Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 490165, 6 pages
http://dx.doi.org/10.1155/2014/490165

Research Article

Symmetries, Associated First Integrals, and Double Reduction of Difference Equations

L. Ndlovu, M. Folly-Gbetoula, A. H. Kara, and A. Love

School of Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

Correspondence should be addressed to A. H. Kara; abdul.kara@wits.ac.za

Received 11 April 2014; Accepted 16 July 2014; Published 23 July 2014

Academic Editor: Igor Leite Freire

We determine the symmetry generators of some ordinary difference equations and proceeded to find the first integral and reduce the order of the difference equations. We show that, in some cases, the symmetry generator and first integral are *associated* via the "invariance condition." That is, the first integral may be invariant under the symmetry of the original difference equation. When this condition is satisfied, we may proceed to double reduction of the difference equation.

1. Introduction

The theory, reasoning, and algebraic structures dealing with the construction of symmetries for differential equations (DEs) are now well established and documented. Moreover, the application of these in the analysis of DEs, in particular, for finding exact solutions, is widely used in a variety of areas from relativity to fluid mechanics (see [1–4]). Secondly, the relationship between symmetries and conservation laws has been a subject of interest since Noether's celebrated work [5] for variational DEs. The extension of this relationship to DEs which may not be variational has been done more recently [6, 7]. The first consequence of this interplay has led to the double reduction of DEs [8–10].

A vast amount of work has been done to extend the ideas and applications of symmetries to difference equations (ΔEs) in a number of ways—see [11–15] and references therein. In some cases, the ΔEs are constructed from the DEs in such a way that the algebra of Lie symmetries remains the same [16]. As far as conservation laws of ΔEs go, the work is more recent—see [12, 17]. Here, we construct symmetries and conservation laws for some ordinary ΔEs, utilise the symmetries to obtain reductions of the equations, and show, in fact, that the notion of "association" between these concepts can be analogously extended to ordinary ΔEs. That is, an association between a symmetry and first integral exists if and only if the first integral is invariant under the symmetry. Thus, a "double reduction" of the ΔE is possible.

2. Preliminaries and Definitions

Consider the following Nth-order OΔE:

$$u_{n+N} = \omega\left(n, u_n, u_{n+1}, \ldots, u_{n+N-1}\right), \tag{1}$$

where ω is a smooth function such that $(\partial\omega/\partial u_n) \neq 0$ and integer n is an independent variable. The general solution of (1) can be written in the form

$$u_n = F\left(n, c_1, \ldots, c_N\right) \tag{2}$$

and depends on N arbitrary independent constants c_i.

Definition 1. We define $\mathcal{S}$ to be the shift operator acting on n as follows:

$$\mathcal{S} : n \longmapsto n + 1. \tag{3}$$

That is, if $u_n = F(n, c_1, \ldots, c_N)$ then

$$\mathcal{S}\left(u_n\right) = u_{n+1}. \tag{4}$$

In the same way,

$$\mathcal{S}\left(u_{n+k}\right) = u_{n+k+1}, \quad k = 0, \ldots, N-2. \tag{5}$$

Definition 2. A symmetry generator, X, of (1) is given by

$$\begin{aligned} X &= Q(n, u_n, \ldots, u_{n+N-1}) \frac{\partial}{\partial u_n} \\ &\quad + (\mathcal{S}Q(n, u_n, \ldots, u_{n+N-1})) \frac{\partial}{\partial u_{n+1}} \\ &\quad + \cdots + \left(\mathcal{S}^{N-1}Q(n, u_n, \ldots, u_{n+N-1})\right) \frac{\partial}{\partial u_{n+N-1}} \end{aligned} \tag{6}$$

and satisfies the symmetry condition

$$S^N Q(n, u_n, \ldots, u_{n+N-1}) - X\omega = 0, \tag{7}$$

where $Q = Q(n, u_n, \ldots, u_{n+N-1})$ is a function called the characteristic of the one-parameter group.

Definition 3. If ϕ is a first integral, then it is constant on the solutions of the OΔE and hence satisfies

$$\begin{aligned} &\mathcal{S}(\phi(n, u_n, \ldots, u_{n+N-1})) = \phi(n, u_n, \ldots, u_{n+N-1}), \\ &\phi(n+1, u_{n+1}, \ldots, \omega(n, u_n, \ldots, u_{n+N-1})) \\ &\quad = \phi(n, u_n, \ldots, u_{n+N-1}), \end{aligned} \tag{8}$$

where $\mathcal{S}$ is the shift operator defined in (3).

2.1. First Integral. In [11], Hydon presents a methodology to construct the first integrals of OΔEs directly. For this method, the symmetries of the OΔE need not be known. Here, we will only consider second-order OΔE's.

We construct first integrals using (8) and an additional condition; that is,

$$\begin{gathered} \phi(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})) = \phi(n, u_n, u_{n+1}), \\ \frac{\partial \phi}{\partial u_{n+1}} \neq 0. \end{gathered} \tag{9}$$

Now let

$$\begin{gathered} P_1(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_n}(n, u_n, u_{n+1}), \\ P_2(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_{n+1}}. \end{gathered} \tag{10}$$

Next we differentiate (9) with respect to u_n; we obtain

$$P_1 = \mathcal{S}P_2 \frac{\partial \omega}{\partial u_n}. \tag{11}$$

Differentiating (9) with respect to u_{n+1} we get

$$P_2 = \mathcal{S}P_1 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S}P_2. \tag{12}$$

Thus, P_2 satisfies the second-order linear functional equation or *first integral condition*,

$$\mathcal{S}\left(\frac{\partial \omega}{\partial u_n}\right) \mathcal{S}^2 P_2 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S}P_2 - P_2 = 0. \tag{13}$$

After solving for P_2 and constructing P_1, we need to check that the *integrability condition*

$$\frac{\partial P_1}{\partial u_{n+1}} = \frac{\partial P_2}{\partial u_n} \tag{14}$$

is satisfied. Hence if (14) holds, the first integral takes the form

$$\phi = \int (P_1 du_n + P_2 du_{n+1}) + G(n). \tag{15}$$

To solve for $G(n)$, we substitute (15) into (9) and solve for the resulting first-order OΔE.

2.2. Using Symmetries to Obtain the General Solution of an OΔE. We begin this section by providing some useful definitions. We consider the theory and example provided by Hydon in [11].

Definition 4. The commutator of two symmetry generators X_N and X_M is denoted by $[X_N, X_M]$ and defined by

$$[X_N, X_M] = X_N X_M - X_M X_N = -[X_M, X_N]. \tag{16}$$

Definition 5. Given a symmetry generator for a second-order OΔE,

$$\begin{aligned} X &= Q(n, u_n, u_{n+1}) \frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})) \\ &\quad \times \frac{\partial}{\partial u_{n+1}}; \end{aligned} \tag{17}$$

there exists an invariant,

$$v_n = v(n, u_n, u_{n+1}), \tag{18}$$

satisfying

$$X v_n = 0, \qquad \frac{\partial v_n}{\partial u_{n+1}} \neq 0. \tag{19}$$

To determine the invariant, we use the *method of characteristics*. Note that the invariant satisfies

$$\left[Q \frac{\partial}{\partial u_n} + \mathcal{S}Q \frac{\partial}{\partial u_{n+1}}\right] v_n = 0. \tag{20}$$

We make the assumption that (18) can be inverted to obtain

$$u_{n+1} = \omega(n, u_n, v_n) \tag{21}$$

for some function ω. Solving (21) requires finding a canonical coordinate

$$s_n = s(n, u_n) \tag{22}$$

which satisfies $X s_n = 1$. The most obvious choice [11] of canonical coordinate is

$$s(n, u_n) = \int \frac{du_n}{Q(n, u_n, \omega(n, u_n, f(n; c_1)))} \tag{23}$$

with a general solution of the form

$$s_n = c_2 + \sum_{k=n_0}^{n-1} g(k, f(k; c_1)), \tag{24}$$

where n_0 is any integer.

3. Application

The aim of this section is to consider two examples and find their symmetries, first integrals, and general solution. We also briefly discuss what is meant by double reduction and association.

3.1. Example 1. Consider the second-order OΔE [11]:

$$\omega = u_{n+2} = \frac{n}{n+1}u_n + \frac{1}{u_{n+1}}. \tag{25}$$

3.1.1. Symmetry Generator. Suppose that we seek characteristics of the form $Q = Q(n, u_n)$. To do this, we use the symmetry condition and solve for $Q = Q(n, u_n)$. Here, the symmetry condition, given by (7), becomes

$$Q(n+2, \omega) + Q(n+1, u_{n+1})\frac{1}{u_{n+1}^2} - Q(n, u_n)\left(\frac{n}{n+1}\right) = 0. \tag{26}$$

Firstly, we differentiate (26) with respect to u_n (keeping ω fixed) and we consider u_{n+1} to be a function of n, u_n, and ω. By the implicit function theorem differentiating u_{n+1} with respect to u_n yields

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{(\partial\omega/\partial u_n)}{(\partial\omega/\partial u_{n+1})} = \frac{nu_{n+1}^2}{n+1}. \tag{27}$$

Secondly, we apply the *differential operator*, given by

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}}, \tag{28}$$

to (26) to get

$$\begin{aligned} &-\frac{2n}{(n+1)u_{n+1}}Q(n+1, u_{n+1}) + \frac{n}{n+1}Q'(n+1, u_{n+1}) \\ &\quad - \frac{n}{n+1}Q'(n, u_n) = 0. \end{aligned} \tag{29}$$

To solve (29), we differentiate it with respect to u_n keeping u_{n+1} fixed. As a result we obtain the ODE:

$$\frac{d}{du_n}\left(\frac{n}{n+1}Q'(n, u_n)\right) = 0 \tag{30}$$

whose solution is given by

$$Q(n, u_n) = \left(\frac{n+1}{n}\right)A(n)u_n + B(n). \tag{31}$$

We suppose that $B(n) = 0$ for ease of computation. Next we substitute (31) into (29) and we simplify the resulting equation to obtain

$$\left[\frac{-n(n+2)}{(n+1)^2}\right]A(n+1) = A(n). \tag{32}$$

Thus,

$$A(n) = \left(\frac{n}{n+1}\right)2c(-1)^{n-1}, \tag{33}$$

where c is a constant. Substituting (33) into (31) leads to

$$Q(n, u_n) = \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right)2c(-1)^{n-1}u_n = 2c(-1)^{n-1}u_n. \tag{34}$$

Therefore, the symmetry generator is given by

$$X = 2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n}. \tag{35}$$

3.1.2. First Integral. Suppose that $P_2 = P_2(n, u_n)$; then (13) can be rewritten to give

$$\begin{aligned} &\left(\frac{n+1}{n+2}\right)P_2(n+2, u_{n+2}) - \frac{1}{u_{n+1}^2}P_2(n+1, u_{n+1}) \\ &\quad - P_2(n, u_n) = 0. \end{aligned} \tag{36}$$

We apply the differential operator L, given by (28), to (36) to get

$$\begin{aligned} &\frac{n}{n+1}\frac{2}{u_{n+1}}P_2(n+1, u_{n+1}) - \frac{n}{n+1}P_2'(n+1, u_{n+1}) \\ &\quad - P_2'(n, u_n) = 0. \end{aligned} \tag{37}$$

Next we differentiate (37) with respect to u_n keeping u_{n+1} constant to obtain $(d/du_n)(P_2'(n, u_n)) = 0$ whose solution is given by

$$P_2(n, u_n) = B(n)u_n + c = B(n)u_n \tag{38}$$

if we take $c = 0$. We substitute (38) into (37) to obtain the difference equation

$$B(n+1) = \frac{n+1}{n}B(n). \tag{39}$$

We choose $B(1) = 1$ to get

$$B(n) = n. \tag{40}$$

The next step consists of substituting (40) into (38) to get

$$P_2(n, u_n) = nu_n. \tag{41}$$

From (11) we get

$$P_1(n, u_n, u_{n+1}) = \mathcal{S}P_2\frac{\partial\omega}{\partial u_n} = nu_{n+1} = P_1(n, u_{n+1}). \tag{42}$$

Since the integrability condition holds, we can calculate the first integral ϕ. From (41) and (42) we have

$$\phi = \int(P_1du_n + P_2du_{n+1}) + G(n) = nu_nu_{n+1} + G(n). \tag{43}$$

To find $G(n)$ we substitute (43) into (9). We obtain

$$G(n+1) - G(n) + n + 1 = 0 \tag{44}$$

whose solution is given by

$$G(n) = -\frac{n(n+1)}{2}. \quad (45)$$

Finally we substitute (45) into (43) to obtain the first integral

$$\phi = nu_n u_{n+1} - \frac{n(n+1)}{2}. \quad (46)$$

Note. The symmetry generator given by (35) acts on the first integral, ϕ, to produce the following equation:

$$\begin{aligned} X\phi &= Q(n, u_n)\frac{\partial\phi}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial\phi}{\partial u_{n+1}} \\ &= 2c(-1)^{n-1}(nu_n nu_{n+1} - nu_n nu_{n+1}) \\ &= 0. \end{aligned} \quad (47)$$

We say X and ϕ are *associated* and this property has far reaching consequences on "further" reduction of the equation.

3.1.3. Symmetry Reduction. Recall that, in Section 3.1.1, we calculated the symmetry generator, X, to be

$$X = 2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n} \quad (48)$$

given by (35). Suppose $v_n = v(n, u_n, u_{n+1})$ is an invariant of X. Then

$$Xv_n = \left(Q(n, u_n)\frac{\partial}{\partial u_n} + \mathcal{S}Q(n, u_n)\frac{\partial}{\partial u_{n+1}}\right)v_n = 0. \quad (49)$$

We can use the characteristics

$$\frac{du_n}{2c(-1)^{n-1}u_n} = \frac{du_{n+1}}{2c(-1)^n u_{n+1}} = \frac{dv_n}{0} \quad (50)$$

to solve for v_n and construct the equation. The independent and dependent variables are given by

$$\alpha = u_n u_{n+1}, \qquad \gamma = v_n, \quad (51)$$

respectively. Therefore by (51), the dependent variable, v_n, is given by

$$v_n = u_n u_{n+1}. \quad (52)$$

Applying the shift operator on v_n and solving the resulting equation we get

$$v_n = \frac{n+1}{2} + \frac{c}{n}, \quad (53)$$

where c is a constant. Then by (52) and (53) and solving for u_{n+1} we obtain

$$u_{n+1} = \frac{n+1}{2u_n} + \frac{c}{nu_n}. \quad (54)$$

Note. Equation (25) has been reduced by one order into (54). Solving (54) for c gives

$$c = nu_n u_{n+1} - \frac{n(n+1)}{2} = \phi. \quad (55)$$

The first integral ϕ, given by (46), and the reduction are the same. This is another indication of a relationship between ϕ and X. In fact, this is the association; that is, ϕ is *invariant* under X.

3.2. Example 2. Consider the following linear difference equation [11]:

$$\omega = u_{n+2} = 2u_{n+1} - u_n. \quad (56)$$

3.2.1. Symmetry. Suppose that $Q = Q(n, u_n)$; then the symmetry condition becomes

$$Q(n+2, \omega) - 2Q(n+1, u_{n+1}) + Q(n, u_n) = 0. \quad (57)$$

Similarly, we apply the operator L to (57) and we differentiate the resulting equation:

$$Q'(n, u_n) - Q'(n+1, u_{n+1}) = 0, \quad (58)$$

with respect to u_n to get $Q''(n, u_n) = 0$. Therefore,

$$Q(n, u_n) = A(n)u_n + B(n). \quad (59)$$

Next we solve for $A(n)$ by substituting (59) into (58). This gives

$$A(n+1) = A(n) = a, \quad (60)$$

where a is a constant. Substituting $A(n) = a$ into (59) yields

$$Q(n, u_n) = au_n + B(n). \quad (61)$$

The substitution of (61) into (57) yields

$$B(n+2) - 2B(n+1) + B(n) = 0. \quad (62)$$

Thus,

$$B(n) = bn + c, \quad (63)$$

where b and c are arbitrary constants. Finally we substitute (63) into (61) and obtain the characteristic

$$Q(n, u_n) = au_n + bn + c. \quad (64)$$

Therefore, the Lie symmetry generators are

$$X_1 = u_n\frac{\partial}{\partial u_n}, \qquad X_2 = n\frac{\partial}{\partial u_n}, \qquad X_3 = \frac{\partial}{\partial u_n}. \quad (65)$$

3.2.2. First Integral. Suppose that $P_2 = P_2(n, u_n)$. The *first integral condition* is given by

$$P_2(n+2, \omega) - 2P_2(n+1, u_{n+1}) + P_2(n, u_n) = 0. \quad (66)$$

The solution to (66) is given by

$$P_2(n, u_n) = ku_n + pn + q, \quad (67)$$

where k, p, and q are constants. Then by (11), we have

$$P_1(n, u_{n+1}) = \mathcal{S}P_2(n, u_n)\frac{\partial\omega}{\partial u_n} = -ku_{n+1} - pn - p - q. \quad (68)$$

Substituting (67) and (68) into (15) we obtain the first integral

$$\phi = pn\left(u_{n+1} - u_n\right) + q\left(u_{n+1} - u_n\right) - pu_n + G(n). \quad (69)$$

Then we have

$$\mathcal{S}\phi = pn\left(u_{n+1} - u_n\right) + q\left(u_{n+1} - u_n\right) - pu_n + G(n+1). \quad (70)$$

To satisfy (8), we equate (69) and (70). This gives

$$G(n+1) = G(n) = r, \quad (71)$$

where r is a constant. We thus write ϕ as

$$\phi = (pn + q)\,u_{n+1} - (pn + p + q)\,u_n + r. \quad (72)$$

Next we check if ϕ is associated with the symmetry generators given in (65).

(i) Consider that $X_1 = u_n \partial/\partial u_n$. One can readily verify that

$$X_1\phi = -u_n\left(pn + q + p\right) + u_{n+1}\left(pn + q\right). \quad (73)$$

Thus ϕ is associated with X_1; that is, $X\phi = 0$, if the following equations are satisfied:

$$pn + q + p = 0, \qquad pn + q = 0. \quad (74)$$

Solving the above equations simultaneously gives $p = q = 0$. Hence, for ϕ to be associated with X_1,

$$\phi = r. \quad (75)$$

(ii) Consider that $X_2 = n\partial/\partial u_n$. We have

$$X_2\phi = n\frac{\partial\phi}{\partial u_n} + (n+1)\frac{\partial\phi}{\partial u_{n+1}} = q. \quad (76)$$

Hence ϕ is associated with X_2 if $q = 0$, that is, if

$$\phi = pnu_{n+1} - (pn + p)\,u_n + r. \quad (77)$$

(iii) Consider that $X_3 = c\partial/\partial u_n$. Then,

$$X_3\phi = c\frac{\partial\phi}{\partial u_n} + c\frac{\partial\phi}{\partial u_{n+1}} = -cp. \quad (78)$$

Here ϕ is associated with X_3 if $p = 0$. Therefore

$$\phi = q\left(u_{n+1} - u_n\right) + r. \quad (79)$$

3.2.3. General Solution. We now find the general solution of (56). We determine the commutators of the symmetries to indicate the order of the symmetries in the reduction procedure.

(i) Since

$$\left[X_1, X_2\right] = -X_2, \quad (80)$$

(56) will be reduced using X_2 first. Suppose that $v_n = v(n, u_n, u_{n+1})$ is the invariant of X_2. Then

$$X_2 v_n = \left[n\frac{\partial v_n}{\partial u_n} + (n+1)\frac{\partial v_n}{\partial u_{n+1}}\right] = 0. \quad (81)$$

Using the method of characteristic we get

$$v_n = nu_{n+1} - (n+1)\,u_n. \quad (82)$$

Applying the shift operator on v_n yields

$$\mathcal{S}\left(v_n\right) = v_{n+1} = v_n; \quad (83)$$

that is,

$$v_{n+1} = v_n = c_1, \quad (84)$$

where c_1 is a constant. Equating (82) and (84) and solving for u_{n+1}, we have

$$u_{n+1} = \frac{c_1}{n} + \left(1 + \frac{1}{n}\right)u_n \quad (85)$$

whose solution is given by

$$u_n = nc_2 + c_1(n-1), \quad (86)$$

where c_2 is an arbitrary constant. Equation (86) is the general solution of (56).

Note that solving for c_1 in (85) yields

$$c_1 = nu_{n+1} - (n+1)\,u_n. \quad (87)$$

Therefore, ϕ (given by (77)) and the reduction are the same if $p = 1$ and $r = 0$. That is, $\phi = c_1$. If this condition holds then ϕ is invariant under X_2.

(ii) We can also find a general solution of (56) by using a different symmetry generator. Here,

$$\left[X_1, X_3\right] = -X_1, \quad (88)$$

so that (56) will be reduced using X_1 first. Again suppose that $v_n = v(n, u_n, u_{n+1})$ is invariant of X_1. Then

$$X_1 v_n = \left[u_n\frac{\partial v_n}{\partial u_n} + u_{n+1}\frac{\partial v_n}{\partial u_{n+1}}\right] = 0. \quad (89)$$

Using the method of characteristics we get

$$v_n = \frac{u_{n+1}}{u_n}. \quad (90)$$

Therefore applying the shift operator on v_n gives

$$v_{n+1} = 2 - \frac{1}{v_n} \quad (91)$$

whose solution is given by

$$v_n = \frac{1 + 2c_1 + nc_1}{1 + c_1 + nc_1}, \tag{92}$$

where c_1 is a constant. Equating (90) and (92) results in

$$u_{n+1} = \left(\frac{1 + 2c_1 + nc_1}{1 + c_1 + nc_1}\right) u_n. \tag{93}$$

Therefore the general solution of (56) is given by

$$u_n = \frac{\left(1 + c_1 + nc_1\right) c_2}{1 + c_1}, \tag{94}$$

where c_2 is a constant.

(iii) Finally we consider the commutator of X_2 and X_3. We have

$$\left[X_2, X_3\right] = 0. \tag{95}$$

Since the commutator is 0, we can first reduce the OΔE with either X_2 or X_3. However, since we have already reduced (56) with X_2, we will use X_3. As before, suppose that $v_n = v(n, u_n, u_{n+1})$ is invariant of X_3. Then

$$X_3 v_n = \left[c\frac{\partial v_n}{\partial u_n} + c\frac{\partial v_n}{\partial u_{n+1}}\right] = 0. \tag{96}$$

Applying the method of characteristics, we have

$$v_n = u_{n+1} - u_n. \tag{97}$$

Applying the shift factor, $\mathcal{S}$, on (97) and solving the resulting equation we get

$$\mathcal{S}\left(v_n\right) = v_{n+1} = v_n = c_1. \tag{98}$$

Equating (97) and (98) gives

$$u_{n+1} = u_n + c_1. \tag{99}$$

We solve (99) and find

$$u_n = nc_1 + c_2 \tag{100}$$

which is a general solution of (56). It has to be noted that (99) is the same as ϕ (given by (75)) if $q = 1$ and $r = 0$. If this is true then ϕ is invariant under X_3.

4. Conclusion

We have recalled the procedure to calculate the symmetry generators of some ordinary difference equations and proceeded to find the first integral and reduce the order of the difference equations. We have shown that, in some cases, the symmetry generator, X, and first integral, ϕ, are associated via the invariance condition $X\phi = 0$. When this condition is satisfied, we may proceed to double reduction of the equation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, vol. 81 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[2] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, Second edition, 1993.

[3] H. Stephani, *Differential Equations: Their Solutions Using Symmetries*, Cambridge University Press, Cambridge, Mass, USA, 1989.

[4] G. W. Bluman, A. F. Cheviakov, and S. C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, vol. 168 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2010.

[5] E. Noether, "Invariante variationsprobleme," *Mathematisch-Physikalische Klasse*, vol. 2, pp. 235–257, 1918.

[6] A. H. Kara and F. M. Mahomed, "Relationship between symmetries and conservation laws," *International Journal of Theoretical Physics*, vol. 39, no. 1, pp. 23–40, 2000.

[7] A. H. Kara and F. M. Mahomed, "A basis of conservation laws for partial differential equations," *Journal of Nonlinear Mathematical Physics*, vol. 9, supplement 2, pp. 60–72, 2002.

[8] W. Sarlet and F. Cantrijn, "Generalizations of Noether's theorem in classical mechanics," *SIAM Review*, vol. 23, no. 4, pp. 467–494, 1981.

[9] A. Biswas, P. Masemola, R. Morris, and A. H. Kara, "On the invariances, conservation laws, and conserved quantities of the damped-driven nonlinear Schrödinger equation," *Canadian Journal of Physics*, vol. 90, no. 2, pp. 199–206, 2012.

[10] R. Morris, A. H. Kara, A. Chowdhury, and A. Biswas, "Soliton solutions, conservation laws, and reductions of certain classes of nonlinearwave equations," *Zeitschrift fur Naturforschung Section A*, vol. 67, no. 10-11, pp. 613–620, 2012.

[11] P. E. Hydon, "Symmetries and first integrals of ordinary difference equations," *Proceedings of the Royal Society A*, vol. 456, no. 2004, pp. 2835–2855, 2000.

[12] O. G. Rasin and P. E. Hydon, "Conservation laws of discrete Korteweg-de Vries equation," *SIGMA*, vol. 1, pp. 1–6, 2005.

[13] D. Levi, L. Vinet, and P. Winternitz, "Lie group formalism for difference equations," *Journal of Physics. A. Mathematical and General*, vol. 30, no. 2, pp. 633–649, 1997.

[14] D. Levi and P. Winternitz, "Symmetrie s of discrete dynamical systems," Tech. Rep. CRM-2312, Centre de recherches mathematiques, Universite de Montreal, 1995.

[15] G. R. W. Quispel and R. Sahadevan, "Lie symmetries and the integration of difference equations," *Physics Letters A*, vol. 184, no. 1, pp. 64–70, 1993.

[16] P. Tempesta, "Integrable maps from Galois differential algebras, Borel transforms and number sequences," *Journal of Differential Equations*, vol. 255, no. 10, pp. 2981–2995, 2013.

[17] P. E. Hydon and E. L. Mansfield, "A variational complex for difference equations," *Foundations of Computational Mathematics*, vol. 4, no. 2, pp. 187–217, 2004.