

## FORMAL BALLS IN FUZZY PARTIAL METRIC SPACES

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ABSTRACT. In this paper, the poset  $BX$  of formal balls is studied in fuzzy partial metric space  $(X, p, *)$ . We introduce the notion of layered complete fuzzy partial metric space and get that the poset  $BX$  of formal balls is a dcpo if and only if  $(X, p, *)$  is layered complete fuzzy partial metric space.

### 1. Introduction

Edalat and Heckmann in [2] studied the domain of formal balls  $BX = X \times [0, \infty)$  and provided a computational model for complete metric spaces. They proved that a metric space  $(X, d)$  is complete if and only if the set of the formal balls  $BX$  ordered by the relation  $(x, r) \sqsubseteq_d (y, s)$  (defined by  $d(x, y) \leq r - s$ ) is a domain. The results of Edalat and Heckman have also been generalized to other kind of metric structures, like for instance quasi-metric spaces [1, 10], fuzzy metric space [8, 9] or lattice-valued partial metric spaces [11].

According to Lawvere's view of metric spaces as  $[0, \infty)^{op}$ -categories, metric spaces can be studied in an enriched categorical method, see [6]. Rutten in [13] studied formal balls in generalized metric spaces by categorical method. It is shown that balls and formal balls (viewed as fuzzy sets) can be related by Isbell conjugation, and the collection of formal balls can be considered as a computational model. In [18], Zhao showed that the process of forming formal balls in a metric space is a special case of tensor completion of  $\Omega$ -categories, the partial order defined on  $BX$  is just the underlying order defined in metric space (take metric space as  $\Omega$ -category).

Fuzzy metric space and its generalization are also important examples in metric spaces. In [5], it is shown that KM-fuzzy metric space is equivalent to fuzzy metric space (defined in [14]). Yue studied fuzzy partial metric as a generalization of both fuzzy metric space and partial metric space by  $\Delta^+$ -valued sets in [17]. The ideas in [13, 18] can be easily extended to enriched categories. Hofmann and Reis studied fuzzy metric spaces viewed as enriched categories in [3].

The aim of this paper is to study formal balls in fuzzy partial metric spaces. Since  $L$ -valued sets are close related to  $\mathcal{Q}$ -category (See [4]), we define a poset  $BX$  of formal balls for fuzzy partial metric space  $(X, p, *)$  by the theory of  $\mathcal{Q}$ -category and prove that  $(X, p, *)$  is layered complete if and only if the formal balls  $BX$  in  $(X, p, *)$  is a dcpo.

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## 2. Preliminaries

In this section, we recall some definitions and properties in fuzzy partial metrics and  $\mathcal{Q}$ -categories.

**Definition 2.1.** (see [7]) A partial metric is a function  $d : X \times X \rightarrow R^+$  such that

- (P1)  $\forall x, y \in X, x = y \Leftrightarrow d(x, x) = d(x, y) = d(y, y)$ ;
- (P2)  $\forall x, y \in X, d(x, x) \leq d(x, y)$ ;
- (P3)  $\forall x, y \in X, d(x, y) = d(y, x)$ ;
- (P4)  $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$ .

The pair  $(X, d)$  is called a partial metric space.

**Definition 2.2.** A binary operation  $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a left-continuous  $t$ -norm if it satisfies the following conditions:

- (1)  $*$  is associative and commutative;
- (2)  $a * 1 = a$  for all  $a \in [0, 1]$ ;
- (3)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ ;
- (4)  $*$  is left-continuous.

For each left-continuous  $t$ -norm  $*$ , the implication  $\rightarrow$  can be determined by  $a \rightarrow b = \bigvee \{c \in [0, 1] \mid a * c \leq b\}$  and we have  $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$  for  $a, b, c \in [0, 1]$ .

Three most commonly used left-continuous  $t$ -norm are the minimum, denoted by  $\wedge$ , the usual product, denoted by  $\cdot$  and the Łukasiewicz  $t$ -norm, denoted by  $\diamond (a \diamond b = \max\{0, a + b - 1\})$ .

$f : [0, +\infty] \rightarrow [0, 1]$  is called a distance distribution function if it satisfies:

- (1)  $f$  is non-decreasing;
- (2)  $f$  is left-continuous on  $(0, +\infty)$ ;
- (3)  $f(0) = 0$  and  $f(+\infty) = 1$ .

Let  $\Delta^+$  denote the set of all distance distribution functions.  $(\Delta^+, \leq)$  will be a complete lattice with the top element  $\varepsilon_0$  and the bottom element  $\varepsilon_{+\infty}$ , where for  $r \in [0, +\infty)$ ,  $\varepsilon_r : [0, +\infty] \rightarrow [0, 1]$  is defined by

$$\varepsilon_r(x) = \begin{cases} 0, & x \leq r, \\ 1, & x > r. \end{cases}$$

Define the commutative semigroup operation  $\oplus_* : \Delta^+ \times \Delta^+ \rightarrow \Delta^+$  by

$$f \oplus_* g(x) = \bigvee_{s+t=x} f(s) * g(t).$$

It is easy to see that  $(\Delta^+, \oplus_*, \varepsilon_0)$  is a quantale and  $\oplus_*$  can induce a binary operation  $\rightarrow_* : \Delta^+ \times \Delta^+ \rightarrow \Delta^+$  by

$$f \rightarrow_* g = \bigvee_{f \oplus_* h \leq g} h,$$

and  $f \oplus_* h \leq g \iff h \leq f \rightarrow_* g$  holds.

**Remark 2.3.** It is easy to check that the following properties are valid:

- (1)  $\varepsilon_{r_1} \oplus_* \varepsilon_{r_2} = \varepsilon_{r_1+r_2}$ ;
- (2)  $\varepsilon_{r_1} \rightarrow_* \varepsilon_{r_2} = \varepsilon_{r_2-r_1}$ , when  $r_1 \leq r_2$ ;
- (3)  $h \leq g \iff h \rightarrow_* g = \varepsilon_0$ .

**Definition 2.4.** (see [17]) Let  $*$  be a left-continuous  $t$ -norm.  $p : X \times X \rightarrow \Delta^+$  is called a fuzzy partial metric if  $p$  satisfies the following axioms:

- (PP1)  $\forall x, y \in X, p(x, y) \leq p(x, x)$ ;
- (PP2)  $\forall x, y \in X, p(x, y) = p(y, x)$ ;
- (PP3)  $\forall x, y, z \in X,$   
 $p(x, y) \oplus_* (p(y, y) \rightarrow_* p(y, z)) \leq p(x, z)$ ;
- (PP4)  $x = y \iff p(x, y) = p(x, x) = p(y, y)$  for  
 $x, y \in X$ .

The triple  $(X, p, *)$  is called a fuzzy partial metric space.

Now we list some basic notions of quantaloid and quantaloid-enriched categories in [12, 16].

A quantaloid is a category  $\mathcal{Q}$  such that the set  $\mathcal{Q}(X, Y)$  of the arrows from  $X$  to  $Y$  is a sup-lattice for all objects  $X, Y \in \mathcal{Q}$ ; and the composition operation  $\circ$  preserves suprema in both variables, that is,

$$f \circ \bigvee_{i \in I} g_i = \bigvee_{i \in I} (f \circ g_i), \quad \bigvee_{j \in J} f_j \circ g = \bigvee_{j \in J} (f_j \circ g)$$

for all  $f, f_j \in \mathcal{Q}(Y, Z)$  and  $g, g_i \in \mathcal{Q}(X, Y)$ . The bottom and top element of  $\mathcal{Q}(X, Y)$  are denoted by  $\perp_{X, Y}$  and  $\top_{X, Y}$  respectively; the identity arrow on an object  $X$  is denoted by  $1_X$ .

Given a quantaloid  $\mathcal{Q}$  and  $\mathcal{Q}$ -arrows  $g \in \mathcal{Q}(Y, Z)$  and  $f \in \mathcal{Q}(X, Y)$ , there are two adjunctions

$$\begin{aligned} - \circ f \dashv - \swarrow f &: \mathcal{Q}(X, Z) \rightarrow \mathcal{Q}(Y, Z), \\ g \circ - \dashv g \searrow - &: \mathcal{Q}(X, Y) \rightarrow \mathcal{Q}(X, Z) \end{aligned}$$

determined by the adjoint property

$$g \circ f \leq h \iff g \leq h \swarrow f \iff f \leq g \searrow h.$$

Let  $\mathcal{Q}$  be a quantaloid. A  $\mathcal{Q}$ -category  $\mathbb{A}$  is a set  $\mathbb{A}_0$  of objects equipped with a type map  $t : \mathbb{A}_0 \rightarrow \text{obj}(\mathcal{Q})$  ( $tx$  is called the type of  $x$ ) and hom-arrows  $\mathbb{A}(x, y) \in \mathcal{Q}(tx, ty)$  such that

- (1)  $1_{tx} \leq \mathbb{A}(x, x)$  for all  $x \in \mathbb{A}_0$ ;
- (2)  $\mathbb{A}(y, z) \circ \mathbb{A}(x, y) \leq \mathbb{A}(x, z)$  for all  $x, y, z \in \mathbb{A}_0$ .

Let  $\mathbb{A}$  and  $\mathbb{B}$  be two  $\mathcal{Q}$ -categories. A map  $F : \mathbb{A} \rightarrow \mathbb{B}$  is called a  $\mathcal{Q}$ -functor if  $F$  satisfies the following conditions:

- (1)  $F$  is type-preserving, i. e.  $tx = tF(x)$ ;

$$(2) \forall x, x' \in \mathbb{A}_0, \mathbb{A}(x, x') \leq \mathbb{B}(F(x), F(x')).$$

A presheaf with type  $t\phi$  on a  $\mathcal{Q}$ -category  $\mathbb{A}$  is a  $\mathcal{Q}$ -distributor  $\phi : \mathbb{A} \dashrightarrow *_t\phi$ , where  $*_t\phi$  is the  $\mathcal{Q}$ -category with exactly one element  $*$  such that  $t* = t\phi$ . For a presheaf  $\phi : \mathbb{A} \dashrightarrow *_t\phi$ , we often write  $\phi(x)$  instead of  $\phi(x, *)$  for short. All presheaves on  $\mathbb{A}$  form a  $\mathcal{Q}$ -category  $\mathcal{P}\mathbb{A}$  with

$$\mathcal{P}\mathbb{A}(\phi, \psi) = \psi \swarrow \phi = \bigwedge_{x \in \mathbb{A}_0} \psi(x) \swarrow \phi(x).$$

For  $x \in \mathbb{A}_0$ ,  $\mathcal{P}(tx)$  denotes all  $\mathcal{Q}$ -arrows with domain  $tx$ . Let  $x \in \mathbb{A}_0$  and  $f \in \mathcal{P}(tx)$ . If there is  $f \otimes x \in \mathbb{A}_0$  with type  $t(f \otimes x) = \text{codomain}(f)$  such that  $\mathbb{A}(f \otimes x, y) = \mathbb{A}(x, y) \swarrow f$  for all  $y \in \mathbb{A}_0$ , then  $f \otimes x$  is called the tensor of  $f$  and  $x$ .  $\mathbb{A}$  is called a tensored  $\mathcal{Q}$ -category if  $f \otimes x$  exists for all  $f$  and  $x$ .

From [15, 16], we know that  $\mathcal{P}\mathbb{A}$  is a tensored  $\mathcal{Q}$ -category, and the tensor of  $f \in \mathcal{P}(t\mu)$  and  $\mu \in \mathcal{P}\mathbb{A}_0$  is  $f \circ \mu$ , i. e. ,  $f \otimes \mu = f \circ \mu$ .  $\mathbb{A}$  can be embedded in  $\mathcal{P}\mathbb{A}$  by the Yoneda embedding  $\mathbb{Y} : \mathbb{A} \rightarrow \mathcal{P}\mathbb{A}$  in the following way:  $\mathbb{Y}(x) = \mathbb{A}(-, x)$ , i. e.  $\mathbb{Y}(x)(y) = \mathbb{A}(y, x)$  for all  $y \in \mathbb{A}_0$ . We denote  $\mathbb{Y}(x)$  by  $\mathbb{Y}_x$ .

Let  $T(\mathbb{A})_0 = \{(x, f) | x \in \mathbb{A}_0, f \in \mathcal{P}(tx)\}$ . Define the type function  $t : T(\mathbb{A})_0 \rightarrow \text{obj}(\mathcal{Q})$  by  $t((x, f)) = t(x)$  and define  $T(\mathbb{A}) : T(\mathbb{A})_0 \times T(\mathbb{A})_0 \rightarrow \mathcal{Q}$  by

$$T(\mathbb{A})((x, f), (y, g)) = \mathcal{P}\mathbb{A}(f \circ \mathbb{Y}_x, g \circ \mathbb{Y}_y).$$

**Lemma 2.5.**  $T(\mathbb{A})$  is a tensored  $\mathcal{Q}$ -category.

**Remark 2.6.** Define a  $\mathcal{Q}$ -functor  $E : \mathbb{A} \rightarrow T(\mathbb{A})$  by  $E(x) = (x, 1_{tx})$ . Since  $T(\mathbb{A})(E(x), E(y)) = T(\mathbb{A})((x, 1_{tx}), (y, 1_{ty})) = \mathbb{A}(x, y)$ ,  $\mathbb{A}$  can be embedded in  $T(\mathbb{A})$  by the full faithful  $\mathcal{Q}$ -functor  $E$ .

**Theorem 2.7.** Let  $\mathbb{C}$  be a tensored  $\mathcal{Q}$ -category and  $F : \mathbb{A} \rightarrow \mathbb{C}$  be a  $\mathcal{Q}$ -functor. Then there is a unique  $\mathcal{Q}$ -functor  $T(F) : T(\mathbb{A}) \rightarrow \mathbb{C}$  preserves tensor and satisfies  $F = T(F) \circ E$ .

*Proof.* Define  $T(F)((x, f)) = f \otimes F(x)$ , where  $f \otimes F(x)$  is the tensor of  $f$  and  $F(x)$  in  $\mathbb{C}$ .

(1)  $F = T(F) \circ E$ : Since

$$T(f) \circ E(x) = T(F)((x, 1_{tx})) = 1_{tx} \otimes F(x) = F(x),$$

this is to say  $F = T(F) \circ E$ .

(2)  $T(F)$  is a  $\mathcal{Q}$ -functor:

$$\begin{aligned} \mathbb{C}(T(F)((x, f)), T(F)((y, g))) &= \mathbb{C}(f \otimes F(x), g \otimes F(y)) \\ &= \mathbb{C}(F(x), g \otimes F(y)) \swarrow f \\ &\geq g \circ \mathbb{C}(F(x), F(y)) \swarrow f \\ &\geq g \circ \mathbb{A}(x, y) \swarrow f \\ &= T(\mathbb{A})((x, f), (y, g)). \end{aligned}$$

(3)  $T(F)$  preserves tensor: On account of

$$T(f)(g \otimes (x, f)) = T(F)((x, g \circ f)) = (g \circ f) \otimes F(x),$$

we have

$$\begin{aligned}
\mathbb{C}((g \circ f) \otimes F(x), -) &= \mathbb{C}(F(x), -) \swarrow (g \circ f) \\
&= (\mathbb{C}(F(x), -) \swarrow f) \swarrow g \\
&= \mathbb{C}(f \otimes F(x), -) \swarrow g \\
&= \mathbb{C}(g \otimes (f \otimes F(x)), -).
\end{aligned}$$

Then  $(g \circ f) \otimes F(x) = g \otimes (f \otimes F(x))$ . It follows that  $T(f)(g \otimes (x, f)) = g \otimes (f \otimes F(x)) = g \otimes T(F)((x, f))$ , as desired.

(4) Uniqueness of  $T(F)$ : Suppose that  $G : T(\mathbb{A}) \rightarrow \mathbb{C}$  is another  $\mathcal{Q}$ -functor preserves tensor and satisfies  $F = G \circ E$ . Then  $G((x, f)) = G(f \otimes (x, 1_{tx})) = f \otimes G(x, 1_{tx}) = f \otimes (G \circ E(x)) = f \otimes F(x) = T(F)(x, f)$ . Therefore,  $G = T(F)$ .  $\square$

**Remark 2.8.** Let  $\mathcal{Q}\text{-cat}$  denote the category of  $\mathcal{Q}$ -categories and  $\mathcal{Q}$ -functors, and  $\mathcal{Q}\text{-Tcat}$  denote the subcategory of  $\mathcal{Q}\text{-cat}$  with objects are all tensored  $\mathcal{Q}$ -category. Theorem 2.7 tells us that  $T : \mathcal{Q}\text{-cat} \rightarrow \mathcal{Q}\text{-Tcat}$  is the right adjoint of the inclusion functor  $i : \mathcal{Q}\text{-Tcat} \rightarrow \mathcal{Q}\text{-cat}$ .

$T(\mathbb{A})$  is called the tensored completion of  $\mathbb{A}$ . According to the underlying order in  $\mathcal{Q}$ -category, we can define  $(x, f) \leq (y, g)$  if and only if  $t(x, f) = t(y, g)$  and  $T(\mathbb{A})((x, f), (y, g)) \geq 1_{t(x, f)}$ , i. e. ,  $(x, f) \leq (y, g)$  if and only if  $tx = ty$  and  $f \leq g \circ \mathbb{A}(x, y)$ .

A quantale is exactly a quantaloid with only one object. From [16], given a quantale  $L = (L, \&, 1)$ , we know there is another way to construct a quantaloid  $\mathcal{D}(L)$ , called the quantaloid of diagonals in  $L$  as follows:

- objects: elements  $a, b, c, \dots$  in  $L$ ;
- morphisms:  $\mathcal{D}(L)(a, b) = \{d \in L : (d \swarrow a) \& a = d = b \& (b \searrow d)\}$  for all objects  $a, b$ .
- composition:  $\beta \circ \alpha = \beta \& (b \searrow \alpha) = (\alpha \swarrow b) \& \beta$  for all  $\alpha \in \mathcal{D}(L)(a, b)$  and  $\beta \in \mathcal{D}(L)(b, c)$ ;
- the unit  $1_a$  of  $\mathcal{D}(L)(a, a)$  is  $a$ ;
- the partial order on  $\mathcal{D}(L)(a, b)$  is inherited from  $L$ .

When  $L$  is a commutative and divisible quantale,  $\searrow$  is in accordance with  $\swarrow$ . Hence we use  $\rightarrow$  instead of  $\searrow$  and  $\swarrow$  in the following discussion. It is easy to see that  $\mathcal{D}(L)(a, b) = \{d \in L : d \leq a \wedge b\}$  and  $\beta \circ \alpha = \beta \& (b \rightarrow \alpha) = \alpha \& (b \rightarrow \beta)$  for  $\alpha \in \mathcal{D}(L)(a, b)$  and  $\beta \in \mathcal{D}(L)(b, c)$ .

**Remark 2.9.** From [17], we know that  $(\Delta^+, \oplus_\wedge, \varepsilon_0)$  is a commutative and divisible quantale. Similar Example 2.14 in [16], we can think of fuzzy partial metric spaces as  $\mathcal{D}(\Delta^+)$ -category with further properties. From the above underlying order of  $T(\mathbb{A})$ , we can define the partial order in fuzzy partial metric spaces: for  $f, g \in \Delta^+$ ,  $(x, f) \leq (y, g)$  if and only if  $p(x, x) = p(y, y)$  and  $f \leq g \circ p(x, y) = g \oplus_\wedge (p(y, y) \rightarrow p(y, x))$ . In the next section, we will use this partial order to study formal balls in fuzzy partial metric spaces.

### 3. Formal Balls in Fuzzy Partial Metric Spaces

A formal ball in a fuzzy partial metric space  $(X, p, *)$  is a pair  $(x, r)$  with  $x$  in  $X$ ,  $r$  in  $R^+$  (i. e. ,  $0 \leq r < \infty$ ). The set of formal balls in  $X$  is denoted by  $BX$ . Formal balls are ordered:  $(x, r) \leq (y, s)$  iff  $(x, \varepsilon_r) \leq (y, \varepsilon_s)$  iff  $p(x, x) \rightarrow_* p(x, y) \geq \varepsilon_{r-s}$  and  $p(x, x) = p(y, y)$ .

**Proposition 3.1.** *For every fuzzy partial metric space  $(X, P, *)$ ,  $BX$  is a poset.*

*Proof.* Reflexivity:  $(x, r) \leq (x, r)$  since  $p(x, x) \rightarrow_* p(x, x) \geq \varepsilon_{r-r} = \varepsilon_0$ .

Transitivity:  $(x, r) \leq (y, s) \leq (z, t)$  implies  $p(x, x) \rightarrow_* p(x, y) \geq \varepsilon_{r-s}$  and  $p(y, y) \rightarrow_* p(y, z) \geq \varepsilon_{t-s}$ , thus

$$\begin{aligned} p(x, x) \rightarrow_* p(x, z) &\geq p(x, x) \rightarrow_* (p(x, y) \oplus_* (p(y, y) \rightarrow_* p(y, z))) \\ &\geq (p(x, x) \rightarrow_* p(x, y) \oplus_* (p(y, y) \rightarrow_* p(y, z))) \\ &\geq (\varepsilon_s \rightarrow_* \varepsilon_r) \oplus_* (\varepsilon_t \rightarrow_* \varepsilon_s) \\ &= \varepsilon_{r-t} \end{aligned}$$

Antisymmetry:  $(x, r) \leq (y, s) \leq (x, r)$  implies  $r \geq s \geq r$ . This gives  $r = s$  and thus,  $p(x, x) \rightarrow_* p(x, y) \geq \varepsilon_0$ ,  $p(y, y) \rightarrow_* p(x, y) \geq \varepsilon_0$  whence  $x = y$ .  $\square$

**Definition 3.2.** Let  $(X, p, *)$  be a fuzzy partial metric space. For every  $h \in \Delta^+$ , we get  $\mathcal{C}(h) = \{x \in X \mid p(x, x) = h\}$ , the  $\mathcal{C}(h)$  is called  $h$ -contour layer on  $(X, p, *)$ .

**Theorem 3.3.** *For every directed subset  $D$  of  $BX$ , there is an ascending sequence  $\{(x_n, r_n)\}$  of elements of  $D$  which has the same upper bounds as  $D$ .*

*Proof.* It is easy to see that there exists  $h \in \Delta^+$  such that  $x_n \in \mathcal{C}(h)$  for all  $n$ . Take  $s = \inf\{r \mid (x, r) \in D\}$ . Then there exists  $(y_n, s_n) \in D$  such that  $s_n \leq s + 1/n$  for each  $n$ . Let  $(x_1, r_1) = (y_1, s_1)$ , and for every  $n > 1$ , since  $D$  is directed, there is  $(x_n, r_n)$  such that it is an upper bound of  $(x_{n-1}, r_{n-1})$  and  $(y_n, s_n)$  in  $D$ . Let  $(z, t)$  be an upper bound of  $\{(x_n, r_n)\}_n$ , and let  $(a, u)$  be an element of  $D$ . Since  $D$  is directed, there are upper bounds  $(b_n, v_n)$  of  $(a, u)$  and  $(x_n, r_n)$  in  $D$ . Then for all  $n$

$$\begin{aligned} &p(a, a) \rightarrow_* p(a, z) \\ &\geq p(a, a) \rightarrow_* (p(a, b_n) \oplus_* (p(b_n, b_n) \rightarrow_* p(b_n, z))) \\ &\geq p(a, a) \rightarrow_* (p(a, b_n) \oplus_* (p(b_n, b_n) \rightarrow_* (p(x_n, b_n) \oplus_* (p(x_n, x_n) \rightarrow_* p(x_n, z)))))) \\ &\geq (p(a, a) \rightarrow_* p(a, b_n)) \oplus_* (p(b_n, b_n) \rightarrow_* p(x_n, b_n)) \oplus_* (p(x_n, x_n) \rightarrow_* p(x_n, z)) \\ &\geq (\varepsilon_{v_n} \rightarrow_* \varepsilon_u) \oplus_* (\varepsilon_{v_n} \rightarrow_* \varepsilon_{r_n}) \oplus_* (\varepsilon_t \rightarrow_* \varepsilon_{r_n}) \\ &= \varepsilon_{(u-v_n)+(r_n-v_n)+(r_n-t)} \\ &\geq \varepsilon_{u-t+2(r_n-s)} \\ &\geq \varepsilon_{u-t+2/n} \end{aligned}$$

Hence,  $p(a, a) \rightarrow_* p(a, z) \geq \varepsilon_{u-t}$ , i. e. ,  $(a, u) \leq (z, t)$  as required.  $\square$

**Definition 3.4.** Let  $(X, p, *)$  be a fuzzy partial metric space,  $\mathcal{C}(h)$  be an  $h$ -contour layer on  $(X, p, *)$  and let  $\{x_n\} \subseteq \mathcal{C}(h)$  be a sequence in  $X$ .

- (1) If  $\forall \delta > 0, \exists N, s.t. \forall n \geq N, P(x, x) \rightarrow_* p(x_n, x) \geq \varepsilon_\delta$ , then  $\{x_n\}$  is called to be layered convergent to  $x$ .
- (2) If  $\forall \delta > 0, \exists N, s.t. \forall n, m \geq N, P(x_n, x_n) \rightarrow_* p(x_n, x_m) \geq \varepsilon_\delta$ , then  $\{x_n\}$  is called a layered Cauchy sequence.

**Proposition 3.5.** *If  $\{(x_n, r_n)\}$  is an ascending sequence in  $BX$ , then  $\{r_n\}$  is descending and convergent, and the sequence  $\{x_n\}$  is a layered Cauchy sequence.*

*Proof.* Assume  $x_n \in \mathcal{C}(h)$  for all  $n$ .  $(x_n, r_n) \leq (x_{n+1}, r_{n+1})$  implies  $r_n \geq r_{n+1}$ . Hence,  $\{r_n\}$  is descending. Since  $r_n \geq 0$  for all  $n$ , the sequence  $\{r_n\}$  is convergent and thus Cauchy. Therefore, for every  $\delta > 0$ , there is  $N$  such that for all  $m, n \geq N$ ,  $r_n - r_m \leq \delta$  and  $\varepsilon_{r_n - r_m} \geq \varepsilon_\delta$  holds. For  $n \geq m \geq N$ ,  $(x_m, r_n) \leq (x_n, r_n)$  holds, whence  $p(x_n, x_n) \rightarrow_* p(x_n, x_m) \geq \varepsilon_{r_n - r_m} \geq \varepsilon_\delta$ . Thus,  $\{x_n\}$  is a layered Cauchy sequence.  $\square$

**Lemma 3.6.** *Every layered Cauchy sequence  $\{x_n\}$  in  $X$  has a subsequence  $\{x_{n_k}\}$  such that  $\{(x_{n_k}, 2^{-k})\}$  is ascending in  $BX$ .*

*Proof.* Assume  $\forall n, x_n \in \mathcal{C}(h)$ . Let  $n_0 = 0$ . When  $k = 1$ , there is  $n_1 > n_0 = 0$  such that  $p(x_i, x_i) \rightarrow_* p(x_i, x_j) \geq \varepsilon_{2^{-2}}$  for all  $i, j \geq n_1$ . When  $k > 1$  there is  $n_k > n_{k+1}$  such that  $p(x_i, x_i) \rightarrow_* p(x_i, x_j) \geq \varepsilon_{2^{-(k+1)}}$  for all  $i, j \geq n_k$ . Hence,  $p(x_{n_k}, x_{n_k}) \rightarrow_* p(x_{n_k}, x_{n_{k+1}}) \geq \varepsilon_{2^{-(k+1)}} = \varepsilon_{2^{-(k+1)} - 2^{-k}}$ , i. e. ,  $(x_{n_k}, 2^{-k}) \leq (x_{n_{k+1}}, 2^{-(k+1)})$ .  $\square$

**Theorem 3.7.** *For an ascending sequence  $\{(x_n, r_n)\}$  in  $BX$  and element  $(y, s)$  of  $BX$ , the following are equivalent:*

- (i)  $(y, s)$  is the least upper bound of  $\{(x_n, r_n)\}$ ;
- (ii)  $(y, s)$  is an upper bound of  $\{(x_n, r_n)\}$ , and  $\lim_{n \rightarrow \infty} r_n = s$ ;
- (iii)  $\lim_{n \rightarrow \infty} x_n = y$  and  $\lim_{n \rightarrow \infty} r_n = s$ ;

*Proof.* (i) $\Rightarrow$ (ii) Since  $\{(x_n, r_n)\}$  is ascending with the upper bound  $(y, s)$ ,  $\{r_n\}$  is descending with the lower bound  $s$ . Assume  $\lim_{n \rightarrow \infty} r_n \neq s$ , i. e. , there is  $\delta > 0$  such that  $r_n \geq \delta + s$  for all  $n$ . Since  $\{x_n\}$  is a layered Cauchy sequence, there is  $N$  such that  $p(x_n, x_n) \rightarrow_* p(x_n, x_m) \geq \varepsilon_{\delta/2}$  for all  $n, m \geq N$ . Hence for all  $n \geq N$ ,  $p(x_n, x_n) \rightarrow_* p(x_n, x_N) \geq \varepsilon_{\delta/2} \geq \varepsilon_{r_n - (s + \delta/2)}$ , or  $(x_n, r_n) \leq (x_N, s + \delta/2)$ . Since the sequence is ascending,  $(x_N, s + \delta)$  is an upper bound of the whole sequence. Thus  $(y, s) \leq (x_N, s + \delta/2)$ , whence  $s \geq s + \delta/2$ , a contradiction.

(ii) $\Rightarrow$ (iii) For every  $n$ ,  $(x_n, r_n) \leq (y, s)$  holds, or  $p(x_n, x_n) \rightarrow_* p(x_n, y) \geq \varepsilon_{r_n - s}$ . Since  $\lim_{n \rightarrow \infty} r_n = s$  holds,  $\lim_{n \rightarrow \infty} x_n = y$  follows.

(iii) $\Rightarrow$ (i) First, we show  $(y, s)$  is an upper bound. Since  $\forall m, n, m \geq n, (x_n, r_n) \leq (x_m, r_m)$ ,  $p(x_n, x_n) \rightarrow_* p(x_n, x_m) \geq \varepsilon_{r_n - r_m}$ . Letting  $m$  tend to infinity, we obtain  $p(x_n, x_n) \rightarrow_* p(x_n, y) \geq \varepsilon_{r_n - s}$ , whence  $(x_n, r_n) \leq (y, s)$ .

If  $(z, t)$  is an arbitrary upper bound of the sequence, then  $(x_n, r_n) \leq (z, t)$  holds for all  $n$ , whence  $p(x_n, x_n) \rightarrow_* p(x_n, z) \geq \varepsilon_{r_n - t}$ . Letting  $n$  tend to infinity,  $p(y, y) \rightarrow_* p(y, z) \geq \varepsilon_{s - t}$  follows, i.e. ,  $(y, s) \leq (z, t)$ .  $\square$

**Definition 3.8.** Let  $(X, p, *)$  be a fuzzy partial metric space,  $\mathcal{C}(h)$  be a  $h$ -contour layer. We call  $(X, p, *)$  is layered complete fuzzy partial metric space, if for every layered Cauchy sequence, there is  $x$  in  $\mathcal{C}(h)$ , such that  $\lim_{n \rightarrow \infty} x_n = x$ .

**Theorem 3.9.** For a fuzzy partial metric space  $X$ , the following are equivalent:

- (i)  $X$  is a layered complete fuzzy partial metric space.
- (ii) In  $BX$ , every ascending sequence has a least upper bound.
- (iii)  $BX$  is dcpo, i. e. every directed set has a least upper bound.

*Proof.* The equivalence of (ii) and (iii) follows from Theorem 3.3.

Let  $X$  be a layered complete fuzzy partial metric space. If  $\{(x_n, r_n)\}$  is an ascending sequence in  $BX$ , then  $\{r_n\}$  converges to some  $s$ , and  $\{x_n\}$  is a layered Cauchy sequence in  $X$ . By completeness,  $\{x_n\}$  converges to some  $y$  in  $X$ . Thus,  $(y, s)$  is the least upper bound of  $\{(x_n, r_n)\}$ .

Assume (ii) holds, and let  $\{x_n\}$  be layered Cauchy sequence in  $X$ . By lemma 3.6, it has a subsequence  $\{x_{n_k}\}$  such that  $\{(x_{n_k}, 2^{-k})\}$  is ascending. By (ii), this sequence has a least upper bound  $(y, s)$ . Thus  $\lim_{n \rightarrow \infty} x_{n_k} = y$ . Since the whole sequence  $\{x_n\}$  is layered Cauchy and a subsequence converges to  $y$ , the whole sequence converges to  $y$ .  $\square$

**Remark 3.10.** Let  $(X, p, *)$  be a fuzzy partial metric space. If for all  $x$  in  $X$ ,  $p(x, x) = \varepsilon_0$ , the fuzzy partial metric space will be reduced to a fuzzy metric space. So we can get the set of the formal balls  $BX$  in a fuzzy metric space. The set of the formal balls  $BX$  are ordered by the relation  $(x, r) \leq (y, s)$  whenever  $p(x, y) \geq \varepsilon_s \rightarrow_* \varepsilon_r$ .

**Question** In this paper, elements in  $BX$  are of the form  $(x, r)$  or  $(x, \varepsilon_r)$ —a special case of  $(x, f)$  where  $f \in \Delta^+$ . When  $BX = \{(x, f) | x \in X, f \in \Delta^+\}$ , we want to know how to define the completeness of fuzzy partial metric space to ensure Theorem 3.9 is also valid? We leave it as a question for future study.

#### 4. Formal Balls in Partial Metric Spaces

Let  $(X, d)$  be a partial metric space and define  $p_d : X \times X \rightarrow \Delta^+$  by  $p_d(x, y) = \varepsilon_{d(x, y)}$ . Then  $p_d$  is a fuzzy partial metric under any left-continuous  $t$ -norm. So we can get the following results about partial metric space.

The partial order on formal balls is define by  $(x, r) \leq (y, s)$  whenever  $p(x, y) - p(x, x) \leq r - s$  and  $p(x, x) = p(y, y)$ . Note that  $(x, r) \leq (y, s)$  implies  $r \geq s$ . For,  $p(x, y) - p(x, x) \leq r - s$ , implies  $0 \leq r - s$ , whence  $s \leq r$ .

**Definition 4.1.** Let  $(X, p)$  be a partial metric space, We call it a layered complete partial metric space, for every  $r \in R^+$ , the metric space  $(X_r, d)$  is a complete metric space, where  $X_r = \{x \in X | d(x, x) = r\}$ ,  $d(x, y) = p(x, y) - p(x, x)$ .

**Theorem 4.2.** For a partial metric space  $X$ , the following are equivalent:

- (i)  $X$  is a layered complete partial metric space;
- (ii) In  $BX$ , every ascending sequence has a least upper bound;
- (iii)  $BX$  is dcpo.



**Example 4.3.** For every complete metric space  $(X, d)$ , the partial metric space  $(X, p)$  is a layered complete partial metric space. Here  $p$  is defined by  $p(x, y) = d(x, y)$ ,  $p(x, x) = 0$ .

**Example 4.4.**  $(X, d)$  is a complete metric,  $BX$  is the set of formal balls in  $(X, d)$  and formal balls are ordered by  $x_r \leq y_s \iff d(x, y) \leq r - s$ . We know that  $BX$  is a post in [1]. We write  $x_r$  instead of  $(x, r)$ . We define a function  $p : BX^2 \rightarrow R^+ :$

$$p(x_r, y_s) = \begin{cases} \max\{2r, 2s\}, & x_r \mid y_s, \\ d(x, y) + r + s, & x_r \nmid y_s. \end{cases}$$

(Here,  $x_r \mid y_s$  implies  $x_r \leq y_s$  or  $y_s \leq x_r$ .  $x_r \nmid y_s$  implies  $x_r \not\leq y_s$  and  $y_s \not\leq x_r$ ). Note that  $p(x_r, y_s) = p(x_r, x_r) \iff x_r \leq y_s$ . We can assert that  $(BX, p)$  is a partial metric space.

*Proof.* In fact, for all  $x_r, y_s, z_t \in BX$ :

(P1)  $x_r = y_s \iff p(x_r, x_r) = p(x_r, y_s)$  and  $p(x_r, y_s) = p(y_s, y_s)$ : Obvious.

(P2)  $p(x_r, x_r) \leq p(x_r, y_s)$ : Obvious.

(P3)  $p(x_r, y_s) = p(y_s, x_r)$ : Obvious.

(p4)  $p(x_r, z_t) \leq p(x_r, y_s) + p(y_s, z_t) - p(y_s, y_s)$ :

(a) Assume  $x_r \mid z_t$ . Since  $x_r \leq z_t$ , we know that  $p(x_r, z_t) = p(x_r, x_r) \leq p(x_r, y_s)$ . Hence  $p(x_r, z_t) \leq p(x_r, y_s) + p(y_s, z_t) - p(y_s, y_s)$ . Similarly, since  $z_t \leq x_r$ , we can get the same result.

(b) Assume  $x_r \nmid z_t$ ,  $x_r \mid y_s$  and  $z_t \mid y_s$ . Since  $x_r \leq y_s$  and  $z_t \leq y_s$ , we know  $p(x_r, y_s) = p(x_r, x_r)$ ,  $p(y_s, z_t) = p(z_t, z_t)$ . Hence  $p(x_r, z_t) = d(x, z) + r + t \leq d(x, y) + d(y, z) + r + t \leq (r - s) + (t - s) + (r + t) = 2t + 2r - 2s = p(x_r, y_s) + p(y_s, z_t) - p(y_s, y_s)$ . Similarly, since  $y_s \leq x_r$  and  $y_s \leq z_t$ , we can get the same result.

(c) Assume  $x_r \nmid z_t$ ,  $x_r \nmid y_s$  and  $z_t \mid y_s$ . Since  $z_t \leq y_s$ , we know  $p(y_s, z_t) = p(z_t, z_t)$ . Hence  $p(x_r, z_t) = d(x, z) + r + t \leq d(x, y) + d(y, z) + r + t \leq d(x, y) + t - s + r + t = d(x, y) + r + s + 2t - 2s = p(x_r, y_s) + p(y_s, z_t) - p(y_s, y_s)$ . Similarly, since  $y_s \leq z_t$ , we can get the same result.

(d) Assume  $x_r \nmid z_t$ ,  $x_r \nmid y_s$  and  $z_t \nmid y_s$ .  $p(x_r, z_t) = d(x, z) + r + t \leq d(x, y) + d(y, z) + r + t \leq d(x, y) + r + s + d(y, z) + t + s - 2s = p(x_r, y_s) + p(y_s, z_t) - p(y_s, y_s)$ .

So  $(BX, p)$  is a partial metric space. It is obvious that  $(BX, p)$  is a layered complete partial metric space too.  $\square$

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