Andrei Smirnov
Department of Mathematics and Mechanics,
Leningrad University,
Leningrad, U.S.S.R.

Free Vibrations of the Rotating Shells of Revolution[1]

1 Introduction

This paper is devoted to the problems of free vibrations of thin rotating shells. The theory of vibration of rotating shells is part of the theory of an arbitrary rotating body and the results which are valid for an arbitrary body are also valid for the shells. Though numerical methods are the main approach to the investigation of the dynamics of rotating bodies, some analytical results have been obtained for some simple bodies like rotating beams and discs. The mathematical theory of nonrotating thin shells is well developed. Several of the most successful are two-dimensional theories of the Kirckhoff-Love type. Using the Novozhilov shell theory, which is of this type, A. L. Goldenveiser, V. B. Lidsky, and P. E. Tovstik have developed the theory of asymptotic integration of the equation of vibration of shells. This theory allows one to estimate and, in some cases, to find analytical solutions for the eigenvalues. The main results of this theory are presented by Goldenveiser et al. (1979).

The aim of this paper is to apply asymptotic methods to the solution of the eigenvalue problem for a rotating shell. We will use Novozhilov's two-dimensional shell theory to obtain the equations for the vibration of the shell and the theory of asymptotic integration of the differential equation to solve the eigenvalue problem for these equations.

For the last few years the analytical approach to the solution of the eigenvalue problem for a rotating shell has become popular. In the list of references we only mention research which deals with the application of two-dimensional theories for obtaining the equation of vibration and papers devoted to the mathematical consideration of these equations. Unfortunately, space does not permit a detailed description of these works.

2 Geometry of a Shell

Consider the thin shell of revolution of constant thickness h. To describe the shell geometry we introduce an orthogonal curvilinear coordinate system, connected with the meridans and parallels of a shell. A position of a point on the neutral surface of a shell is defined by a longitudinal angle $\alpha (0 \le \alpha < 2\pi)$ and the length of the arc of meridan $s (s_1 \le s \le s_2)$. The shell is limited to two parallels $s = s_1$ and $s = s_2$. As a particular case, a shell may be closed at the top (sphere, cupola).

The geometry of the shell is characterized by the function $B(s)$, which is the distance between the axis of symmetry and the neutral surface (see Fig. 1). We will also use functions R_1 and R_2, which are the main radii of a curvature and function $\theta(s)$, which is the angle between the initial normal to the neutral surface and the axis of symmetry. These functions are expressed through the function B:

$$\frac{1}{R_1} = \frac{d\theta}{ds} = -\frac{B''}{\sqrt{1-B'^2}}, \quad \frac{1}{R_2} = \frac{\sin\theta}{B} = \frac{\sqrt{1-B'^2}}{B},$$

$$\cos\theta = B', \; 0 \le \theta < \pi.$$

At each point of the neutral surface we introduce a local system of cartesian coordinates, the axes of which are the tangent to the meridan, the tangent to the parallel, and the initial normal. **U** is a displacement vector with components u, v, w in the local coordinate system.

The shell rotates with the constant angular velocity Ω around the axis of symmetry.

We use the following notation:

E = Young's modulus
σ = Poisson's ratio
ρ = density of the shell material
l = the length of the cylindric shell
ϵ_k, ω = components of the tangential deformation and the angle of the elastic rotations
T,Π = kinetic and potential energies
t = time

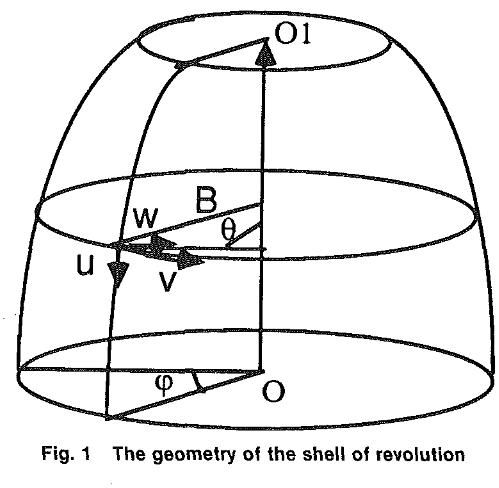

Fig. 1 The geometry of the shell of revolution

[1]This paper was prepared by Dr. Smirnov during his participation in the official exchange between Carleton University, Ottawa, Ontario, Canada and the State University of Leningrad.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for presentation at the Joint ASCE/ASME Applied Mechanics, Biomechanics, and Fluids Engineering Conference, San Diego, Calif., July 9 to 12, 1989.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by ASME Applied Mechanics Division, January 13, 1988; final revision, May 1, 1988.

Paper No. 89-APM-23.

We will nondimensionalize the variables as follows:

$$(\mathbf{U},u,v,w,R_k,B,s,l)=R^{-1}(\mathbf{U}^*,u^*,v^*,w^*,R_k^*,B^*,s^*,l^*),$$

$$(\epsilon_k,\omega)=(\epsilon_k^*,\omega^*),\ t=k^{-1}t^*,\ k=R\sqrt{\frac{\rho(1-\nu^2)}{E}},$$

$$h=(R\sqrt{12})^{-1}h^*,\ (\omega,\Omega)=k(\omega^*,\Omega^*),$$

$$(T,\Pi)=(1-\nu^2)(Eh^*)^{-1}(T^*,\Pi^*),$$

where R is a characteristic radius of a shell. For example, if $B(s_1)\neq 0$ we can assume $R=B(s_1)$.

3 The Equation of a Motion

There are various ways to get the equations of vibration of a rotating shell. We will use Hamilton's principle. The mathematical formulation of this principle is

$$\delta(T-\Pi)=0, \tag{1}$$

where T is the kinetic energy of the shell and Π is the potential energy of the shell. δ is a symbol of variation. If $\mathbf{V}$ is the velocity of a shell element, the kinetic energy of the shell will be

$$T=\frac{1}{2}\int_0^{2\pi}\int_{s_1}^{s_2}\mathbf{V}^2 Bdsd\varphi,\ \mathbf{V}=\mathbf{\Omega}\times(\mathbf{r}+\mathbf{U})+\dot{\mathbf{U}}, \tag{2}$$

where $\mathbf{r}$ is a radius vector to a point on the neutral surface. Here we take only the linear displacements terms. In the work by Vorobiov and Detistov (1981a) some nonlinear terms were retained.

The potential energy Π may be expressed as

$$\Pi=\frac{1}{2}\int_0^{2\pi}\int_{s_1}^{s_2}\Pi Bdsd\varphi, \tag{3}$$

Here the second Π is a strain energy density. We will consider it as a function of the displacements and their first derivatives.

If we substitute expressions (2) and (3) into (1) we obtain the equation of vibration

$$\int_0^{2\pi}\int_{s_1}^{s_2}(\rho\ddot{\mathbf{U}}+\rho\mathbf{\Omega}\times[\mathbf{\Omega}\times(\mathbf{r}+\mathbf{U})]+2\rho\mathbf{\Omega}\times\dot{\mathbf{U}}$$

$$+\nabla\Pi[\mathbf{U}])\delta\mathbf{U}Bdsd\varphi=0, \tag{4}$$

where

$$\nabla\Pi=\frac{\partial\Pi}{\partial\mathbf{U}}-\frac{1}{B}\frac{\partial}{\partial s}\left(B\frac{\partial\Pi}{\partial\mathbf{U}_s'}\right)-\frac{\partial}{\partial\varphi}\frac{\partial\Pi}{\partial\mathbf{U}_\varphi'}.$$

For arbitrary elastic bodies this equation has been obtained by Vilke (1986), using the Lagrange-D'Alembert's principle. The precise form of the density Π depends on the type of shell theory used. We will discuss this problem next. The only condition, which the shell theory must satisfy, is that the reciprocity principle of Betti is valid. More details about the application of Betti's principle in the theory of shells can be found in Goldenveizer's monograph (1961).

Equation (4) is valid for all theories of shells in which the strain energy density is only a function of the displacements and their derivatives, for example, theories of the Kirchhoff-Love type. Theories of other types may include additional independent variables. For example, in Reisner's shell theory, the two angles of rotation of the normal to shell element are independent quantities. Using this theory of shells the equation for the vibrations of a cone was obtained by Vorobiov and Detistov (1981a).

We will investigate small vibrations of shells about the axisymmetric equilibrium state generated by centrifugal forces. We represent the displacement $\mathbf{U}(s,\varphi,t)$ as a combination of an initial axisymmetrical displacement $\mathbf{U}^0(s)$ and an additional displacement $\mathbf{U}^1(s,\varphi,t)$

$$\mathbf{U}(s,\varphi,t)=\mathbf{U}^0(s)+\mathbf{U}^1(s,\varphi,t). \tag{5}$$

The expressions for the vectors $\mathbf{r}$ and $\mathbf{\Omega}$ in a local system of coordinates will be

$$\mathbf{r}=(B\cos\theta,0,-B\sin\theta),\ \mathbf{\Omega}=(-\Omega\sin\theta,0,-\Omega\cos\theta). \tag{6}$$

Since strain energy density Π is positive and we are only considering small deformations, we may assume that this density is a quadratic form of the deformations and the angles of rotations. The actual expressions for the deformations and angles of rotation will be introduced next. For now, we write them in the form

$$x_i=\sum_j a_{ij}y_i+\sum_j b_{ij}y_i^2$$

Here x is any deformation or angle of rotation and y is a displacement or its derivative. Geometrical linear shell theory assumes $b_{ij}=0$.

Now we represent the strain energy density in the next form

$$\Pi[\mathbf{U}^1,\mathbf{U}^0]=\Pi[\mathbf{U}^1]+\Pi_F[\mathbf{U}^1,U^0]+\Pi_0[\mathbf{U}^0]$$

$$+\Pi_\Omega[\mathbf{U}^1,\mathbf{U}^0]+\Pi_*[\mathbf{U}^1,\mathbf{U}^0]. \tag{7}$$

Here $\Pi[\mathbf{U}^1]$ is the quadratic form of the displacements $\mathbf{U}$. $\Pi_F[\mathbf{U}^1,\mathbf{U}^0]$ is the linear form with respect to both $\mathbf{U}^1$ and $\mathbf{U}^0$. $\Pi_0[\mathbf{U}^0]$ is a term depending only on $\mathbf{U}^0$. $\Pi_\Omega[\mathbf{U}^1,\mathbf{U}^0]$ is the quadratic form with respect to $\mathbf{U}^1$ and the linear form with respect to $\mathbf{U}^0$, in $\Pi_*[\mathbf{U}^1,\mathbf{U}^0]$ we include all other items. In our consideration we will neglect the last item. We will also omit the index "1" for the displacement.

Substituting expressions (5–7) into equation (4), and taking into account the independence of the coordinates, we get two vector equations:

$$L_F(\mathbf{U}^0)=\mathbf{F}, \tag{8}$$

$$L(\mathbf{U})+\Omega^2L_\Omega[\mathbf{U}^0,\mathbf{U}]=\ddot{\mathbf{U}}+2\Omega L_C\dot{\mathbf{U}}+\Omega^2L_e\mathbf{U}, \tag{9}$$

where

$$L_C=\begin{pmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & \sin\theta\\ 0 & -\sin\theta & 0\end{pmatrix},$$

$$L_e=\begin{pmatrix}-\cos^2\theta & 0 & \sin\theta\cos\theta\\ 0 & -1 & 0\\ \sin\theta\cos\theta & 0 & -\sin^2\theta\end{pmatrix},$$

$$L_F(\mathbf{U}^0)=-\nabla\Pi_F[\mathbf{U}^0,\mathbf{U}],\quad L_\Omega(\mathbf{U}^0,\mathbf{U})=-\frac{1}{\Omega^2}\nabla\Pi_\Omega[\mathbf{U}^0,\mathbf{U}],$$

$$L(\mathbf{U})=-\nabla\Pi[\mathbf{U}],\quad \mathbf{F}=\Omega^2\mathbf{r}=\Omega^2(B\cos\theta,0,-B\sin\theta). \tag{10}$$

From equation (8) we can find the initial displacements and stresses in a shell, which we will substitute into the equation of free vibration of a shell (9). Operator L corresponds to the nonrotating shell. Operator L_Ω describes the change of the geometry of shell and the existence of the initial stresses. The equation of vibration of rotating shells was obtained in a number of papers for different kinds of shell geometry (most often for cylinders) under different assumptions for different types of shell theories. Some of these papers are listed in the references. In these papers the equation of vibration has the different operators L and L_Ω and depend on the type of shell theory and assumptions about the initial equilibrium. However, for all theories of the Kirckhoff-Love type, the differences in operators L will be only in terms of order h^2 and higher.

The scalar product of the vectors $\mathbf{U}_k$ and $\mathbf{U}_l$ are determined by the formula

$$(\mathbf{U}_k, \mathbf{U}_l) = \int_{s_1}^{s_2} (u_k u_l + v_k v_l + w_k w_l) B ds.$$

For a large class of boundary conditions all operators included in equation (9) are self-adjoint, i.e.,

$$(L\mathbf{U}_k, \mathbf{U}_l) = (L\mathbf{U}_l, \mathbf{U}_k). \tag{11}$$

For that it is only necessary for the principle of Betti to be valid. We will use only those boundary conditions, which are usually called "idealized." For example, the boundary conditions of free, clamped, and freely-supported edges are of that kind. All idealized boundary conditions are linear, hence,

$$\Gamma(\mathbf{U}) = 0, \quad \Gamma(\mathbf{U}^0) = 0$$

are boundary conditions for the initial and additional displacements.

4 The Form of Solution

For linear and homogeneous boundary conditions we can search the solution of equation (9) in a form of "running waves," that is

$$u(s,\varphi,t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=1}^{2} u_{mk}(s) C_{mk}^l \cos(m\varphi + \omega_{mk} t - \alpha^l)$$

$$v(s,\varphi,t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=1}^{2} v_{mk}(s) C_{mk}^l \sin(m\varphi + \omega_{mk} t - \alpha^l),$$

$$w(s,\varphi,t) = \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=1}^{2} w_{mk}(s) C_{mk}^l \cos(m\varphi + \omega_{mk} t - \alpha^l),$$

$$\alpha^1 = 0, \quad \alpha^2 = \frac{\pi}{2}. \tag{12}$$

where α^l is the phase, m is the wave number along a parallel, and ω_{mk} is the frequency of the free vibrations. Since the boundary conditions and equations of vibrations are linear, we can consider the vibrations separately for each wave number m. The equation of vibration of a shell for fixed m is

$$L_m(\mathbf{U}_m) + \Omega^2 L_{\Omega m}(\mathbf{U}_m) + \omega_m^2 \mathbf{U}_m + 2\omega_m \Omega L_{Cm} \mathbf{U}_m + \Omega^2 L_{em} \mathbf{U}_m = 0, \quad \Gamma(\mathbf{U}_m) = 0, \tag{13}$$

where

$$L_{Cm} = \begin{pmatrix} 0 & -\cos\theta & 0 \\ -\cos\theta & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix},$$

$$L_{em} = \begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix}.$$

The operators L_m and $L_{\Omega m}$ are operators L and L_Ω for fixed m. Further, we will consider the problem for fixed m and omit the index "m" for both variables and operators.

It is useful to write the energy equation corresponding to vector equation (9). Multiplying it by $\mathbf{U}$, we obtain:

$$\omega^2 T + 2\omega\Omega T_C + \Omega^2 T_e = \Pi + \Pi_\Omega, \tag{14}$$

where

$$(\mathbf{U},\mathbf{U}) = \int_{s_1}^{s_2} (u^2 + v^2 + w^2) B ds = 2T,$$

$$(L(\mathbf{U}),\mathbf{U}) = -(\nabla\Pi[\mathbf{U}],\mathbf{U}) = -2\int_{s_1}^{s_2} \Pi B ds = -2\Pi,$$

$$(L_\Omega(\mathbf{U}),\mathbf{U}) = = -(\nabla\Pi_\Omega[\mathbf{U}_0,\mathbf{U}],\mathbf{U})$$

$$= -\frac{2}{\Omega^2}\int_{s_1}^{s_2} \Pi_\Omega B ds = -2\Pi_\Omega,$$

$$(L_C(\mathbf{U}),\mathbf{U}) = -2\int_{s_1}^{s_2} (\cos\theta u - \sin\theta w) v B ds = 2T_C,$$

$$(L_e(\mathbf{U}),\mathbf{U}) = \int_{s_1}^{s_2} ((\cos\theta u - \sin\theta w)^2 + v^2) B ds = 2T_e.$$

Here Π is the potential energy of the additional displacements, Π_Ω is the potential energy of the initial stresses and displacements, and T is the kinetic energy of the relative motion. All energies are computed for the fixed mode.

It is known that the nonrotating shell of revolution has a series of frequencies of free vibrations and modes, for which

$$\omega_k = \omega_{-k}, \quad \mathbf{U}_k = \mathbf{U}_{-k}. \tag{15}$$

That is, two running waves propagate in opposite directions along the parallel with equal angular speeds. The relations (15) can be used to transform expressions (12). We consider the sum of two waves, running in opposite directions for fixed k, with the arbitrary constant in a form

$$C_k^1 = C_{-k}^1, \quad C_k^2 = C_{-k}^2 = 0. \tag{16}$$

Thus, for the displacement u, we get

$$u = C u_k \cos m\varphi \sin\omega_k t. \tag{17}$$

This is a standing wave, oscillating with a frequency ω_k.

Now consider the case of a rotating shell. For the rotating shell the expressions (15) are not valid. We introduce the parameters α and β as

$$\alpha = \frac{1}{2}(\omega_k - \omega_{-k}), \quad \beta = \frac{1}{2}(\omega_k + \omega_{-k})$$

and amplitude vectors as

$$\mathbf{U}_k^1 = \mathbf{U}_k + \mathbf{U}_{-k}, \mathbf{U}_k^2 = \mathbf{U}_k - \mathbf{U}_{-k}.$$

The arbitrary constants are determined by expression (16). Making the same transformations we get

$$u = C(u_k^1 \cos(m\varphi + \beta t)\cos\alpha t + u_k^2 \sin(m\varphi + \beta t)\sin\alpha t). \tag{18}$$

This is a superposition of the two standing waves, oscillating with a frequency α and precessing with the angular velocity β/m. It is clear that for nonrotating shell $\alpha = \omega_k$, $\beta = 0$, $\mathbf{U}^2 = 0$, and expression (18) transforms into expression (17). The reader has to pay attention that the rotation generates both precession of the modes and shift of the frequency of oscillation.

5 Application of a Small Disturbance Method

Let Ω be the small parameter. The correctness of such an assumption has been discussed by Smirnov and Tovstik (1981) and Lidsky and Tovstik (1984). We expand frequencies ω_k and amplitude vectors $\mathbf{U}_k$ in a series of the parameter Ω, that is

$$\omega_k = \alpha(\Omega) + \beta(\Omega), \; \omega_{-k} = -\alpha(\Omega) + \beta(\Omega)$$

$$\alpha(\Omega) = \omega_0 + \sum_{i=1}^{\infty} \Omega^{2i}\alpha_{2i}, \beta(\Omega) = \sum_{i=0}^{\infty} \Omega^{2i+1}\beta_{2i+1},$$

$$\mathbf{U}_k(\Omega) = \sum_{i=0}^{\infty} (-1)^i \mathbf{U}_i \Omega^i. \tag{19}$$

The frequency ω_0 is the corresponding frequency of the nonrotating shell. It is obvious that $\alpha(0) = \omega_0$ and $\beta(0) = 0$. Substituting these expressions (19) into equation (13), and equating the coefficients for equal powers of Ω, we get

$$L\mathbf{U}_0 + \omega_0^2\mathbf{U}_0 = 0, \quad \Gamma(\mathbf{U}_0) = 0$$

$$L\mathbf{U}_1 + \omega_0^2\mathbf{U}_1 = -2\omega_0(L_C\mathbf{U}_0 + \beta_1\mathbf{U}_0), \quad \Gamma(\mathbf{U}_1) = 0$$

$$L\mathbf{U}_2 + \omega_0^2\mathbf{U}_2 = -2\omega_0(L_C\mathbf{U}_1 + \beta_1\mathbf{U}_1) - (2\omega_0\alpha_2 + \beta_1^2)\mathbf{U}_0 - -2\beta_1 L_C\mathbf{U}_0 - L_e\mathbf{U}_0 - L_\Omega\mathbf{U}_0, \quad \Gamma(\mathbf{U}_2) = 0. \qquad (20)$$

These equations determine the amplitude vectors $\mathbf{U}$ and free frequencies ω. The first equation gives the free frequencies and modes of the nonrotating shell. To obtain the coefficient β_1, we scalar multiply the second equation by $\mathbf{U}_0$ and apply the property of self-adjoint operators (11). Hence,

$$\beta = \beta_1\Omega + O(\Omega^3), \quad \beta_1 = -\frac{T_C(\mathbf{U}_0)}{T(\mathbf{U}_0)}. \qquad (21)$$

The speed of precession is determined by β/m.

Applying the same method we can find all the coefficients in the series α and β. We write only the second term for α.

$$\alpha_2 = \frac{\beta_1^2}{2\omega_0} - \frac{(L_C\mathbf{U}_1, \mathbf{U}_0) + \beta_1(\mathbf{U}_1, \mathbf{U}_0)}{2T(\mathbf{U}_0)} + \frac{T_e(\mathbf{U}_0) + \Pi_\Omega(\mathbf{U}_0)}{2\omega_0 T(\mathbf{U}_0)}. \qquad (22)$$

Later we will call β_1 the coefficient of bifurcation and α_2 the coefficient of shift. We will assume that it is possible to neglect the second term in formula (22). This assumption has been verified by Smirnov and Tovstik (1982) for some special cases, for example, the low frequency vibrations of shells of zero curvature. However, in the general case, the error of this assumption has not yet been established. With this assumption, the formula for α may be rewritten in the form

$$\alpha = \alpha_2\Omega^2(1 + O(\Omega^2)), \ \alpha_2 = \frac{1}{2\omega_0}\left(\beta_1^2 + \frac{T_e(\mathbf{U}_0) + \Pi_\Omega(\mathbf{U}_0)}{T(\mathbf{U}_0)}\right). \qquad (23)$$

Estimating the coefficients of bifurcation and shift requires the amplitude vectors of the nonrotating shell $\mathbf{U}_0$. This is possible only if we know the operator L or the density Π. The form of this density depends on the type of the shell theory. In the next section we will apply Novozhilov's shell theory, which is of the Kirkhoff-Love type.

It is possible to make some conclusions about the coefficient of bifurcation. Vilke (1986) has shown that $-1 \le \beta \le 0$. Egarmin (1986) has shown that $\beta = -1$ only for $m = 1$ and only for rings or cylinders where axial displacements are zero; i.e., $u = 0$.

It is clear now why the effect of precession of modes of the rotating solid body was ignored for many years. First, this effect is rather small for low speeds of rotation and, secondly, it could not be discovered using a one-dimensional model. This effect only appears in two- or three-dimensional models. Finally, this phenomenon does not appear in axisymmetric vibrations. Indeed, it is known that axisymmetric vibrations may be separated into twisting modes with amplitude vector $\mathbf{U}_1 = (0, v, 0)$ and longitudinal-bending modes with amplitude vector $\mathbf{U}_2 = (u, 0, w)$. It is clear that the scalar products $(L_C\mathbf{U}_1, \mathbf{U}_1)$ and $(L_C\mathbf{U}_2, \mathbf{U}_2)$ and the coefficient of bifurcation $\beta = 0$ are all equal to zero.

The effect of precession is well known in physics and is a consequence of the theorem of conservation of angular momentum. In all cases, when an arbitrary body oscillates in a field of gyroscopic forces $\mathbf{F}$, such that

$$\mathbf{F} = \mathbf{a} \times \mathbf{V},$$

where $\mathbf{a}$ is a field vector and $\mathbf{V}$ is the speed of a body element, we have the precession of a mode. The effect of bifuration of the frequencies for an oscillation of a particle in a field of magnetic forces is known as Larmore precession. Another example of this effect is the vibration of the Foucault pendulum.

To determine the strain energy density Π we have to choose the shell theory. We will use the Novozhilov's theory. Smirnov and Tovstik (1981) and Smirnov (1981a, 1981b) have shown that for Novozhilov's shell theory the expresssions for items proportional to Ω^2 in equation (14) have the next form

$$\Pi_0(\mathbf{U}_0) + P\Pi_P(\mathbf{U}_0) = \Pi_\Omega(\mathbf{U}_0) - \Omega^2 T_e(\mathbf{U}_0), \quad \Pi_0 = m^2T - 2mT_C. \qquad (24)$$

Constant P depends on the boundary conditions and not equal to zero only when both edges are clamped, i.e., $u(s_1) = u(s_2) = 0$. We will not consider this case.

If P is equal to zero, the formulas for the shift coefficients will be rewritten as:

$$\alpha = \frac{1}{2\omega_0}\ (m + \beta_1)^2\Omega^2(1 + O(\Omega^2)), \qquad (25)$$

If we assume $\mathbf{U} = \mathbf{U}_0$, and submit the expression (24) into (14), we immediately obtain

$$\omega_\pm = \beta_1\Omega \pm (\omega_0 + (m + \beta_1)^2\Omega^2)^{1/2}. \qquad (26)$$

The last expression is more precise for lower eigenvalues, but from an asymptotic point of view the relative error will be the same as in formula (25), that is $O(\Omega^2)$.

6 Partition Method

Most of the results of Sections 3, 4, and 5 are valid for an arbitrary elastic body, rotating around an axis of symmetry. For example, for any elastic body

$$\beta = \frac{T_C(\mathbf{U}_0)}{T(\mathbf{U}_0)}\ \Omega(1 + O(\Omega^2)),$$

describes the gyroscopic effect, where $\mathbf{U}_0$ is the vector of displacements for a nonrotating body. We will omit the index $_0$.

Now we will use the specific property of the shell, small thickness h, to find the approximate analytical expression for the modes of a nonrotating shell. For the quantities of the same order as h, we will use the symbol $\sim$. If we manage to construct the modes in the form

$$\mathbf{U} = \mathbf{U}^0 + \mathbf{U}^1, \ \|\mathbf{U}^1\| \sim h^\alpha \|\mathbf{U}^0\|, \ \alpha > 0,$$

we then get an approximation for the coefficient β in a form

$$\beta = \frac{T(\mathbf{U}^0)}{T_C(\mathbf{U}^0)}\ \Omega(1 + O(\Omega^2) + O(h^\alpha)).$$

The solution $\mathbf{U}^0$ is usually called the general solution and $\mathbf{U}^1$ is called the additional solution. If h is a small parameter, it is possible to represent the modes as a power series in h. The character of both the general and additional solutions and, hence, the actual form of the series depends on the frequency, geometry of the shell, boundary conditions, and value of the wave number m. For example, the general solution of the equations of vibration with lower frequency is often the bending mode and the additional solution has the character of a boundary effect. The main results of the theory of asymptotic integration of the equations of vibration of shells can be found in a monograph (Goldenveiser et al., 1979). In the next sections we will use the results of this work.

7 Rayleigh's Vibrations

From a technical point of view, the quality of a structure is often determined by the lower eigenvalues. In this paper we are interested in the lower part of the spectrum of the frequencies of free vibration. Among the lower frequencies we are mostly interested in superlow frequencies. These frequencies are of the order of h^α, where $\alpha>0$ and, hence, decrease as the thickness decreases. If the thickness is small enough, all lower frequencies are superlow, if they exist. In a paper by Smirnov and Tovstik (1982), it was shown that the maximum effect of the rotation is on these frequencies. The results of numerical investigations proving this conclusion can be found in a paper by Shih-sen Wang and Chen Yu (1974).

Let us consider the Rayleigh formula for frequencies

$$\omega^2=\frac{\Pi_\epsilon(\mathbf{U})+h^2\Pi_\chi(\mathbf{U})}{T(\mathbf{U})}.$$

Here Π_ϵ is the elongation-shear energy and Π_χ is the bending-twisting energy.

It is clear that the eigenvalues are superlow if we consider the modes for which the potential energy of the elongation (membrane energy) is small, i.e.,

$$\Pi_\epsilon\sim h^\beta,\quad \beta>0.$$

The limiting case $\Pi_\epsilon = 0$, represents pure bending modes. These modes have been investigated by Lord Rayleigh.

The condition $\Pi_\epsilon = 0$ is equivalent to the system of equations

$$\epsilon_1=\epsilon_2=\omega=0,$$

or, using the definition for the tangential deformations (Goldenveizer, 1961), to a system of differential equations

$$u'-\frac{w}{R_1}=0,$$

$$\frac{B'u}{B}+\frac{mv}{B}-\frac{w}{R_2}=0,$$

$$B\left(\frac{v}{B}\right)'-\frac{mu}{B}=0. \tag{27}$$

This system has two solutions which may be found for any given shell type.

8 Shells of Zero Curvature

In this section we will consider shells of zero curvature, i.e., cones and cylinders. For cones the radii of curvature and function B can be determined from:

$$\frac{1}{R_1}=0,\quad \frac{1}{R_2}=\frac{\cos\alpha}{B},\quad B=s\sin\alpha,\quad B'=\sin\alpha,$$

where α is the cone semi-angle. The case $\alpha=0$ represents cylinder.

For a cone, the system of equations (27) has the solutions

$$\mathbf{U}_1=\left(0,s,\frac{ms}{\cos\alpha}\right),\ \mathbf{U}_2=\left(\sin\alpha,-m,\frac{\sin^2\alpha-m^2}{\cos\alpha}\right), \tag{28}$$

and for a cylinder the system of equations (27) has the solution

$$\mathbf{U}_1=(0,1,m),\quad \mathbf{U}_2=(1,ms,m^2s). \tag{29}$$

It is possible to show that the first mode is symmetric to the middle of the shell and the second mode is in an antisymmetric mode. The frequency corresponding to the first mode is always lower than the frequency corresponding to the second mode.

Now, if we have a boundary condition, which allows the existence of pure bending, the formulas (28) and (29) give us an approximation for the exact modes. The error of this approximation is proportional to the influence of boundary effects, that is, $O(\sqrt{h})$.

We substitute the modes (28) and (29) into (21) and (26). For the cone we get

$$\omega_\pm=\beta_1\Omega\pm(\Omega_0^2+(m-\beta_1{}^2)^{1/2}, \tag{30}$$

where, for the first mode

$$\beta_1=-\frac{2m\cos^2\alpha}{\cos^2\alpha+m^2},$$

and for the second mode

$$\beta_1=-\frac{2m^3}{A_m},\quad A_m=\sin^2\alpha+m^2+\left(\frac{\sin^2\alpha-m^2}{\cos\alpha}\right)^2.$$

Here, ω_0 is the eigenvalue of the nonrotating shell.

For the first mode of the cylinder we get

$$\beta_1=-\frac{2m}{1+m^2},\quad \omega_\pm=\beta_1\omega\pm\left(\Omega_0^2+m^2\left(\frac{m^2-1}{m^2+1}\right)^2\Omega^2\right)^{1/2}. \tag{31}$$

For the second mode we may use the same expression, but with an error of the order $O(m^{-2}l^{-2})$.

If m is a large parameter, the formulas for the first and second modes converge and we obtain the next approximation

$$\beta_1=-\frac{2\cos^2\alpha}{m}(1+O(m^{-2})),\quad \omega_\pm=\beta_1\Omega\pm(\omega_0^2+m^2\Omega^2)^{1/2}. \tag{32}$$

All these results are valid for pure bending vibration, but not all shells and boundary conditions produce pure bending. For some cases, the shell has modes very similar to bending modes. These modes are called pseudobending modes. The reader can find theoretical results about pseudobending modes in a monograph by Goldenveiser et al. (1979).

We will consider only two cases of vibration of shells with zero curvature, where pseudobending modes exist. The first is the vibration of a shell of medium length, i.e., $l\sim R$. It was shown (Goldenveiser et al., 1979) that for the medium length shells of zero curvature, pseudobending modes exist for any type of the boundary condition. For these modes the wave number is large ($m\sim h^{-1/4}$) and the lowest frequency is of the order of $h^{1/2}$. To determine the frequencies of vibrations of these modes, we can use the formula (32). The error of this formula increases when we consider large or small values of m. The error will rise if we consider a rigid boundary condition. For example, this formula gives better results for simply-supported edges than for clamped edges. Nevertheless, (32) gives a good approximation for low frequencies. Figure 2 compare the eigenvalues of a cylindrical shell determined by numerical method (points on the graph) by using (32) (lines on the graph) for $m = 5$, $l = 2$, $h = 0.01$. Endo et al. (1984) and Saito and Endo (1986) obtained good agreement between numerical, experimental, and theoretical results for a wide range of parameters of thickness and wave number, and various boundary conditions.

The influence of rotation on the eigenvalues, determined with formula (32), does not depend on the boundary condition. Using the asymptotic representation found by Goldenveiser et al. (1979), it is possible to get the next term in the series for the eigenvalue, taking into account the effect of the boundary condition. For a cylinder, Goldenveiser et al. (1979) have found that the first term for the displacement has the form

$$\mathbf{U}^0=\left(\frac{1}{m^2}(w^0)',\frac{1}{m}w^0,w^0\right)(1+O(\sqrt{h})), \tag{33}$$

where w^0 is a solution of the equation

$$(1-\nu^2)(w^0)^{iv}+(m^8h^2-\omega_0^2m^4)w^0=0.$$

For the simply-supported edges, the corresponding boundary conditions are

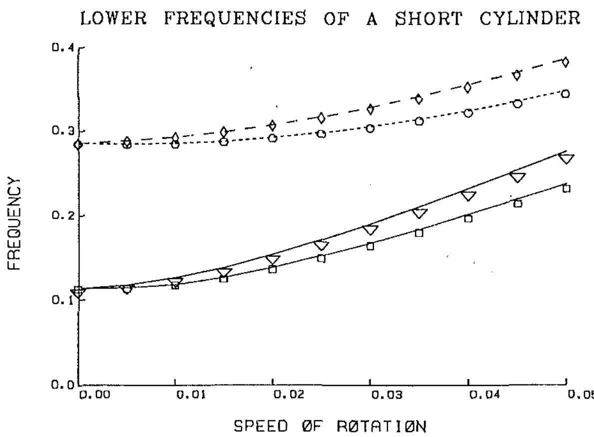

Fig. 2 The lower eigenvalues versus speed of rotation for a short cylindrical shell

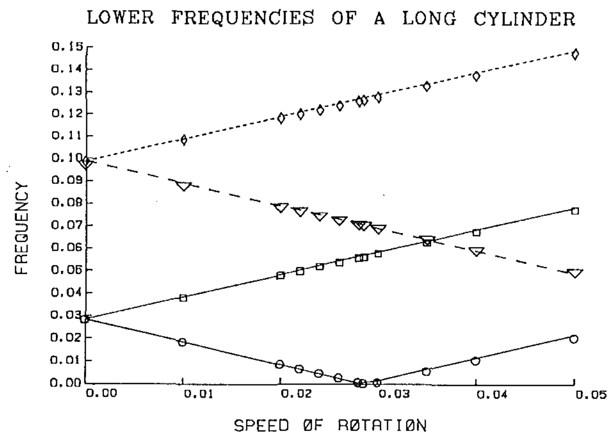

Fig. 3 The lower eigenvalues versus speed of rotation for a long shell

$$(w^0)''(0)=w^0(0)=(w^0)''(l)=w^0(l)=0.$$

It is clear that if m is a large parameter, for example, $(m\sim h^{-1/4})$, equation (32) is still valid. The error due to neglecting terms of the order of m^{-2}, is of the same order as that due to neglecting the boundary effect. If m is not large; we must use the more precise formula.

The second case we will consider is long cylindrical shells. It was shown that for the sufficiently long shells for each wave number, m pseudobending modes exist. In this case, the error in frequency due to (32) is proportional to $m^{-2}l^{-2}$. Egarmin (1986), did not require the assumption $\epsilon_1 = 0$. It allowed him to obtain the next term in the expression for the coefficient of bifurcation

$$\beta_1 = -\frac{2m}{m^2+1+D^k\dfrac{1}{m^2l^2}}, \tag{34}$$

where coefficient D^k depends on the boundary conditions and the wave number (k) in the direction of the meridian. D^k increases with k, so the influence of the boundary condition increases for higher modes. The coefficient D^k is larger for more rigid boundary conditions.

The same result can be obtained using (33). For example, for a simply-supported shell,

$$\mathbf{U}^0=\left(\frac{\pi k}{lm^2}\cos\frac{\pi ks}{l},\ \frac{1}{m}\sin\frac{\pi ks}{l},\ \sin\frac{\pi ks}{l}\right)$$

and the coefficient of bifurcation is equal to:

$$\beta_1=-\frac{2m}{m^2+1+\dfrac{\pi^2k^2}{m^2l^2}}.$$

In this case, $D^k = \pi^2k^2$. For higher modes we must use (34), but for the case $k = 1$ and for long shells, (32) provides adequate results.

The influence of rotation is maximum for $m = 1$. This is the only case when the eigenvalue may be equal to zero for some rotation speed. For a long cylinder with $m = 1$, (34) transforms into

$$\omega_+=\omega_0-\Omega,\quad \omega_-=-\omega_0-\Omega_0. \tag{35}$$

The comparison of the numerical (points on the graph) and asymptotic (lines on the graph) results for the lower frequencies of the long cylinder can be seen in Fig. 3 for the following parameters $m = 1$, $h = 0.01$, and $l = 15$.

We see that for certain specific values of the speed of rotation, the eigenvalue is equal to zero and the shell becomes unstable. The behavior of a long shell is thus similar to the behavior of a long beam, which also becomes unstable at critical speeds of rotation.

9 Vibrations of Spherical Shells

In this section we will consider the vibration of spherical cupolas, which were first examined by Zhuravliev and Klimov (1985). The geometry of these shells is described by function $B = \sin\theta$, $0 \le \theta \le \theta_2$, $R_1 = R_2 = 1$. We start by investigating pure bending modes of vibration. Rayleigh has solved the system (27) for a spherical shell. There are two solutions. We consider the one which is limited in the top of the cupola:

$$\mathbf{U}=\left(\sin\theta\left(\tan\frac{\theta}{2}\right)^m,\ \sin\theta\left(\tan\frac{\theta}{2}\right)^m,\right.$$
$$\left.(m+\cos\theta)\left(\tan\frac{\theta}{2}\right)^m\right),\ 0\le\theta\le\theta_2. \tag{36}$$

Now if we substitute this solution into (21) and (26), we get the formulas for the coefficients of bifurcation and shift

$$\omega\pm=\beta_1\Omega\pm(\omega_0+(m+\beta_1)^2\Omega^2)^{1/2},\ \beta_1=-2m\frac{I_m^1}{I_m^2+I_m^3+I_m^4}, \tag{37}$$

where

$$q=\cos^2\frac{\theta_2}{2},$$

$$I_m^1=4\int_q^1(1-t)^{m+1}t^{1-m}dt,\ I_m^2=(m-1)^2\int_q^1(1-t)^mt^{-m}dt,$$

$$I_m^2=4(m+1)\int_q^1(1-t)^mt^{1-m}dt,\ I_m^3=-4\int_q^1(1-t)^mt^{2-m}dt.$$

For each value of m we can compute these integrals. For $m=2$, the value of the coefficient of bifurcation has been determined by Egarmin (1986). In our notation

$$\beta_1=-2\frac{24\ln q-72q+36q^2-8q^3+44}{3q^{-1}-30\ln q+81q-30q^2+4q^3-58}. \tag{38}$$

It is interesting to investigate the value of the coefficient of bifurcation as a function of the angle θ_2, which is a coordinate of the free edge of the spherical shell. In Fig. 4 we can see the value of the modulus of the coefficient of bifurcation as a function of the angle θ_2. The maximum $(\beta_1=0.76)$ corresponds to a cupola with an angle of 135 deg. It is interesting

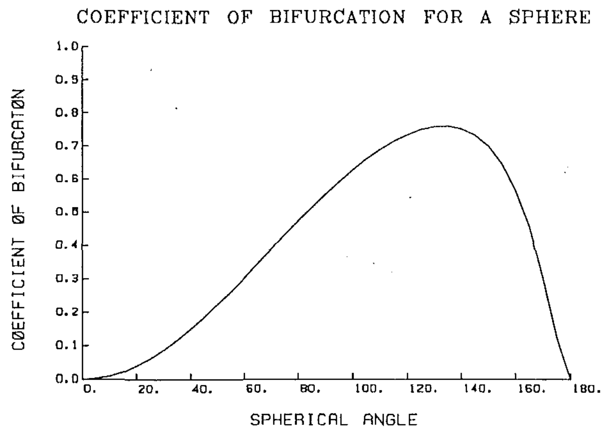

Fig. 4 The relationship between the coefficient of bifurcation and the spherical angle for a spherical shell

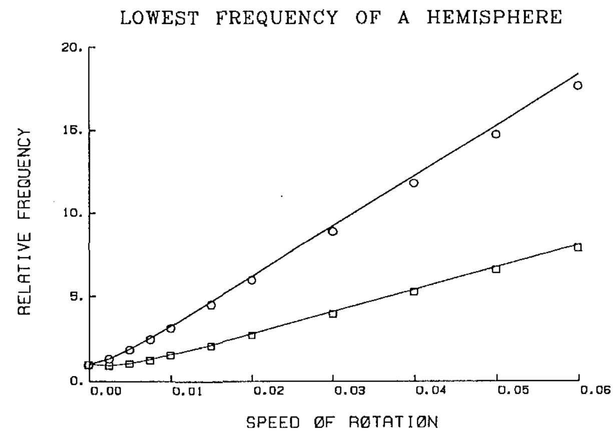

Fig. 5 The lowest eigenvalue versus speed of rotation for a hemisphere

to compare the results for the sphere with the results for the cylinder. For $m=2$, the maximum value of the modulus of the coefficient of bifurcation for cylindrical shell, according to (34), is 0.8. Since the influence of rotation on the shells of different geometry is maximal for cylinders, the last number is an upper limit for the coefficient of bifurication of a spherical shell. In the case of a spherical shell, the bifurcation will be more for the shell, which is more "similar" to the cylinder.

For a hemisphere ($\theta_2=\pi/2$) with $m=2$, we obtain $\beta_1=-0.554$ and the speed of precession $\beta/m=-0.227$. This result was first obtained by Zhuravliev and Klimov (1985). Evaluating the limit of expression (38) for θ_2 converging to 0 and to π, it is not difficult to show that β_1 converges to zero.

The comparison of the numerical (points on the graph) and asymptotic (lines on the graph) results for the lowest frequency of the hemisphere with free edge can be seen in Fig. 5 for the following parameters $m=2$, $h=0.01$. The frequency for the nonrotational shell is assumed to be equal to 1.

These results are valid for spheres with a clamped top and free edge. The error in the mode and, hence, for frequencies is determined by the boundary effect and is $O(\sqrt{h})$. For other boundary conditions at the top, or if $\theta_1>0$, the error in using (36) increases. Egarmin (1986) estimated the influence of the boundary conditions on the eigenvalues. He has shown that this effect is much less important than the effect of the influence of the free edge. This seems reasonable since the main vibration for these shells is concentrated at the free edge.

Acknowledgment

The opportunity for the appearance of this paper is provided by the program of scientific exchange between Carleton University and Leningrad University. The author expresses special gratitude to Prof. J. Goldak, for very fruitful discussions and help. I also thank my friends who helped me with advice and support.

References

DiTaranto, R. A., and Lessen, M., 1964, "Coriolis acceleration effect on the vibration of a rotating thin-walled circular cylinder," ASME JOURNAL OF APPLIED MECHANICS, Vol. 31, pp. 700–701.

Egarmin, N. E., 1986, "O precessii stoiachikh voln kolebanii vrashchaiushcheisia osesimmetrichnoi obolochki," *Mekhanika tverdogo tela*, Vol. 21, pp. 142–148.

Endo, M., et al., 1984, "Flexural vibration of a thin rotating ring," *Journal of Sound and Vibration*, Vol. 92, pp. 261–272.

Goldenveizer, A. L., 1961, *Theory of Elastic Thin Shell*, Pergamon Press, London.

Goldenveizer, A. L., Lidsky, V. B., and Tovstik, P. E., 1979, *Svobodnye kolebania tonkikh uprugikh obolochek*, Nauka, Moscow.

Lidsky, V. B., and Tovstik, P. E., 1984, "Spektry v teorii obolochek," *Advances in Mechanics*, Vol. 7, pp. 25–54.

Padovan, J., 1973, "Natural frequencies of rotating prestressed cylinders," *Journal of Sound and Vibration*, Vol. 31, pp. 469–482.

Padovan, J., 1974, "Orthogonality principle for the vibrational modes of anisotropic composite domain problem," ASME JOURNAL OF APPLIED MECHANICS, Vol. 41, pp. 832–834.

Padovan, J., 1975a, "Numerical analysis of asymmetric frequency and buckling eigenvalues of prestressed rotating anisotropic shells of revolution," *Computers and Structures*, Vol. 5, pp. 145–154.

Padovan, J., 1975b, "Travelling wave vibrations and buckling of rotating anisotropic shells of revolution by finite elements," *International Journal of Solids and Structures*, Vol. 11, pp. 1367–1380.

Padovan, J., 1978, "On gyroscopic problems in elasticity," *International Journal of Engineering Science*, Vol. 16, pp. 1061–1073.

Saito, T., and Endo, M., 1986, "Vibration of finite length rotating cylindrical shells," *Journal of Sound and Vibration*, Vol. 107, pp. 17–28.

Shih-sen Wang, and Chen Yu, 1974, "Effect of rotation on vibrations of circular cylindrical shells," *Journal of the Acoustic Society of America*, Vol. 55, pp. 1340–1342.

Smirnov, A. L., 1981a, "Kolebania vrashchaiushchikhsia obolochek vrashchenia," *Pricladnaia mekhanica*, vyp. 5, Leningrad University Publishers, pp. 176–186.

Smirnov, A. L., 1981b, "Kolebania vrashchaiushchikhsia obolochek vrashchenia," Ph.D. Thesis, Leningrad University, Leningrad.

Smirnov, A. L., 1981c, "Integraly uravnenii kolebanii vrashchaiushchikhsia obolochek vrashchenia," *Vestnik Leningradskogo Universiteta*, No. 13, pp. 114–117.

Smirnov, A. L., and Tovstik, P. E., 1982, "Kachestvennoe issledovanie dinamiki vrashchaiushchikhsia obolochek vrashchenia," *Sovremennye problemy mekhaniki i aviacii*, Mashinostroenie, Moscow, pp. 280–290.

Srinivasan, A. V., and Lauterbach, G. F., 1971, "Travelling waves in rotating cylindrical shell," ASME *Journal of Engineering for Industry*, Vol. 93, pp. 1229–1232.

Vilke, V. G., 1986, "Ob inercialnykh svoistvakh sobstvennykh form osesymmetrichnogo uprugogo tela," *Vestnik Moscovskogo universiteta*, ser. 1, No. 2, pp. 66–72.

Vorobiev, Yu.S., and Detistov, S. I., 1981a, "K vyvodu uravnenii kolebanii vrashchaiushcheisia ortotropnoi konicheskoi obolochki," *Problemy mashinostroenia*, No. 13, pp. 12–17.

Vorobiev, Yu.S., and Detistov, S. I., 1981b, "O vliianii centrobezhnykh sil na sobstvennye chastoty krugovykh konicheskikh obolochek," *Problemy mashinostroenia*, No. 14, pp. 27–29.

Zhuravliev, V. F., and Klimov, A. L., 1985, "O precessii sobstvennoi formy kolebanii sfericheskoi obolochki pri eie vrashchenii," *Mechanica tverdogo tela*, Vol. 20, pp. 147–151.

Zohar, A., and Aboudi, J., 1973, "The free vibrations of a thin circular finite rotating cylinder," *International Journal of Mechanical Science*, Vol. 15, pp. 269–278.