, \theta_2)]\left\{\begin{array}{ll} \frac{1}{2}\,erfc\left\{-\frac{\hat{\delta}_p^{i1,i2}(\theta_1, \theta_2)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_p^{i1,i2}(\theta_1, \theta_2) \leq 0 \\ 1 - \frac{1}{2}\,erfc\left\{\frac{\hat{\delta}_p^{i1,i2}(\theta_1, \theta_2)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_p^{i1,i2}(\theta_1, \theta_2) > 0 \end{array}\right\} \\ + G(\theta_1 | \theta_1, \theta_2)\left\{\begin{array}{ll} \frac{1}{2}\,erfc\left\{-\frac{\hat{\delta}_{p,m}^{i2}(\theta_2)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_{p,m}^{i2}(\theta_2) \leq 0 \\ 1 - \frac{1}{2}\,erfc\left\{\frac{\hat{\delta}_{p,m}^{i2}(\theta_2)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_{p,m}^{i2}(\theta_2) > 0 \end{array}\right\} \\ + G(\theta_2 | \theta_1, \theta_2)\left\{\begin{array}{ll} \frac{1}{2}\,erfc\left\{-\frac{\hat{\delta}_{p,m}^{i1}(\theta_1)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_{p,m}^{i1}(\theta_1) \leq 0 \\ 1 - \frac{1}{2}\,erfc\left\{\frac{\hat{\delta}_{p,m}^{i1}(\theta_1)}{\sqrt{2}\,\hat{\sigma}_p}\right\} & \hat{\delta}_{p,m}^{i1}(\theta_1) > 0 \end{array}\right\} \end{array}\right. \quad (25)$$

However

$$\left\{\begin{array}{ll} G(\theta_1, \theta_2 | \theta_1, \theta_2) = 1 & when\ \omega_1|\theta_1| + \omega_2|\theta_2| \geq \xi;\ |\theta_2| \geq \kappa_2;\ |\theta_1| \geq \kappa_1 \\ G(\theta_1 | \theta_1, \theta_2)) = 1 & when\ \omega_1|\theta_1| + \omega_2|\theta_2| \geq \xi\ and\ |\theta_2| \leq \kappa_2 \\ G(\theta_2 | \theta_1, \theta_2)) = 1 & when\ \omega_1|\theta_1| + \omega_2|\theta_2| \geq \xi\ and\ |\theta_1| \leq \kappa_1 \end{array}\right. \quad (26)$$

Table 1. Failure Probabilities as a Function of the Number of Sensors*

n	Φ^0		Φ^1		Φ^2	
	$\lambda_i = 50$	$\lambda_i = 12.5$	$\lambda_i = 50$	$\lambda_i = 12.5$	$\lambda_i = 50$	$\lambda_i = 12.5$
1	0.9804	0.9259	0.0196	0.0741		
2	0.9612	0.8573	0.0384	0.1372	0.0004	0.0055
3	0.9423	0.7938	0.0565	0.1905	0.0011	0.0152
5	0.9057	0.6806	0.0906	0.2722	0.0036	0.0436
10	0.8203	0.4632	0.1641	0.3706	0.0148	0.1334
20	0.6730	0.2145	0.2692	0.3433	0.0511	0.2609
50	0.3715	0.0213	0.3715	0.0853	0.1820	0.1672

*All instruments with the same failure probability.

Indeed, the MT is given by: $\mathbf{Z}_k^{MP} = (1/\sqrt{W_{kk}})|W_{ki1}\theta_{i1} + W_{ki2}\theta_{i2}|$, where $\boldsymbol{W} = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^T)^{-1}\boldsymbol{A}$. Then, it is clear that $\mathbf{Z}_k^{MP} = (1/\sqrt{W_{kk}})|W_{ki1}\theta_{i1} + W_{ki2}\theta_{i2}| = 1.96$ constitutes the line below which the test will not flag positive for a 95% confidence level.

Now, we may have that $(1/\sqrt{W_{i2i2}})|W_{i2i1}\theta_{i1} + W_{i2i2}\theta_{i2}| = s > 1.96$, but $(1/\sqrt{W'_{kk}})|W'_{ki1}\theta_{i1}| < 1.96$, where W'_{kk} corresponds to the system where the measurement in *i2* was eliminated. A similar equation can be derived for $G(\theta_{i2})$.

It is cumbersome to integrate the above expressions analytically, so one has to resort to a numerical scheme. We now turn our attention to the probabilities of failure.

Effect of the failure frequency

Now, if all instruments have the same failure and repair rate, we have

$$\Phi^j = \sum_i \Phi_i^j = \binom{n}{j}\left(\frac{1}{1+\lambda}\right)^j\left(\frac{1}{1+\lambda}\right)^{n-j} \tag{27}$$

where λ is the ratio of repair rate to failure rate. Because the probabilities of failure depend on the time elapsed since the last repair, $P\{\hat{m}_p \geq m_p^*\}$ is a function of time, which makes matters more complicated. Indeed, as time goes by, the above probability should increase until a failure occurs, because the probability $f_i(t)$ is not constant. We will assume a bathtub curve and ignore the burn in period. This is discussed, for example, by Dhillon,[9] and specifically as applied to instruments by Bagajewicz.[2] Thus, we will assume that such rate is constant.

If the sensors are never repaired (or seldom repaired/calibrated), then $f(t)$ is related to the service reliability $R_i^s(t)$ as follows:[3]

$$f_i(t) = 1 - R_i^s(t) = 1 - e^{-r_i t} \tag{28}$$

where r_i is the failure rate. However, when corrective maintenance is performed, $f(t)$ relates to the service availability function $A_i^s(t)$, as follows[2]:

$$f(t) = 1 - A_i^s(t) = \frac{r_i}{r_i + \mu_i}(1 - e^{-(r_i+\mu_i)t}) \tag{29}$$

where μ_i is the repair rate. In both cases, we start with a zero probability of failure and build up to level off at one if no repair is made, or at $(r_i/r_i + \mu_i)$ in case of repairs. When corrective maintenance is performed, the usual case, the ratio of time to reach 99% of the asymptotic value ($t_i^{0.99}$) to the repair time $(1/\mu_i)$, is given by:

$$\chi_i^{0.99} = \frac{t_i^{0.99}}{(1/\mu_i)} = -\lambda_i\frac{\ln 0.01}{(1+\lambda_i)} \cong 4.61\frac{\lambda_i}{(1+\lambda_i)} \tag{30}$$

where $\lambda_i = (\mu_i/r_i)$, the ratio of repair rate to failure rate. Thus, for small values of λ_i, the probability of failure levels off very quickly, whereas for $\lambda_i \gg 1$ the probability of failure levels off at about 4.6 times the inverse of the repair rate to level off. Notice that the failure rate for flowmeters (r_i) ranges from 0.1 to 10 failures/10^6 hours.[10] However, these are referred to as functional failure. When a noticeable loss of calibration is also referred to as failure, which is the case we are interested in, the numbers are higher. For simplicity, we will use here the asymptotic constant value for the probability of failure. This puts us in the worst condition and makes the analysis conservative.

To get an idea of what range the values are, consider two values of λ_i, 50 and 25. To get an idea of what these numbers represent, assume that the failure rate of an instrument is one failure per year, and that the repair rate is close to 20 repairs a week, then λ_i=50. For the same repair rate, λ_i=12.5 corresponds to a failure rate of 3.5 failures per year. Table 1 shows the results of the sum of failure probabilities Φ^j when all instruments have the same failure and repair rate.

For 10 instruments or less, we have that $\Phi^0 + \Phi^1 > 0.99$, which indicates that the rest can be ignored. At larger plants, like the refinery included in the article by Bagajewicz et al.,[1,3] which has 49 instruments, the ratio of repair to failure rate is critical. For example, when λ_i=50, we have that $\Phi^0 + \Phi^1 + \Phi^2 > 0.99$ for $n = 20$ and larger than 0.92 for $n = 50$. The situation is not so good for the case of λ_i=12.5. In this case, the probability of eight sensors or less failing is larger than 0.99 (0.97 for seven sensors or less). In addition, notice that for 50 instruments, one has to go to very large repair rates to have a sizable probability of no failure. This is costly and, therefore, suggests that better instrumentation should be bought to reduce the failure rate and also to reduce the financial loss through larger accuracy, as we shall discuss below.

Downside Expected Financial Loss. In the absence of biases, the downside expected financial loss is[3]:

$$DEFL^0(\hat{\sigma}_p, \sigma_p) = \int_{-\infty}^{m_p^*} g_p(m_p^*, \sigma_p, m_p) \times \left\{ K_S T \int_{-\infty}^{m_p^*} (m_p^* - \hat{m}_p)\, g_M(\hat{m}_p; m_p, \hat{\sigma}_p) d\hat{m}_p \right\} dm_p \quad (31)$$

which resulted in the following expression for normal distributions:

$$DEFL^0(\hat{\sigma}_p, \sigma_p) = \gamma K_S T \hat{\sigma}_p \beta(\sigma_p/\hat{\sigma}_p) \quad (32)$$

where $\gamma = (1/2\sqrt{2\pi}) \int_0^\infty \xi e^{-\xi^2} d\xi = 0.19947$ and $\beta(x) = \{(1/\sqrt{x^2+1}) + x1/\sqrt{(1/x+1)}\}$.

Now, when there is one gross error present, the expected financial loss is:

$$DEFL^1|i = \left\{ \left[\int_{-\infty}^{m_p^*} K_S T \left\{ \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi; m_p, \hat{\sigma}_{p,i}^R) d\xi \right\} g_P(m_p; m_p^*, \sigma_p) dm_p \right] \left[\int_{-\infty}^{-\tilde{\delta}_i^p} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta + \int_{\tilde{\delta}_i^p}^{\infty} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \right] + \int_{-\tilde{\delta}_i^p}^{\tilde{\delta}_i^p} \left[\int_{-\infty}^{m_p^*} K_S T \left\{ \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi; m_p + \hat{\delta}_p^i(\theta), \hat{\sigma}_p) d\xi \right\} g_P(m_p; m_p^*, \sigma_p) dm_p \right] h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \right\} \quad (33)$$

When the distributions are normal and $\sigma_p/\hat{\sigma}_p \to 0$, we have

$$DEFL^1 \Big| i = DEFL^0 \left[\frac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] \left[\int_{-\infty}^{-\tilde{\delta}_i^p} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta + \int_{\tilde{\delta}_i^p}^{\infty} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \right] + \frac{1}{2} K_S T \int_{-\tilde{\delta}_i^p}^{\tilde{\delta}_i^p} \left\{ \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi; m_p^* + \hat{\delta}_p^i(\theta), \hat{\sigma}_p) d\xi \right\} h_i(\theta; \bar{\delta}_i, \rho_i) d\theta \quad (34)$$

which with normal distributions of biases gives:

$$DEFL^1|i = DEFL^0 \Omega_1 + \frac{1}{2} K_S T(\Omega_2 - \Omega_3) \quad (35)$$

where

$$\Omega_1 = \begin{cases} \frac{1}{2} erfc\left\{ \frac{(\tilde{\delta}_i^p + \bar{\delta}_i)}{\sqrt{2}\,\rho_i} \right\} + 1 - \frac{1}{2} erfc\left\{ \frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i} \right\} & \tilde{\delta}_i^p \le \bar{\delta}_i < \infty \\ \frac{1}{2} erfc\left\{ \frac{(\tilde{\delta}_i^p + \bar{\delta}_i)}{\sqrt{2}\,\rho_i} \right\} + \frac{1}{2} erfc\left\{ \frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i} \right\} & -\tilde{\delta}_i^p \le \bar{\delta}_i < \tilde{\delta}_i^p \\ 1 - \frac{1}{2} erfc\left\{ -\frac{(\bar{\delta}_i + \tilde{\delta}_i^p)}{\sqrt{2}\,\rho_i} \right\} + \frac{1}{2} erfc\left\{ \frac{(\tilde{\delta}_i^p - \bar{\delta}_i)}{\sqrt{2}\,\rho_i} \right\} & \bar{\delta}_i \le -\tilde{\delta}_i^p \end{cases} \quad (36)$$

$$\Omega_2 = \frac{1}{2} K_S T \frac{\hat{\sigma}_p e^{(\bar{\delta}_i^2/2\rho_i^2)[1-(1/2\rho_i^2 r)]}}{4\rho_i \sqrt{\pi r}} \left\{ \begin{array}{ll} 2 - erfc\{\sqrt{r}(e + \tilde{\delta}_i^p)\} - erfc\{\sqrt{r}(\tilde{\delta}_i^p - e)\} & -\tilde{\delta}_i^p \le e \le \tilde{\delta}_i^p \\ erfc\{\sqrt{r}(-\tilde{\delta}_i^p - e)\} - erfc\{\sqrt{r}(\tilde{\delta}_i^p - e)\} & -\tilde{\delta}_i^p \ge e \\ erfc\{\sqrt{r}(e - \tilde{\delta}_i^p)\} - erfc\{\sqrt{r}(e + \tilde{\delta}_i^p)\} & e \ge \tilde{\delta}_i^p \end{array} \right\} \tag{37}$$

$$\Omega_3 = \begin{cases} \dfrac{\alpha_p^i}{2\sqrt{2\pi}\,\rho_i} \displaystyle\int_{-\tilde{\delta}_i^p}^{0} \theta \left[2 - erfc\left\{ -\frac{\alpha_p^i \theta}{\sqrt{2}\,\hat{\sigma}_p} \right\} \right] e^{-[(\theta-\bar{\delta}_i)^2/2\rho_i^2]} d\theta + \dfrac{\alpha_p^i}{2\sqrt{2\pi}\,\rho_i} \displaystyle\int_{0}^{\tilde{\delta}_i^p} \theta\, erfc\left\{ \frac{\alpha_p^i \theta}{\sqrt{2}\,\hat{\sigma}_p} \right\} e^{-[(\theta-\bar{\delta}_i)^2/2\rho_i^2]} d\theta & \alpha_p^i > 0 \\ \dfrac{\alpha_p^i}{2\sqrt{2\pi}\,\rho_i} \displaystyle\int_{-\tilde{\delta}_i^p}^{0} \theta erfc\left\{ \frac{\alpha_p^i \theta}{\sqrt{2}\,\hat{\sigma}_p} \right\} e^{[(\theta-\bar{\delta}_i)^2/2\rho_i^2]} d\theta + \dfrac{\alpha_p^i}{2\sqrt{2\pi}\,\rho_i} \displaystyle\int_{0}^{\tilde{\delta}_i^p} \theta \left[2 - erfc\left\{ -\frac{\alpha_p^i \theta}{\sqrt{2}\,\hat{\sigma}_p} \right\} \right] e^{-[(\theta\bar{\delta}_i)^2/2\rho_i^2]} d\theta & \alpha_p^i < 0 \end{cases} \tag{38}$$

where $r = (1/2\hat{\sigma}_p^2\rho_i^2)[[\alpha_p^i]^2\rho_i^2 + \hat{\sigma}_p^2]$ and $e = (\bar{\delta}_i/2\rho_i^2 r)$. For zero mean in the biases ($\bar{\delta}= 0$), this expression reduces to:

$$DEFL^1|i = DEFL^0 \left\{ \left[\frac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] erfc\left\{ \frac{\tilde{\delta}_i^p}{\sqrt{2}\,\rho_i} \right\} + \frac{1}{4\gamma\rho_i\sqrt{\pi r}} \times [1 - erfc\{\sqrt{r}\,\tilde{\delta}_i^p\}] + \frac{|\alpha_p^i|\rho_i}{2\gamma\hat{\sigma}_p\sqrt{2\pi}} [1 - e^{-[(\tilde{\delta}_i^p)^2/2\rho_i^2]}] \right\} \tag{39}$$

Now, if the variance of the bias ρ_i is small compared to $\tilde{\delta}_i^p$ (good instruments as compared to the existing redundancy), we write:

$$DEFL^1|i = DEFL^0 \left[\frac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] - O(e^{-(\tilde{\delta}_i^p/\sqrt{2}\rho_i)}\tilde{\delta}_i^p/\rho_i) \quad \rho_i \ll \tilde{\delta}_i^p \tag{40}$$

reflecting the fact that almost all errors are detected. The formula also indicates that when good accuracy is dependent on only a few instruments, then the residual precision can be a large number and, therefore, the financial loss increases considerably. For example, if one has a system with orifice meters (about 2% precision) and decides to install a coriolis meter in stream p, then even if this meter has small biases, its frequency of failure needs to be also very small. Otherwise, the financial loss can make a big jump because the residual precision increases considerably. It is, therefore, better to install more instruments so that the residual precision does not deteriorate under one instrument failure so much.

We also recognize that in the case where no data reconciliation is performed, $\tilde{\delta}_i^p = 0\ \forall i \neq p = 0$ and $\tilde{\delta}_p^p = \delta_p^\#$ (usually very large). In addition, $\hat{\sigma}_p^{R,i} \to \infty$ (the estimator is lost), so we use $\sigma_p^\#$, the variance of the best estimate one can make.

$$DEFL^1|p = DEFL^0 \left[\frac{\sigma_p^\#}{\sigma_p} \right] \tag{41}$$

which indicates that the downside financial loss can be large.

Returning to Eq. 39, when variable p has very high accuracy, then $\tilde{\delta}_i^p$ is small compared to p_i. Thus, in the limit we get a different expression from Eq. 40:

$$DEFL^1|i = DEFL^0 \left\{ \begin{array}{l} \left[\dfrac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] + \sqrt{\dfrac{2}{\pi}} \left\{ \sqrt{2} - \left[\dfrac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] \right\} x + \\ -\dfrac{\alpha_p^i \rho_i}{2\hat{\sigma}_p} x^2 + \dfrac{2}{3}\sqrt{\dfrac{2}{\pi}} \left\{ \dfrac{1}{4} \left[\dfrac{\hat{\sigma}_p^{R,i}}{\hat{\sigma}_p} \right] - \rho_i^2 r \right\} x^3 \end{array} \right\} + O(x^4) \quad \tilde{\delta}_i^p \ll \rho_i \tag{42}$$

The first term reflects the loss in precision due to the fact that large gross errors have been detected and their measurement eliminated. A large value of variance of the biases (ρ_i^2) appears to make matters better. This is because a smaller portion of those biases will have value smaller than $\hat{\delta}_i^p$, and the rest will be detected. However, one needs to remember that $\tilde{\delta}_i^p \ll \rho_i$ and, therefore, this case would correspond to precise instruments that are prone to have large biases, something not very common.

In the case of no data reconciliation we get:

$$DEFL^1|p = DEFL^0 \left\{ \left[\frac{\sigma_p^\#}{\sigma_p} \right] \left[1 - \sqrt{\frac{2}{\pi}} \frac{\delta_p^\#}{\rho_p} \right] + \frac{2}{\sqrt{\pi}} \frac{\delta_p^\#}{\rho_p} + O([\delta_p^\#/\rho_p]^2) \right\} \quad \delta_p^\# \ll \rho_p \tag{43}$$

And since $\sigma_p^\# > \sigma_p$, this number can be very large.

A similar expression can be written for the case of two gross errors starting from

$$DEFL^2|i1, i2\} = \int_{-\infty}^{m_p^*} \left\{ \begin{array}{l} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p,\, \hat{\sigma}^R_{p,i1,i2}) d\xi \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\theta_1,\, \theta_2|\theta_1,\, \theta_2) h_{i1}(\theta_1;\, \bar{\delta}_{i1},\, \rho_{i1}) h_{i2}(\theta_2;\, \bar{\delta}_{i2},\, \rho_{i2}) d\theta_1 d\theta_2 + \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p + \hat{\delta}_p^{i1,i2}(\theta_1,\, \theta_2),\, \hat{\sigma}_{p,m}) d\xi \right] \\ [1 - G(\theta_1,\, \theta_2|\theta_1,\, \theta_2)] h_{i1}(\theta_1;\, \bar{\delta}_{i1},\, \rho_{i1}) h_{i2}(\theta_2;\, \bar{\delta}_{i2},\, \rho_{i2}) d\theta_1 d\theta_2 \\ + \int_{-\infty}^{\infty} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p + \hat{\delta}^{i2}_{p,m}(\theta_2),\, \hat{\sigma}^R_{p,i1}) d\xi \\ h_{i2}(\theta_2;\, \bar{\delta}_{i2},\, \rho_{i2}) d\theta_2 \int_{-\infty}^{\infty} G(\theta_1|\theta_1,\, \theta_2) h_{i1}(\theta_1;\, \bar{\delta}_{i1},\, \rho_{i1}) d\theta_1 \\ + \int_{-\infty}^{\infty} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p + \hat{\delta}^{i1}_p(\theta_1),\, \hat{\sigma}^R_{p,i2}) d\xi \\ h_{i1}(\theta_1;\, \bar{\delta}_{i1},\, \rho_{i1}) d\theta_1 \int_{-\infty}^{\infty} G(\theta_2|\theta_1,\, \theta_2) h_{i2}(\theta_2;\, \bar{\delta}_{i2},\, \rho_{i2}) d\theta_2 \end{array} \right\} g_P(m_p^R;\, m_p^*,\, \sigma_p) dm_p \quad (44)$$

which for the case of $\sigma_p/\hat{\sigma}_p \to 0$ reduces to:

$$DEFL^2|i1, i2\} = \frac{1}{2} \left\{ \begin{array}{l} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p^*,\, \hat{\sigma}^R_{p,i1,i2}) d\xi \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\theta_1,\, \theta_2|\theta_1,\, \theta_2) h_{i1}(\theta_1,\, \rho_{i1}) h_{i2}(\theta_2,\, \rho_{i2}) d\theta_1 d\theta_2 + \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p^* + \hat{\delta}_p^{i1,i2}(\theta_1,\, \theta_2),\, \hat{\sigma}_{p,m}) d\xi \right] \\ [1 - G(\theta_1,\, \theta_2|\theta_1,\, \theta_2) h_{i1}(\theta_1,\, \rho_{i1}) h_{i2}(\theta_2,\, \rho_{i2}) d\theta_1 d\theta_2 \\ + \int_{-\infty}^{\infty} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p^* + \hat{\delta}^{i2}_{p,m}(\theta_2),\, \hat{\sigma}^R_{p,i1}) d\xi \\ h_{i2}(\theta_2,\, \rho_{i2}) d\theta_2 \int_{-\infty}^{\infty} G(\theta_1|\theta_1,\, \theta_2) h_{i1}(\theta_1,\, \rho_{i1}) d\theta_1 \\ + \int_{-\infty}^{\infty} \int_{-\infty}^{m_p^*} (m_p^* - \xi)\, g_M(\xi;\, m_p^* + \hat{\delta}^{i1}_p(\theta_1),\, \hat{\sigma}^R_{p,i2}) d\xi \\ h_{i1}(\theta_1,\, \rho_{i1}) d\theta_1 \int_{-\infty}^{\infty} G(\theta_2|\theta_1,\, \theta_2) h_{i2}(\theta_2,\, \rho_{i2}) d\theta_2 \end{array} \right\} \quad (45)$$

The integral can be calculated by using the properties of the functions $G(\theta_1,\ \theta_2|\theta_1,\ \theta_2)$, $G(\theta_1|\theta_1,\ \theta_2)$, and $G(\theta_2|\theta_1,\ \theta_2)$. This needs to be done numerically. The expression can be generalized for a larger number of biases in a similar fashion.

Complete expression for the downside financial loss

To obtain a complete expression for the financial loss, we add the financial loss corresponding to the different mutually exclusive events multiplied by their frequency.

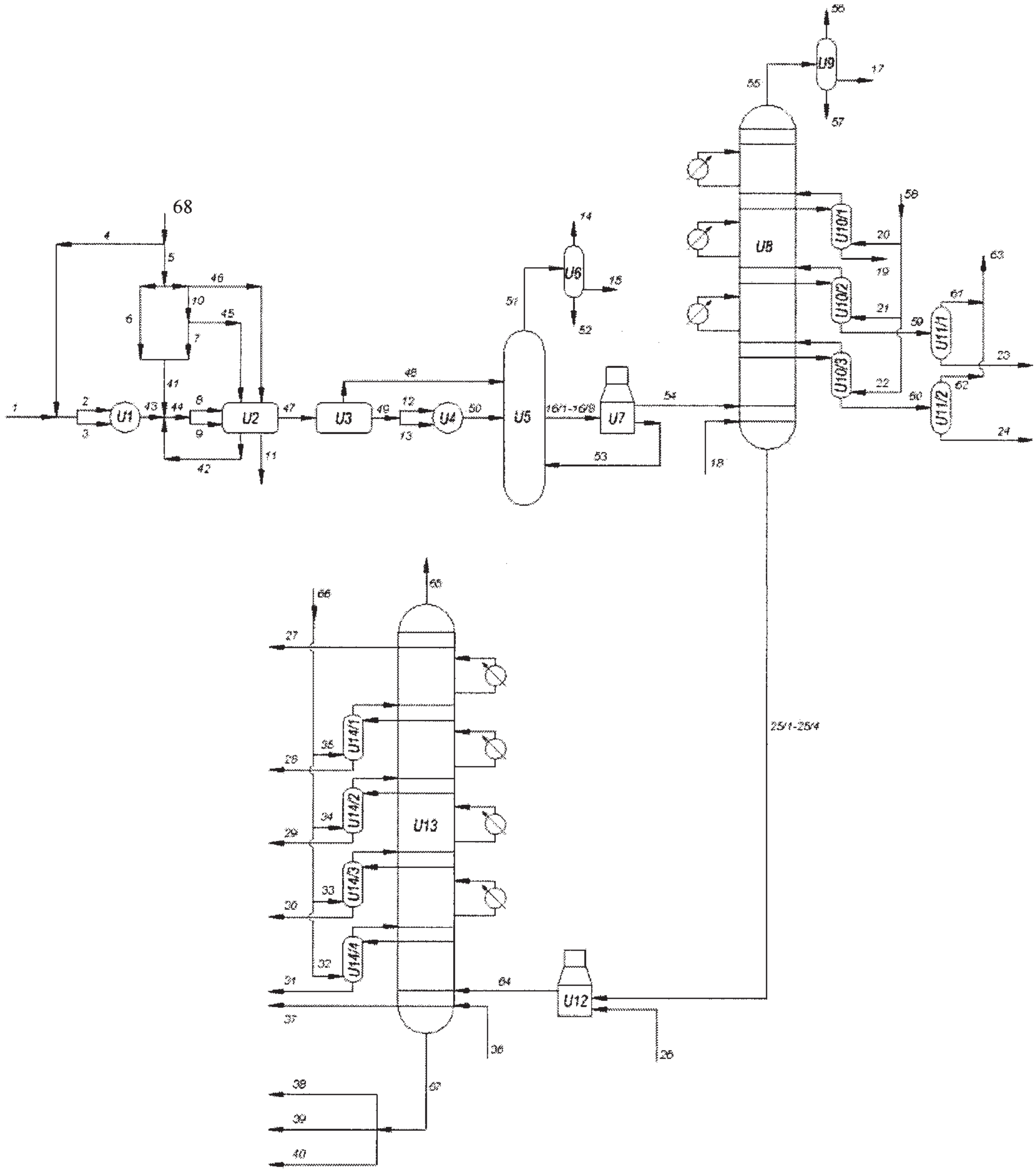

Figure 2. Flow sheet for crude distillation unit.

Units: U1 –HEN; U2 –desalting unit; U3 –crude vessel; U4 –HEN; U5 –pre-fractionation tower; U6 –condenser; U7 –furnace; U8 –atmospheric tower; U9 –condenser; U10/1÷ U10/3 –pre-flash column; U11/1, U11/2 –atmospheric product dryer; U12 –furnace; U13 –vacuum tower; U14/1÷ U14/4 –pre-flash column.

Streams: crude oil: 1,2,3,8,9,43,44 –desalted crude oil: 12,13,47,49,50 –water 4,5,6,7,10,41,45,46,68 –hydrocarbon vapors: 48,56,61,62,63,65 –sour water: 11,42,48 –oily water: 52,57 –pre-fractionation products: 14,15 –atmospheric products: 17,19,23,24 –vacuum products: 27,28,29,30,31,37,67 –steam: 18,20,21,22,26,32,33,34,35,36,58 –

$$DEFL = \Psi^0 DEFL^0 + \sum_i \Psi_i^1 DEFL^0 \Big|_i + \sum_{i1,i2} \Psi_{i1,i2}^2 \cdot DEFL^2 \Big|_{i1,\,i2} + \ \ldots \qquad (46)$$

In the above expression, Ψ^0 are the average fraction of time the system is in the state without biases, Ψ_i^1 the average fraction of time the system has only one undetected bias only in stream i, etc. These values are in fact equal to the probabilities of each state.

As in our previous article,[3] we recall that the assumption was made that the operator will not introduce corrective actions when the measurement is above the target. We discussed that in the case where corrections to setpoints are made when readings indicate that targeted productions are exceeded, the result would be even worse. We leave the discussion of this and many other simplifying assumptions for future work.

Trade-Offs Between Value and Cost

We briefly discuss here the trade-offs, although they are not the object of this article. In the case of buying a data reconciliation package, one would write

$$NPV = d_n\{\text{Change in } DEFL\} - \textit{Change in Cost} \quad (47)$$

where d_n is the sum of discount factors for n years. The change in cost includes now the cost of the license and/or the cost of new instrumentation plus the increased maintenance cost.

The cost of maintenance is a function of the expected number of repairs. For a given instrument, this is given by[2]:

$$\Lambda_i(t) = \mu_i\left[\frac{1}{(1+\lambda_i)}\,t + \frac{1}{r_i[1+\lambda_i]^2}\,(e^{-(1+\lambda_i)r_it} - 1)\right] \quad (48)$$

Using this function, one can construct the net present cost of maintenance for new instrumentation. In fact, larger maintenance will reduce *DEFL* by reducing the frequency of failure

Example

It is quite clear that plants with a large number of instruments, especially if the rate of repair is low, require the evaluation of a fairly large number of terms of Eq. 46. We will now illustrate the results for a crude distillation unit (CDU) (Figure 2), which was used in our previous article.[3,2] We will assume that the instruments are fairly well maintained. The values of mass flow for process streams and the reconciled values as well as the costs for the products were given in our previous article.

Value of Performing Data Reconciliation. In our previous article, we concluded that the net present value of performing data reconciliation in this plant was \$236,817, based on comparisons of $DEFL^0$ only. Assume $\lambda_i = 200$, which is fairly high. In such a case, we can assume that the likelihood of two failures at the same time is smaller than 2.5%, rendering $\Psi^0 = 0.78$ and $\Psi_i^1 = 0.195$. To estimate the financial loss when there is no data reconciliation performed, we use Eq. 43 assuming that $\sigma_p^{\#} = 3\%$ (twice that of the measurement). For $(\delta_p^{\#}/\rho_p) = 0.25$, we get $\Sigma_i\, DEFL^1|i = 2.2363\, DEFL^0$, which needs to be added for all the instruments (a total of 15). Thus, a lower bound for DEFL when no data reconciliation is made is about \$23.82 million (using a horizon of 5 years only).

If reconciliation is used, we use Eq. 40 to evaluate the financial loss due to one gross error (we assume that the variance of the bias is very small, that is, we have good instruments and/or good maintenance). In this case, residual precision for single faults is about $\hat{\sigma}_p^{R,i} = 1.002\hat{\sigma}_p$ (there are several redundant measurements). This renders DEFL = 7.38 million. Thus, the NPV of data reconciliation is 16.44 million. This proves that in plants of this size, the financial loss due to biases is far larger than the one due to precision.

Conclusions

In this article, the new concept of economic value of precision was presented. The concept was illustrated for the case of the value added of data reconciliation as well as instrument upgrade. Maintenance cost, however, is related to reliability of instruments, or more precisely, the rate of failure audit is related to the preventive maintenance policies. Failure of instrumentation reduces precision, and therefore reduces value added. This feature as well as the relaxation of the simplifying assumption used will be added in future work. This will also be used in techniques to upgrade instrumentation.

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Notation

A = incidence matrix
$A_i^s(t)$ = service availability of instrument i
d_n = sum of discount factors for n years
$DEFL^0(\hat{\sigma}_p, \sigma_p)$ = downside expected financial loss when no gross error is present
$DEFL^1|i$ = downside expected financial loss when one gross error is present in stream i
$f_i(t)$ = Probability of failure of instrument i at time t.
$g_M(\xi;\, m_p,\, \hat{\sigma}_p)$ = probability distribution of measurements around the true value
$g_P(m_p;\, m_p^*,\, \sigma_p)$ = probability distribution of the real value around the targeted mean.
$G(\delta_{i1}, \ldots \delta_{ip}|\delta_{i1}, \ldots \delta_{in_b})$ = 0-1 function indicating that the MP test has detected errors $\delta_{i1}, \ldots \delta_{ip}$ out of n_b errors $\delta_{i1}, \ldots \delta_{in_b}$
$H_S(T)$ = tank hold up at the end of the time window T
H_S^* = target hold up value
$H_s(\theta_{k_1},\, \theta_{k_2} \ldots \theta_{k_s})$ = auxiliary function
$h_i(\theta;\, \bar{\delta}_i,\, \rho_i)$ = probability distribution of failure
m_r = feed streams
m_I = intermediate streams
m_p = product stream
m_p^* = target flowrate
$\hat{m}_p$ = estimator of the true value of m_p
NPV = net present value
$P\{\hat{m}_p \geq m_p^*|i\}$ = probability of measured rate being larger than the target in the presence of one gross error in stream i
$P\{\hat{m}_p \geq m_p^*|n_b = 1\}$ = probability of measured rate being larger than the target in the presence of one gross error in the whole system
$R_i^s(t)$ = service reliability of instrument i
r_i = failure rate
T = window of time under consideration
W = auxiliary matrix ($W = A^T(A\Sigma A^T)^{-1}A$)
Z_k^{MP}: = maximum power Z statistics for variable k

Greek letters

$\beta(x)$ = correction function to account for process variability
γ = constant (= 0.19947)
δ_i = bias in variable i
$\hat{\delta}_{p,\max}^{i1,\ldots,in_b}$ = maximum induced bias that goes undetected by the maximum power measurement test when there are n_b gross errors
$\delta_p^{\#}$ = maximum expected undetected gross error in the absence of redundancy
$\tilde{\delta}_i^p$ = absolute value of the error in stream i that corresponds to the maximum undetectable (or minimum detectable) induced bias in stream p ($\hat{\delta}_{p,\max}^i$)
$\hat{\delta}_p^i$ = induced bias in the estimator of m_p
$\bar{\delta}_i$ = mean value of failure
Φ_i^1 = probability of instrument i failing and the others not
$\Phi_{i1,i2}^2$ = probability of instruments $i1$ and $i2$ failing and the others not

$\Phi^{nT}_{i1,i2,\ldots,in_b}$ = probability of instruments $i1$, $i2$, . . . and in_b failing and the others not

Φ^j = frequency of j instruments failing

$\lambda_i = (\mu_i/r_i)$ = ratio of repair rate to failure rate

μ_i = repair rate

Ω_i = auxiliary terms

Ψ^0 = the average fraction of time the system is in the state without biases

Ψ^k_i = the average fraction of time the system has only k undetected biases only in stream i

ρ^2_i = variance of failure

$\hat{\sigma}^R_{p,i}$ = residual precision left after the measurement of variable i has been eliminated

$\sigma^\#_p$ = variance of the best estimate one can make without redundancy

$\hat{\sigma}^R_{p,\overline{\{k\}}}$ = residual precision when all but the gross error in stream k is eliminated

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