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Classification of Atoms

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Abstract: This article is devoted to give a self-contained presentation of classification of atoms of probability space as equivalent or non-equivalent. It will be established that an event, i.e., a member of a σ-field of a probability space can contains uncountable many equivalent atoms. We will show that the relation of being equivalent atoms is an equivalence relation. An independent proof will enable us to state that an event of a probability space with σ-finite probability measure can contains at most countable many non-equivalent atoms. We will also establish that for a purely atomic probability space with σ-finite probability measure, probability measure of every event is equal to the sum of the probability measures of its non-equivalent atoms. We will also justify that in some of the results, the probability space and respective probability measure can be replaced as measure space and respective measure.

Key words: Equivalent atoms, σ-field, σ-finite measure, equivalence relation, probability measure, atomic probability space.

INTRODUCTION

The basic mathematical object we are to study is an atom of probability space. There is a vast literature available on measure theory, but the first systematic presentation of measure theory appeared in Halmos (1950). The first account of measure theory specifically oriented towards probability was given by Loeve (1955). Several useful refinements were made by Royden (1963), Neveu, (1965) Rudin (1966) and Burrill (1967).

Ash (1972) has given readable introductory treatment about atoms of measure space and has related conditional expectation to atoms. Halmos (1950) achieved slightly greater generality at the expense of technical complication by replacing σ-field by σ-ring. He has defined non-atomic and atomic measure rings but nothing else has been said there except an exercise problem "The metric space of a σ-finite measure ring is convex if and only if the measure ring is non-atomic". Taylor (1973) has associated atoms with set functions. Feller (1986) has associated atoms with distribution. The theory of atoms and conditional atoms introduces the Entropy theory (Hanen, 1966), which itself has significant applications in Information theory (Shannon, 1959 Khintchine (1957) and Wolfowitz (1964), Statistical Mechanics (Ruelle) and Ergotic theory (Katok, 1957; Sinai, Jacobs, 1963; Ornstein, 1974 and Parry, 1969).

Mathematics Subject Classification 2000: 60b05:

This paper represents a modest attempt on our part to sketch some of the measure theoretic ideas about atoms of probability and measure spaces. As we go along, the development of the theory of atoms will be facilitated if we classify atoms as equivalent or non-equivalent. This classification of atoms will enable us to establish that the relation of being equivalent atoms in fact is an equivalence relation. It will also be shown that an event, i.e., a member of the σ-field may have uncountable many equivalent atoms but restricting the probability measure to be σ-finite, we will be able to show that an event can contains at most countable many non-equivalent atoms. We will also establish that for a purely atomic probability space with σ-finite probability measure, probability measure of every event is equal to the sum of the probability measures of its non-equivalent atoms. We will also justify that in some of the results, the probability space and respective probability measure can be replaced as measure space and respective measure.

MATERIAL AND METHODS

A collection F of subsets of a non-empty set S is a σ-field of sets in S if;

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i S is a member of F.
ii F is closed under complementation.
iii F is closed under countable union.

The set S will be called fundamental probability set or sample space and members of S are said to be elementary events. A member of the σ-field F is termed as an event.

Let S be a non-empty set. A function P on S is called extended real-valued set function if its domain is a collection say F of subsets of S and range is contained in the extended real number system. We say P is non-negative if P (A) ≥0 for each member A of F. A set function P defined on a σ-field F is said to be additive (also σ-additive) if;

i P(Φ = empty set) = 0

If $\{E_i\}$ is a sequence of disjoint events of F then $P(\bigcup_i E_i) = \sum_i P(E_i)$

A non-negative additive set function defined on a σ-field will be called a measure. Given a σ-field F of subsets of a non-empty set S, any measure P with P(S) = 1 is called probability measure on F. A probability space is a triple (S, F, P) consisting of a non-empty set S as sample space, a σ-field F in S, and a probability measure P defined on F.

The condition P(S) = 1 serves only for norming purposes and nothing essentially changes if it is replaced by P(S) < ∞(Feller[3]). Consequently, we will confuse probability space with measure space unless otherwise stated (the case where P (event) = ∞). The condition P(S) < ∞ may be weakened further by requiring only that S be the union of countable many parts E_i such that $P(E_i) < \infty$ one speaks then of a σ-finite measure (Feller[3]). We will say that a probability measure P is σ-finite if for all E of σ-field F, there exist E_i in F such that $E \subset \cup E_i$ *and* $P(E_i) < \infty$ where $\cup$ stands for disjoint union.

Let (S, F, P) be a probability space. An event E, i.e., a member E of F is called an atom if P (E) > 0 and if for all B belonging to F such that B ⊂ E, either P (B) = 0 or P (B) = P (E).

Two atoms A and B are said to be equivalent if P (A) = P (B) and A ⊂ B or B ⊂A, otherwise we will call them non-equivalent. A probability space (S, F, P) will be called purely atomic if and only if every set of positive probability measure contains an atom.

RESULTS AND DISCUSSION

Example:

Consider the sample space as set of Real and σ-field as Borel sets. If E is an event then defined P as

$$P(E) = \begin{cases} 0 & if\ 0 \in E \\ 1 & otherwise \end{cases}$$

Clearly P is a probability measure and we can have un-countable many equivalent atoms.

Theorem 1:

The relation of being equivalent atoms is an equivalence relation.

Proof:

Validity of reflexive and symmetric properties is obvious.

Transitive Property:

Let E_1, E_2 and E_3 are atoms of a probability space (S, F, P) such that E_1 is equivalent to E_2 and E_3 is equivalent to E_3. By definition;

$$(E_2) = P(E_1 \cap E_2) \ \ and \ P(E_2) = P(E_3) = P(E_2 \cap E_3) \Rightarrow P(E_1) = P(E_3) \qquad \text{(i)}$$

Since $\subset E_3 \Rightarrow P(E_1 \cap E_3) \leq P(E_3)$ (ii)

Similarly

$$P(E_1 \cap E_3) \geq P(E_1 \cap E_2 \cap E_3) \geq P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3^c) = P(E_1 \cap E_2) = P(E_3) \quad \text{(iii)}$$

Combining (i), (ii) and (iii), we have $P(E_1) = P(E_3) = P(E_1 \cap E_3)$ Hence, E_1 is equivalent to E_1. This completes the proof.

Theorem 2:

Let (S, F, P) be a probability space with σ-finite probability measure P then an event of F can contain at most countable many non-equivalent atoms.

Proof:

Since P is σ-finite probability measure, atoms can be assumed to be disjoint sets of P-measure greater than zero. We claim that if P is σ-finite probability measure then F cannot have un-countable many disjoint sets of P-measure greater than zero. Consequently, if E is an event then E cannot have un-countable many non-equivalent atoms.

Let $\{B_i : i \in I\}$ be disjoint collection of events such that $P.(B_i) > 0 \ \forall i \in I$ Let A be an event such

thatP(A) < ∞ . If $i_1, i_2, i_3, ---, i_n$ are distinct indices such that.

$$P(A \cap B_{ij}) \geq m > 0 \ \ then \ \ nm \leq \sum_{j=1} P(A \cap B_{ij}) \leq P(A) \Rightarrow n \leq \frac{P(A)}{m}$$

Hence, the index set $\{i : P(A \cap B_i) \geq m\}$ is finite. Therefore, $\{i : P(A \cap B_i) > 0\}$ is finite

(taking union over positive rationals).

Since P is σ- finite probability measure, therefore, $S = \bigcup_k A_k$ such that $P(A_k) < \infty$ but then

$I_k = \{i : P(A_k \cap B_i) > 0\}$ is countable for each k, if $P(A_k \cap B_i) = 0 \ \forall k$ then

$P(B_i) \leq \sum_k P(A_k \cap B_i) = 0$ Hence, $I = \bigcup_k I_k$ is countable. This completes the proof.

Theorem 3:

Let (S, F, P) be a purely atomic probability space with σ-finite probability measure P then for each event

E of F $P(E) = \sum_{A \in T} P(A)$ where T is a set of non-equivalent atoms of E.

Proof:

Let E be an event and T be the set of all non-equivalent atoms of E. By theorem 2, T is countable.

Since P is countable additive, hence, $\sum_{A \in T} P(A) = P(\bigcup_{A \in T} A)$. $P(E - \bigcup_{A \in T} A) = 0$, because if

$P(E - \bigcup_{A \in T} A) > 0 \Rightarrow E - \bigcup_{A \in T} A$ contains an atom say A* , but for all $A \in T., A \cap A^* = \phi$ Hence,

a contradiction. Therefore,.

$$0 = P(E - \bigcup_{A \in T} A) = P(E) - P(\bigcup_{A \in T} A) \Rightarrow P(E) = \sum_{A \in T} P(A)$$

$A^* \notin T$ $A^* \notin T$ $A^* \notin T$

This completes the proof.
In the next theorem, we will extend the notion of theorem 3 by considering σ-finite probability measure P as a measure, i.e., P (E) = ∞.

Theorem 4:

Let (S, F, P) be a purely atomic measure space with σ-finite measure P then for each member E of F $P(E) = \sum_{A \in T} P(A)$ where T is a set of non-equivalent atoms of E.

Proof:

Since P is σ-finite measure, therefore, for each E of F, there exists E_i in F such that $E \subset \dot{\cup} E_i$ *and* $P(E_i) < \infty$ *where* $\dot{\cup}$ stands for disjoint union. Let $A \subset E$ be an atom then $0 < P(A) = P(A \cap E) \le P(A \cap \dot{\cup} E_i) = P(\dot{\cup}(A \cap E_i)) = \sum P(A \cap E_i) \Rightarrow$ we can pick an i_0 such that $P(A \cap E_{i_0}) \ne 0$ and $P(A) = P(A \cap E_{i_0})$ $\left(Possibly\ A = E_{i_0}\right)$ but then A is equivalent to $A \cap E_{i_0}$. Note that i_0 have to be unique otherwise if there exists $i_0^* \ne i_0$ then $P(A) \ge P(A \cap (E_{i_0} \dot{\cup} E_{i_0}^*) = P(A \cap E_{i_0}) + P(A \cap E_{i_0}^*) = 2P(A)$, which is not possible. Hence, for an atom $A \subset E$ such that P(A) > 0, there exist unique E_{i_0} in F such that A is equivalent to $A \cap E_{i_0}$ Now using theorem 2, if T is a set of non-equivalent atoms in E and T_i be a set of non-equivalent atoms in E_i then $T = \dot{\cup} T_i$, hence as in theorem 3, we have; $P(E) = \sum P(E_i) = \sum_{i,\, A \in T_i} .\ P(A) = \sum_{A \in T} P(A)$ This completes the proof.

REFERENCES

Ash, R.B., 1972. Real Analysis and Probability, Academic Press, N.Y.

Burrill, C.W., 1967. Measure, integration, and Probability, McGraw-Hill.

Feller, W., 1986. An Introduction to Probability Theory and its Applications, Vol. II, Wiley Eastern Limited.

Halmos, P.R., 1950. Measure Theory, Van Nostrand-Reinhold

Hanen, A. and J. Neveu, 1966. Atoms Conditionnels d'un Espace de Probabilite, Acta Math. Acad. Sci., Hung, 17: 443-449.

Jacobs, K., 1963. Ergadic Theory, Lecture Notes, Aarhus.

Katok, A.B., Ja. G. Sinai, and A.M. Stepin, 1975. Theory of Dynamical Systems and General Groups of Mappings with Invariant Measure, Mat. Anal., 13: 129-264.

Khintchine, A.I., 1957. Mathematical Foundations of Information Theory, Dover.

Loeve, M., 1955. Probability Theory, Nostrand-Reinhold.

Neveu, J., 1965. Mathematical Foundations of the Calculus of Probability, Holden-Day.

Ornsteen, D., 1974. Ergodic Theory, Randomness and Dynamical Systems, Yale University Press.

Parry, W., 1969. Entropy and Generators in Ergodic Theory, Benjamin.

Royden, H.L., 1963. Real Analysis, Mc-Millan.

Rudin, W., 1966. Real and Complex Analysis, Mc-Graw-Hill.

Ruelle, D., 1969. Statistical Mechanics;Rigorous Results, Benjamin.

Shannon, C. and W. Weaver, 1959. The Mathematical Theory of Communication, Univ of Illinois Press (Urbana).

Sinai, Ja.G., 1970. Theory of Dynamic Systems, Lecture Notes, Aarhus.

Taylor, S.J., 1973. Introduction to Measure and Integration, Cambridge University Press.

Wolfowitz, J., 1964. Coding Theory of Information Theory, Springer-Verlag.