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Stability of an Annular Plate Reinforced With a Surrounding Edge Beam

In this paper, the buckling load for a radially compressed annular plate surrounded on its outer edge with an edge beam is determined. A uniform, ring-type load is applied to the edge beam along its entire length, and the inner edge of the annular plate is free from tractions. In the solution to this problem, all possible buckled shapes are considered to obtain the critical buckling load. Use of an edge beam provides a practical way to vary the fixity condition at the plate's outer edge.

Introduction

The purpose of this paper is to determine the buckling load for a radially compressed annular plate surrounded on its outer edge with an edge beam. The mathematical analysis that is performed applies to any edge beam whose center line describes a plane curve (zero torsion) with constant curvature (a circular curve), and whose cross-sectional geometry is such that the cross section is symmetric about the horizontal centroidal axis of the beam. The edge beam is subjected to a ring-type uniform loading along its entire length, and the inner edge of the annular plate is free from tractions.

In past research, such as in the work done by Olsson [1],[2] Schubert [2], and Meissner [3], most annular plate buckling problems have been analyzed with the assumption that the buckling mode is radially symmetric [4]. These results, assuming radial symmetry, provide values of the critical load which are unrealistic in many cases. Yamaki [5] has shown that in the buckling of an annular plate subjected to *equal* compressive loadings at both edges, a radially symmetric mode often does not correspond to the lowest buckling load.

Approximate analyses of the asymmetric buckling of an annular plate with the outer edge clamped, the inner edge free, and loaded with a uniform radial compressive force applied at the outer edge have been published by Rozsa [6] and Majumdar [7]. In this paper, the exact solution to the subject problem is presented for the general case of asymmetric buckling.

The use of an edge beam provides a practical means of varying the fixity condition at the outer edge of the plate between the extremes of simply supported and clamped boundary conditions. The type of edge beam used is similar to the edge beam employed by Amon and Widera [8], who further developed the work done by Reismann [9]. These authors used solid circular plates and considered only the radially symmetric buckling mode. The results of this paper degenerate to the results found by Reismann, as modified by Amon and Widera.

Physical Setup and Assumptions

The plate with its edge beam is shown in Fig. 1. The simple support used under the edge beam constrains it to remain in its original plane. The simple support is also used to indicate, in a physical way, that the plate and edge beam are free to expand inward and outward in the plane of the plate. It is assumed that the plate is integral with the edge beam at the outer edge of the plate.

The external load applied to the structure, P_0, is a uniformly distributed load per unit of beam length applied to the outer side of the edge beam, as shown in Fig. 1. The annular plate has a thickness, t, an inner radius, b, and an outer radius, a. The other material properties pertinent to this analysis are A_B, the edge beam cross-sectional area; E_B, the modulus of elasticity of the beam; I_B, the moment of inertia of the beam cross section about its horizontal centroidal axis; I_{20}, the moment of inertia of the beam cross section about its vertical centroidal axis; E, the modulus of elasticity of the plate material; σ, the Poisson's ratio for the plate material. The inner edge of this annular plate is traction-free.

Throughout this analysis, the following assumptions have been made:

1 The plate and edge beam material are constructed from isotropic elastic solids.

2 The plate is thin and of constant thickness.

3 There are no body forces present.

4 The plate deflections are small in comparison to the thickness of the plate so that classical plate theory as discussed by Timoshenko [10] applies.

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[2] Numbers in brackets designate References at end of paper.

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General Solution

The theory involved in solving a plate buckling problem is well known [4, 10]. The initial step in finding a plate buckling solution is to solve the plane-stress problem, yielding the in-plane stress and displacement fields. Plate bending theory is then employed to solve for the transverse deflection of the plate. Once the general solution for the transverse deflections is obtained, the appropriate boundary conditions are applied to this general solution. These define an eigenvalue problem, and the buckling loads are found for the various modes of buckling that are generated by the P_0 loading.

The plane-stress portion of the plate stability problem is represented by the expression

$$\nabla^4\phi = 0 \tag{1}$$

where ϕ denotes the scalar variable Airy stress function and ∇^4 is the biharmonic operator.

The Airy stress function, ϕ, is related to the stresses in the plate by the following equations:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \tag{3}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{4}$$

where σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are the radial, tangential, and shearing stresses, respectively.

The general solution to equation (1) was developed by Michell [11, 12]. It is necessary to eliminate those terms in the Michell solution which produce multivalued stresses and/or displacements. The stress and displacement fields obtained are, in view of the symmetry of the plane-stress problem, functions of the variable r only.

The plane-stress boundary conditions along the inside hole of the plate are

$$\sigma_{rr}(b, \theta) = 0 \quad \text{and} \quad \sigma_{r\theta}(b, \theta) = 0 \tag{5}$$

Since the plate is integrally attached to the edge beam at $r = a$ as shown in Fig. 1, the radial displacement of the plate is equal in magnitude to the radial displacement of the edge beam, and the tangential plate displacement is equal in magnitude to the tangential displacement of the edge beam. To obtain general expressions for the edge beam radial and tangential displacements, the theory of curved beams as developed by Rakowski and Solecki [13] was employed.

In what follows, only an edge beam of rectangular cross section is illustrated and discussed, however, the theory and the applications that are used could be modified to include any edge beam possessing the previously discussed property of cross-sectional symmetry.

Using equations (5) and equating edge beam-plate displacements as previously described, leads to the following stress field:

$$\sigma_{rr} = A_0 r^{-2} + 2B_0, \quad \sigma_{\theta\theta} = -A_0 r^{-2} + 2B_0 \quad \text{and} \quad \sigma_{r\theta} = 0 \tag{6}$$

where A_0 and B_0 are given by

$$A_0 = \frac{P_0 b^2}{t}\left(\frac{1}{\alpha[(1-\sigma) + (b/a)^2(1+\sigma)] + 1 - (b/a)^2}\right) \tag{7}$$

$$B_0 = -\frac{A_0}{2b^2} \tag{8}$$

and α is defined as

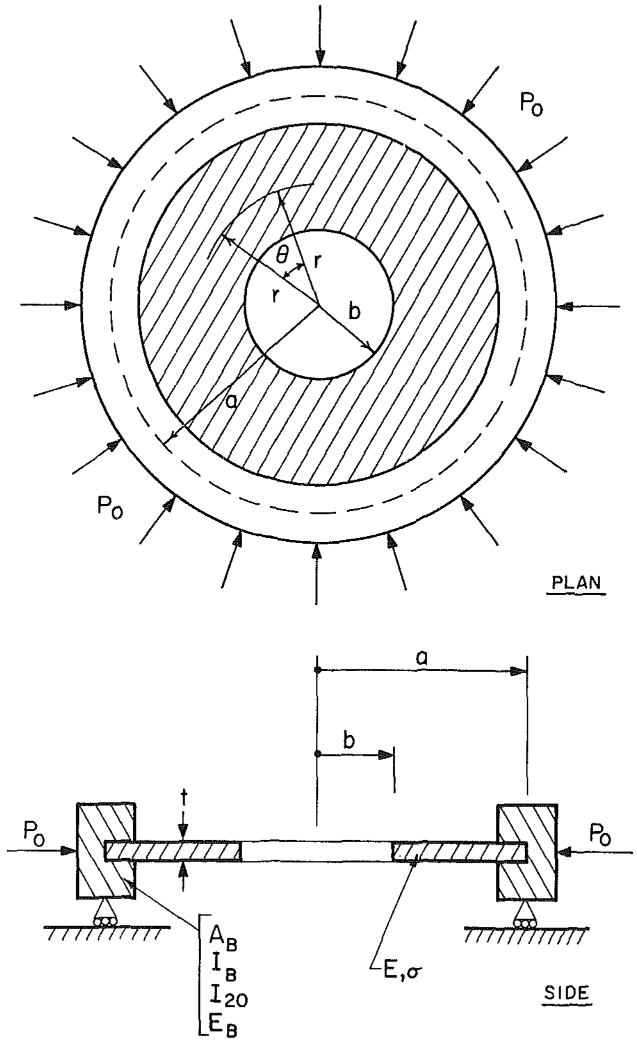

Fig. 1 Plate-edge beam configuration

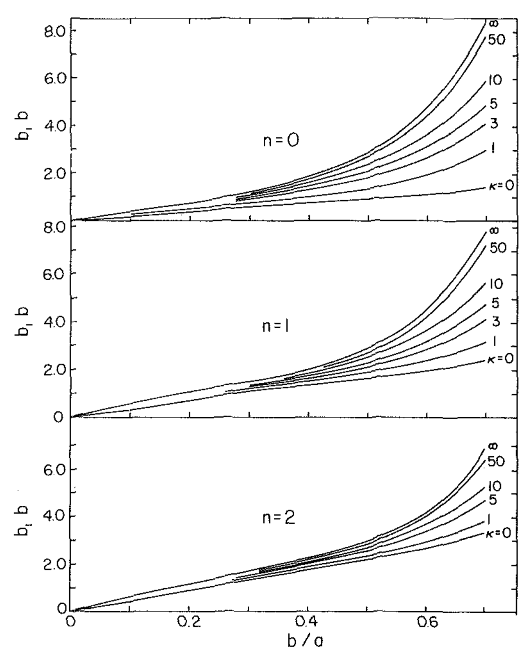

Fig. 2(a) Dimensionless buckling coefficient (b_1b) versus b/a

$$\alpha \equiv \frac{E_B A_B'}{Eat}$$

where (9)

$$A_B' \equiv A_B + \frac{I_{20}}{a^2}$$

and E is the modulus of elasticity of the plate.

The in-plane forces per unit length are found by multiplying the stresses given by equation (6) by the constant plate thickness, t. These forces are then employed in the well-known partial differential plate equation containing in-plane forces [10], which must be solved in order to obtain the transverse deflection of the plate, $\omega(r, \theta)$. The resulting differential equation may be written as

$$\frac{\partial^4\omega}{\partial r^4} + \frac{2}{r}\frac{\partial^3\omega}{\partial r^3} + [-1/r^2 + 2\beta^2 t/D(1 - (b/r)^2)]\frac{\partial^2\omega}{\partial r^2} + \left[1/r^3 + \frac{2\beta^2 t}{Dr}(1 + (b/r)^2)\right]\frac{\partial\omega}{\partial r} + \left[4/r^4 + \frac{2\beta^2 t}{Dr^2}(1 + (b/r)^2)\right]\frac{\partial^2\omega}{\partial\theta^2} - 2/r^3\frac{\partial^3\omega}{\partial r\partial\theta^2} + 2/r^2\frac{\partial^4\omega}{\partial r^2\partial\theta^2} + 1/r^4\frac{\partial^4\omega}{\partial\theta^4} = 0 \quad (10)$$

where

$$\beta^2 \equiv -B_0$$

and (11)

$$D \equiv Et^3/[12(1 - \sigma^2)]$$

Equation (10) may be transformed by means of finite Fourier transformations. The resulting transformed ordinary differential equations were solved using the method of Frobenius [14, 15]. The very lengthy calculations finally yield the solution of equation (10) in the form:

$$\omega(r, \theta) = C_{0,0} + C_{0,1}\int J_\nu(b_1r)dr + C_{0,2}\int Y_\nu(b_1r)dr + C_{0,3}\int\left[-J_\nu(b_1r)\int Y_\nu(b_1r)dr + Y_\nu(b_1r)\int J_\nu(b_1r)dr\right]dr$$
$$+ (C_{1,0}\cos\theta + C_{1,0}'\sin\theta)r + (C_{1,1}\cos\theta + C_{1,1}'\sin\theta)r\int\frac{J_{\nu^*}(b_1r)}{r}dr + (C_{1,2}\cos\theta + C_{1,2}'\sin\theta)r\int\frac{Y_{\nu^*}(b_1r)}{r}dr$$
$$+ (C_{1,3}\cos\theta + C_{1,3}'\sin\theta)r\int\left[-\frac{J_{\nu^*}(b_1r)}{r}\int\frac{Y_{\nu^*}(b_1r)}{r}dr + \frac{Y_{\nu^*}(b_1r)}{r}\int\frac{J_{\nu^*}(b_1r)}{r}dr\right]dr$$
$$+ \sum_{n=2}^{\infty}\Bigg\{\left(\sum_{p=0}^{\infty}\frac{(-1)^p\left(p + \frac{\nu_1 + n - 1}{2}\right)!\left(p + \frac{\nu_1 - n - 1}{2}\right)!}{(p)!\,(p + \nu_1)!\left(p + \frac{\nu_1 - \nu_2}{2}\right)!\left(p + \frac{\nu_1 + \nu_2}{2}\right)!}\left(\frac{b_1r}{2}\right)^{2p+\nu_1+1}\right)(C_{n,0}\cos n\theta + C_{n,0}'\sin n\theta)$$
$$+ \left(\sum_{p=0}^{\infty}\frac{(-1)^p\left(p + \frac{-\nu_1 + n - 1}{2}\right)!\left(p + \frac{-\nu_1 - n - 1}{2}\right)!}{(p)!\,(p - \nu_1)!\left(p + \frac{-\nu_1 - \nu_2}{2}\right)!\left(p + \frac{-\nu_1 + \nu_2}{2}\right)!}\left(\frac{b_1r}{2}\right)^{2p-\nu_1+1}\right)(C_{n,1}\cos n\theta + C_{n,1}'\sin n\theta)$$
$$+ \left(\sum_{p=0}^{\infty}\frac{(-1)^p\left(p + \frac{\nu_2 + n - 1}{2}\right)!\left(p + \frac{\nu_2 - n - 1}{2}\right)!}{(p)!\,(p + \nu_2)!\left(p + \frac{\nu_2 - \nu_1}{2}\right)!\left(p + \frac{\nu_2 + \nu_1}{2}\right)!}\left(\frac{b_1r}{2}\right)^{2p+\nu_2+1}\right)(C_{n,2}\cos n\theta + C_{n,2}'\sin n\theta)$$
$$+ \left(\sum_{p=0}^{\infty}\frac{(-1)^p\left(p + \frac{-\nu_2 + n - 1}{2}\right)!\left(p + \frac{-\nu_2 - n - 1}{2}\right)!}{(p)!\,(p - \nu_2)!\left(p + \frac{-\nu_2 - \nu_1}{2}\right)!\left(p + \frac{-\nu_2 + \nu_1}{2}\right)!}\left(\frac{b_1r}{2}\right)^{2p-\nu_2+1}\right)(C_{n,3}\cos n\theta + C_{n,3}'\sin n\theta)\Bigg\} \quad (12)$$

where p is an indexing parameter, $J_\nu(b_1r)$ is Bessel's function of the first kind of order ν and of argument b_1r, $Y_\nu(b_1r)$ is Bessel's function of the second kind, Weber's form, of order ν and of argument b_1r, and the functions $J_{\nu^*}(b_1r)$ and $Y_{\nu^*}(b_1r)$ are similar Bessel functions of order ν^* where

$$\nu^2 \equiv 1 + (b_1b)^2 \quad \text{and} \quad (\nu^*)^2 \equiv 4 + (b_1b)^2 \quad (13)$$

$$b_1^2 \equiv \frac{2\beta^2 t}{D} \quad (14)$$

$$\nu_1 \equiv \left(1 + n^2 + \frac{(b_1b)^2}{2} + \left(4n^2 + 2n^2(b_1b)^2 + \frac{(b_1b)^4}{4}\right)^{1/2}\right)^{1/2} \quad (15)$$

$$\nu_2 \equiv \left(1 + n^2 + \frac{(b_1b)^2}{2} - \left(4n^2 + 2n^2(b_1b)^2 + \frac{(b_1b)^4}{4}\right)^{1/2}\right)^{1/2} \quad (16)$$

The four plate-bending boundary conditions that must be satisfied by $\omega(r, \theta)$ are

$$\omega(a, \theta) = 0 \quad (17)$$

$$M_R(b, \theta) = -D\left[\frac{\partial^2\omega(r,\theta)}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial\omega(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\omega(r,\theta)}{\partial\theta^2}\right)\right]\Bigg|_{r=b} = 0 \quad (18)$$

$$V_R(b, \theta) = -D\left[\frac{\partial^3\omega(r,\theta)}{\partial r^3} + \frac{1}{r}\frac{\partial^2\omega(r,\theta)}{\partial r^2} - \frac{1}{r^2}\frac{\partial\omega(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^3\omega(r,\theta)}{\partial r\partial\theta^2} - \frac{2}{r^3}\frac{\partial^2\omega(r,\theta)}{\partial\theta^2}\right]\Bigg|_{r=b} - \left[\frac{(1-\sigma)D}{r}\left(\frac{1}{r}\frac{\partial^3\omega(r,\theta)}{\partial r\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2\omega(r,\theta)}{\partial\theta^2}\right)\right]\Bigg|_{r=b} = 0 \quad (19)$$

$$\theta^* + \frac{\partial\omega(r,\theta)}{\partial r}\Bigg|_{r=a} = 0 \quad (20)$$

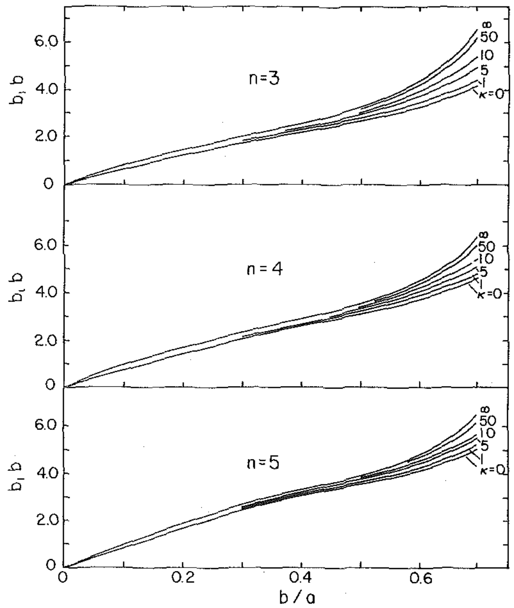

Fig. 2(b) Dimensionless buckling coefficients (b_1b) versus b/a

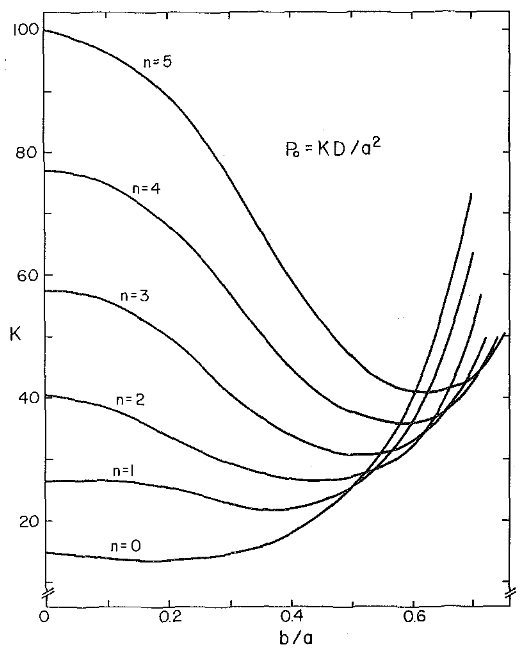

Fig. 3 Dimensionless buckling coefficient (K) versus b/a for clamped edge case

where M_R and V_R represent, respectively, the radial moment per unit length and the Kirchhoff shearing force per unit length in the plate. The fourth boundary condition is due to the radial slope continuity condition between the plate and edge beam at $r = a$, where θ^* is the radial edge beam slope that is found from solving the appropriate ordinary differential equation developed in curved beam theory [13].

Upon finding θ^*, it is substituted, together with $\omega(r, \theta)$ as given by equation (12), into equations (17)–(20), leading to, for each n, a 4×4 determinant that must be set equal to zero so as to yield nontrivial solutions for the P_0 load. The values of these determinants are functions of b_1b, b_1a, various material properties and the parameter n.

The components of the determinants formed can be found in reference [16]. The value of n specifies the mode of plate buckling and for each corresponding secular equation there are an infinite number of eigenvalues. It is of interest to find the lowest value of P_0 necessary to buckle the plate.

It was found mathematically that the series appearing in the components of the determinants [16], and in the deflection function, equation (12), are all absolutely convergent.

Computer methods were used to evaluate the infinite series which are utilized throughout this analysis and to evaluate the roots, b_1b, of the secular equations formed, where

$$(b_1b)^2 = \frac{2\beta^2b^2t}{D} \tag{21}$$

Once b_1b has been determined, it follows from equations (11), (8), and (7) that

$$P_0 = \left[\frac{(b_1b)^2\{\alpha[(1-\sigma)+(b/a)^2(1+\sigma)]+1-(b/a)^2\}}{(b/a)^2}\right] D/a^2 \tag{22}$$

The minimum P_0-value found for a specific plate, admitting all n-values, is the critical buckling load.

Results and Concluding Remarks

The results of this investigation are represented in a series of graphs shown in Figs. 2(a, b) and 3 for a Poisson's ratio of 1/3. In Figs. 2(a, b), the dimensionless variable b_1b is plotted against the ratio b/a for n varying from 0–5. For a given problem, the appropriate value of b_1b is needed for use in equation (22). The critical buckling load corresponds to that mode (n) associated with the lowest b_1b-value. The other variable appearing in Figs. 2(a, b) is the dimensionless stiffness parameter of the edge beam, defined as

$$\kappa = n^2\frac{G_BI_T}{Da} + \frac{E_BI_B}{Da} \tag{23}$$

where G_B is the shearing modulus of elasticity of the edge beam, and I_T is the torsional constant of the edge beam cross section.

That an edge beam presents a practical means of increasing the stability of an annular plate may best be shown by an example. Consider an annular plate with the following properties: $a = 20$ in. (50.8 cm), $b = 14$ in. (35.6 cm), $t = 0.45$ in. (1.14 cm), $E_B = E$, $\sigma = 1/3$, edge beam width = 1 in. (2.54 cm), edge beam height = 6 in. (15.2 cm). It follows that $I_B = 18$ in.^{4} (748 cm^4), $I_{20} = 1/2$ in.^{4} (20.8 cm^4), $A_B' = 6$ in.^{2} (38.6 cm^2), $b/a = 0.7$, $\alpha = 2/3$, $I_T = 1.79$ in.^{4} (74.5 cm^4), and

$$\kappa = 4n^2 + 105$$

With this value of κ, we must now check the six graphs presented in Figs. 2(a, b) to determine the critical (lowest) value of b_1b. For this example, the lowest value of b_1b is 6.30 and is associated with the $n = 4$ mode. Equation (22) then yields a critical buckling load of $P_0 = 112\ D/a^2$. The corresponding critical buckling load for the same plate with no edge beam and clamped along its outside boundary can be found to be only $P_0 = 42.5\ D/a^2$, also associated with mode $n = 4$. This illustrates the contribution of the edge beam in resisting the applied loading. It is interesting to note that to assume that this plate and its edge beam will buckle in the symmetric mode leads to an incorrect "critical" buckling load of $P_0 = 182\ D/a^2$.

Two special cases that are of particular interest in this discussion are the cases of a plate with no edge beam and of a plate with an edge beam that provides the effect of clamping the plate's outer edge. The plate with no edge beam represents the case of a simply supported plate on its outer edge. For this case $A_B = I_B = I_{20} = E_B = G_B = \alpha = 0$, and $\kappa = 0$, and the critical roots, $b_1 b$, are found from Figs. 2(a, b) along the curves $\kappa = 0$. A check of Figs. 2 will show that the axisymmetric mode ($n = 0$) controls for the simply supported case for all values of b/a. To properly represent the clamped support case, a special type of edge beam is used. The effect described by the $E_B A_B'$-value for the edge beam is a stiffness contribution which provides a direct resistance to the ring load applied to the beam's outer edge. Thus, to provide an edge beam that is "equivalent" to the clamped support case, the $E_B A_B'$-value is taken to be zero so that P_0 is not resisted by the beam's $E_B A_B'$ stiffness. At the same time a zero slope is required at the plate's outer edge. The edge beam, therefore, must possess a very large value of $E_B I_B$, thus providing the inertial restraint at the outer edge to keep the radial slope equal to zero. It follows therefore from equations (9) and (23) that $\alpha = 0$ and $\kappa = \infty$ corresponds to an edge beam equivalent to the clamped case.

Because the clamped outside edge case represents an interesting practical problem, the results for this boundary condition are shown separately in Fig. 3 (they could be obtained from Figs. 2(a, b)). In Fig. 3, the buckling parameter K which is defined as the bracketed portion of equation (22), is plotted versus b/a. Note that the axisymmetric mode controls for $0 \leq b/a \leq 0.50$. As b/a continues to increase, however, higher modes control. In the limit, the results (not shown) correspond to the buckling load of an axially loaded long and narrow rectangular plate with one long side clamped and the other free [4].

Acknowledgment

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