

# Episodic Liquidity Crises: Cooperative and Predatory Trading

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## ABSTRACT

We describe how episodic illiquidity arises from a breakdown in cooperation between market participants. We first solve a one-period trading game in continuous-time, using an asset pricing equation that accounts for the price impact of trading. Then, in a multi-period framework, we describe an equilibrium in which traders cooperate most of the time through repeated interaction and provide ‘apparent liquidity’ to each other. Cooperation breaks down when the stakes are high, leading to predatory trading and episodic illiquidity. Equilibrium strategies involving cooperation across markets lead to less frequent episodic illiquidity, but cause contagion when cooperation breaks down.

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Why is illiquidity rare and episodic? Pastor and Stambaugh (2003) detect only 14 aggregate low-liquidity months in the time period 1962-1999. Despite being of significant magnitude, most of those episodes were short-lived and were followed by long-periods of liquidity.<sup>1</sup> The origin of this empirical observation still remains a puzzle. In this paper, we attempt to contribute to the explanation of this puzzle by developing a theoretical model in which a breakdown in cooperation between traders in the market manifests itself in predatory trading. This mechanism leads to sudden and short-lived illiquidity.

We develop a dynamic model of trading based on liquidity needs. During each period, a *liquidity event* may occur in which a trader is required to liquidate a large block of an asset in a relatively short time period. This need for liquidity is observed by a tight oligopoly, whose members may choose to predate or cooperate. Predation involves racing and fading the *distressed* trader to the market, causing an adverse price impact for the trader<sup>2</sup>. Cooperation involves refraining from predation and allows the distressed trader to transact at more favorable prices. In our model, traders cooperate most of the time through repeated interaction, providing ‘apparent liquidity’ to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity.

The following quote provides a recent example of an episodic breakdown in cooperation between cooperative periods in the European debt market (New York Times Sept. 15, 2004):

*“...The bond sale, executed Aug. 2, caused widespread concern in Europe’s markets. Citigroup sold 11 billion euros of European government debt within minutes, mainly through electronic trades, then bought some of it back at lower prices less than an hour later, rival traders say. Though the trades were not illegal, they angered other bond houses, which said the bank violated an unspoken agreement not to flood the market to*

*drive down prices.*"<sup>3</sup>

This suggests that market participants cooperate, though there is episodic predation which leads to acute changes in prices. Note that predatory behavior can involve either exploiting a distressed trader's liquidity requirements or inducing another trader to be distressed.

There exists some empirical evidence that cooperation affects price evolution and liquidity in financial markets. Cooperation and reputation have been shown to affect liquidity costs on the floor of the New York Stock Exchange (NYSE). Battalio, Ellul, and Jennings (2005) show an increase in liquidity costs in the trading days surrounding a stock's relocation on the floor of the exchange.<sup>4</sup> They find that brokers who simultaneously relocate with the stock and continue their long-term cooperation with the specialist obtain a lower cost of liquidity, which manifests in a smaller bid-ask spread. Cocco, Gomes, and Martins (2003) detect evidence in the Interbank market that banks provide liquidity to each other in times of financial stress. They find that banks establish lending relationships in this market to provide insurance against the risk of shortage or excess of funds during the reserve maintenance period.<sup>5</sup>

We model the effects of cooperation and predation on liquidity as follows. We start by establishing a *predatory stage game* in continuous time, and then model cooperation by embedding it in a repeated game framework. In the stage game, each trader faces a differential game with other strategic traders, in which trading has both a temporary and a permanent impact on the price of the asset.<sup>6</sup> That is, the price of the asset is affected by the current rate of trading (temporary price impact) and by the total cumulative quantity traded over time (permanent price impact). Because we use a pricing equation that accounts for the effect of trading pressure on price (in contrast to the model of Brunnermeier and Pedersen (2005)), the strategic traders, as a group, suffer a surplus loss when predatory trading is present. This surplus loss motivates the traders to cooperate and

provide liquidity to each other in our repeated game.<sup>78</sup>

In the equilibrium of our stage game, traders ‘race’ to market, selling quickly in the beginning of the period, at an exponentially decreasing rate. Also in equilibrium, predators initially race the distressed traders to market, but eventually ‘fade’ them and buy back. This racing and fading behavior is well-known in the trading industry and has been previously modeled by Foster and Viswanathan (1996).

We model cooperation by embedding the predatory stage game in a dynamic game. We first consider an infinitely-repeated game in which the magnitude of the liquidity event is fixed. In this framework, there exists an extremal equilibrium which is Pareto superior for the traders. We extend the model to episodic illiquidity by allowing the exogenous magnitude of the liquidity event in the repeated game to be stochastic. Given such stochastic liquidity shocks, we provide predictions as to the magnitude of liquidity event required to trigger liquidity crises and describe how a breakdown in cooperation leads to price volatility. Finally, we allow for multimarket contact in the stochastic version of our dynamic game. This increases cooperation across markets, but leads to contagion of predation and liquidity crises across all markets.<sup>9</sup>

We note a few empirical implications of our model. We show that the need for liquidity over time needs to be sufficiently symmetric for the traders to cooperate. Asymmetric distress probabilities lead to the abandonment of cooperation in equilibrium. We also show that traders are more likely to cooperate in markets where the permanent price impact of trading is high and the temporary price impact of trading is low. These are also the markets in which the predatory equilibrium strategies are the most aggressive. We would expect liquidity in these markets to be smooth most of the time and the episodes of illiquidity to be the most marked. If the degree of asymmetric information associated with an asset is a good predictor of the permanent price impact of trading,

then securities that have a concentrated ownership, that have a high insider ownership, or that are from high-growth, research intensive firms should exhibit this type of pattern and have the most marked spikes in illiquidity. In contrast, securities with diffuse ownership of shares from mature (value) firms should exhibit spikes in illiquidity with lower magnitude. If a higher average daily trading volume and higher number of outstanding shares are good predictors of a lower temporary impact of trading, then securities with large number of outstanding shares that are traded in large volumes should also have more steady apparent liquidity, but more marked illiquidity episodes. To our knowledge, even though this empirical typology seems natural, securities in the market have not been grouped into these categories before and studied for their permanent and transitory illiquidity, which would be required to test the implications of our model.<sup>10</sup>

The paper is organized as follows. Section I introduces the price equation and sets up the stage game. We derive closed-form solutions for the trading dynamics and quantify the surplus loss due to competitive trading. Section II uses the stage game with one predator and one distressed trader as the basis for a multi-period game. We consider an infinitely-repeated game with the magnitude of the liquidity shocks fixed in Section II A. Section II A also provides a model for the relationship between insiders and outsiders in these markets. Section II B models episodic illiquidity by having the magnitude of the liquidity shocks be stochastic. Contagion of illiquidity across markets is also addressed in this section. Section III concludes. Appendix A contains proofs. While the stage-game solution in Section I is based on the equilibrium over *open-loop* strategies, in Appendix B we consider the equilibrium over *closed-loop* strategies and argue that the results are not qualitatively different.

# I. Trading and Predation

## A. Asset price model

The economy consists of two types of participants. The first type are the *strategic traders*, which we index with  $i = 1, 2, \dots, n$ . These traders are risk-neutral and maximize trading profits. They form a tight oligopoly over order flow in the financial market. Large traders are usually present in markets as proprietary trading desks, trading both on their own accounts as well as for others. The strategic traders observe the order flow and have inside information regarding transient liquidity needs in the market. They attempt to generate profits through their ability to forecast price moves, and to affect asset prices.

The second type of players are the *long-term investors* who form the competitive fringe. The long-term investors usually trade in the interest of mutual funds or private clients and exhibit less aggressive trading strategies. Long-term investors are more likely to follow a ‘buy and hold strategy’, limit the number of transactions that they undertake, and avoid taking over-leveraged positions. They trade according to fundamentals. The primary difference between the two types of traders is that the long-term investors are not aware of transient liquidity needs in the economy.

There exists a risk-free asset and a risky asset, traded in continuous-time. The aggregate supply  $S > 0$  of the risky asset at any time  $t$  is divided between the strategic investors’ holdings  $X_t$  and the long-term investors’ holdings  $Z_t$  such that  $S = X_t + Z_t$ . The return on the risky asset is stochastic. The yield on the risk-free asset is zero.

The asset is traded at the price

$$P_t = U_t + \gamma X_t + \lambda Y_t, \tag{1}$$

where

$$dX_t = Y_t dt, \tag{2}$$

and  $U_t$  is the stochastic process  $dU_t = \sigma(t, U_t)dB_t$ , with  $B_t$  some one-dimensional Brownian motion on  $(\Omega, \mathcal{F}, P)$ .

A similar pricing relationship was previously derived by Vayanos (1998), and also by Gennotte and Kyle (1991) who show that it arises from the equilibrium strategies between a market maker and an informed trader when the position of the noise traders follows a smoothed Brownian motion. Likewise, Pritsker (2004) obtains a similar relationship for the price impact of large trades when institutional investors transact in the market. Huberman and Stanzl (2004b) consider a similar relationship in discrete time.<sup>1112</sup>

The pricing equation is composed of three parts.  $U_t$  represents the expected value of future dividends and is modeled as a martingale stochastic diffusion process. The diffusion does not include a drift term, which is justified by the short-term nature of the events modeled.<sup>13</sup>

The second and third terms partition the price impact of trading into permanent and temporary components. Such a decomposition is justified on theoretical grounds, as noted above, and seems reasonable in light of the empirical work on block trades by Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1990) and Cheng and Madhavan (1997) that find large permanent and temporary effects for block trades on the New York Stock Exchange. Cheng and Madhavan (1997) estimate the temporary and permanent price impacts of block trades exceeding 10,000 shares in the ‘downstairs’ and ‘upstairs’ markets at the NYSE. The permanent price impact, measured from the price at the trade immediately prior to the block trade to the price of the 20th trade after the block trade, is  $-6.66$  in the downstairs market and  $-7.59$  upstairs, both in basis points of

the logarithmic return averaged over blocks of all sizes. The temporary price impact, measured from the price at the 20th trade after the block trade to the price at which the block was traded, is  $-5.28$  downstairs and  $-5.81$  upstairs, again in basis points of the logarithmic return averaged over blocks of all sizes. Cheng and Madhavan show that both the permanent and temporary price impact increase in magnitude with the size of the traded block. For instance, the permanent price impact in the downstairs market is  $-5.63$  for blocks of size 10 to 20 (thousands),  $-8.77$  for blocks of size 20 to 50, and  $-15.09$  for blocks over 50. Importantly, Sadka (2005) finds that the correlation between the temporary component and the permanent component is around 0.28, which suggests that there is significant variation in the ratio of the temporary price impact to the permanent price impact.<sup>14</sup>

In the second term,  $X_t = \sum_{i=1}^n X_t^i$  is the inventory variable, which measures the aggregate amount of the asset that the strategic traders hold at time  $t$ . As  $X_t$  increases, the supply available to the long-term investors decreases and the price at which they can access the asset increases. The model parameter  $\gamma$  measures the permanent liquidity effects of trading. That is, it measures the change in price of the asset which is independent of the rate at which the asset is traded. Note that the level of asymmetric information in an asset is likely to be a major determinant of  $\gamma$ , as demand for the asset will then play a more important role in price formation. For instance, we expect a AAA-rated corporate bond to have lower asymmetric information associated with it than a B-rated bond. In our model, the AAA-rated bond should then have a lower  $\gamma$  than the B-rated bond. Likewise, an asset with concentrated ownership should have a higher level of asymmetric information (and therefore a higher  $\gamma$ ) than an asset with a more dispersed ownership structure. For an asset with more asymmetric information, the market will more strongly adjust the asset price based on the net change in the supply of the asset.



The third term measures the instantaneous, reversible price pressure that occurs as a result of trading.  $Y_t = \sum_{i=1}^n Y_t^i$  is the aggregate rate of trading of the asset by the strategic traders. The faster the traders sell, the lower a price they will realize. This leads to surplus loss effects, which are discussed in the following subsections. The price-impact parameter  $\lambda$  measures the temporary, reversible asset price change that occurs during trading. Trading volume and shares outstanding are likely to play a role in the level of  $\lambda$ .

### B. Stage Game: Trading Dynamics

Our stage game is one of complete information. Many assets that are prone to illiquidity are traded in non-anonymous markets in which a few large dealers dominate order flow.<sup>15</sup> Further, roughly half of the trading volume at the New York Stock Exchange is traded in blocks over 10,000 shares (Seppi 1990) and much of that occurs in the ‘upstairs’ market, which is non-anonymous. As a result, the liquidity needs of large traders are usually observed quickly by others.

For an example where this game structure is a natural choice, consider a thinly-traded corporate bond issue that is traded by a small number of broker-dealers. Trading occurs either by direct negotiation over the phone, or by ‘sunshine trading’ in which a mini-auction is held. The players are well-known to each other because each deals repeatedly with the others. Their trading habits and strategies are common knowledge.<sup>16</sup> When one trader needs to trade a large block of shares of an asset, this need is observed by others in the market and the optimal trading strategies solve a game of complete information.

In the stage game, strategic traders are either *distressed* or are *predators*. A liquidity event occurs at time  $t = 0$ , whereby the distressed traders are required to buy or sell a large block of the asset  $\Delta x$  in a short time horizon  $T$  (say, by the end of the trading day). Forced liquidation usually

arises because of the need to offset another cash-constrained position such as an over-leveraged position, or it occurs as a result of a risk management maneuver. The predators are informed of the trading requirement of the distressed traders and compete strategically in the market to exploit the price impact of the distressed trader's selling. For clarity of exposition, we assume that the opportunistic traders must return to their original positions in the asset by the end of the trading period, and that the distressed traders are informed of this requirement.<sup>17</sup> Further, we assume that, except for their trading targets, all strategic traders are identical. That is, the only difference between the two types of strategic traders is that the trading target is  $\Delta x$  for the distressed traders, and zero for the predatory traders.

At the start of the stage game, every trader chooses a trading schedule  $Y_t^i$  over the period  $[0, T]$  to maximize their own expected value, assuming the other traders will do likewise. Subject to their respective initial and terminal holding constraints, they solve the following dynamic program

$$\begin{aligned}
& \underset{Y_t^i \in \mathcal{Y}}{\text{maximize}} && \mathbf{E} \left[ \int_0^T -P_t Y_t^i dt \right] \\
& \text{subject to} && X_0^i = x_{0i} \\
& && X_T^i = x_{Ti},
\end{aligned} \tag{3}$$

where the maximization is over  $Y_t^i$ ,  $t \in [0, T]$ , with the strategy space  $\mathcal{Y}$  restricted so that, for each  $i$ ,  $Y_t^i$  is continuous and admissible, *i.e.*, satisfies the integrability condition

$$\mathbf{E} \left[ \int_0^T -P_t Y_t^i dt \right] < \infty. \tag{4}$$

Note that  $P_t$  depends on  $Y_t^i$  and on the trading strategies of other traders as in Equation 1. The restriction  $Y_t^i \in L^2$ , (*i.e.*,  $\int_0^T (Y_t^i)^2 < \infty$ ), among others, ensures admissibility. Note that the

position  $X_t^i$  is a differentiable function with continuous derivatives.<sup>18,19</sup>

We restrict our analysis in the paper to the open-loop Nash equilibrium solutions to this differential game.<sup>20</sup> In equilibrium, each trader chooses *ex ante* a time-dependent trading strategy that is the best response to the other traders' expected actions. As noted in the proof in Appendix A, the restriction to smooth functionals that are admissible and the concavity of the objective function in  $X_t^i$  and  $Y_t^i$  ensures that the problem is well-posed.<sup>21</sup>

The equilibrium solution of the open-loop problem is 'weakly time consistent'. That is, the solution to the subproblem over the interval  $[t_1, T]$  (with initial conditions as given by the solution of the  $[0, T]$ -problem at time  $t_1$ ) is the truncation of the  $[0, T]$ -solution over that sub-interval.<sup>22</sup> Along the equilibrium path, a trader does follow a continuation strategy that maximizes his utility given his equilibrium play until that point.<sup>23</sup>

The following result outlines the unique Nash equilibrium solution for the traders. This formulation will serve as a basis for deriving the equilibrium strategies when several distressed traders are present without opportunism and when there are both opportunistic and distressed traders present in the economy. It will also allow for analysis of trader surplus, which will motivate cooperation between traders in the repeated game.

**RESULT 1** (*General Solution*) *Consider  $N$  traders that choose a time-dependent trading rate  $Y_t^i$  to solve the optimization problem in Equation 3, subject to the asset price given by Equation 1. Then, the unique open-loop Nash equilibrium (among the class of continuous admissible functionals) in this game is for trader  $i$  to trade according to the function*

$$Y_t^i = a e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} t} + b_i e^{\frac{\gamma}{\lambda} t}, \quad (5)$$

with  $a \in \mathbf{R}$ , and  $b_i \in \mathbf{R}$ ,  $i = 1, \dots, n$ , such that  $\sum_{i=1}^n b_i = 0$ . The coefficients  $a$  and  $b_i$  are uniquely determined from the boundary conditions to be

$$\begin{aligned} a &= \frac{n-1}{n+1} \frac{\gamma}{\lambda} \left(1 - e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} T}\right)^{-1} \frac{\sum_{i=1}^n \Delta x_i}{n} \\ b_i &= \frac{\gamma}{\lambda} \left(e^{\frac{\gamma}{\lambda} T} - 1\right)^{-1} \left(\Delta x_i - \frac{\sum_{j=1}^n \Delta x_j}{n}\right), \end{aligned} \tag{6}$$

where  $\Delta x_i = x_{Ti} - x_{0i}$ .

**Proof.** See Appendix A. ■

The equilibrium trading strategy in Equation (5) is composed of two parts. For small  $t$ , the first component dominates the trading strategy and for larger  $t$  within the interval  $T$ , the second part dominates. The first term,  $a e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} t}$ , describes how fast traders ‘race’ to the market during a sell-off or a buying frenzy. The second term,  $b_i e^{\frac{\gamma}{\lambda} t}$ , describes the magnitude of ‘fading’ by each trader. Fading refers to traders reversing the direction in which they are trading and, for the cases we consider in our stage game, only occurs when opportunistic traders are present. For example, consider the case where a trader needs to sell a block of shares of an asset and there are predatory traders present in the market. The first component describes the rate at which they all trade when they initially race each other to the market, and the second component describes the trading dynamic when the predators buy back.

Note that the constant  $a$  in Equation (6) is a function of the average trading target over all traders. All traders race to the market in similar fashion, based on the common knowledge of their overall trading target. Towards the end of the period, traders ‘fade’ based on their particular trading targets. The constants  $b_i$  are a function of how each trader’s trading target is different from

the average. A ‘distressed trader’, in the sense that he has a higher-than-average trading target, towards the end of the period will trade in the same direction as the racing. A ‘predatory trader’, in the sense that she has a smaller-than-average trading target, will ‘fade’ in the opposite direction, that is reverse the direction in which she is trading.

To develop more intuition regarding Result 1, we evaluate Equation (5) for special cases, which we will use when we consider the games of repeated interaction in Section II.

**Case 1 (Symmetric Distressed Traders).** First, consider the optimal trading policy when a trader has monopoly power and buys or sells in the absence of other strategic traders. For  $n = 1$ , the optimal trading policy (5) for a single trader is to trade at a constant rate,  $Y_t = a = \frac{\Delta x}{T}$ , where  $\Delta x$  is the block of shares that the trader needs to buy or sell. This result is consistent with Bertsimas and Lo (1998) and Huberman and Stanzl (2004b).

Now, consider  $n$  symmetric traders, each needing to sell an identical amount of shares  $\frac{\Delta x}{n}$ . From Equation (5), the unique equilibrium smooth trading strategy is

$$Y_t^i = a e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} t}, \quad i = 1, \dots, n, \quad (7)$$

where  $a$  is as in (6) with  $\sum_{i=1}^n \Delta x_i = \Delta x$ . Figure 1 plots these trading policies with  $\Delta x = 1$ ,  $T = 1$ ,  $\gamma = 10$  and  $\lambda = 1$ . If the permanent price impact of trading is lower (small  $\gamma$ ), trading will occur comparatively later. If the temporary price impact of trading is smaller (small  $\lambda$ ), trading will occur comparatively earlier. For  $n = 1$  we obtain the constant selling rate (*solid horizontal line*). If there are more traders, everybody will trade earlier. Note that the rate of trading goes to  $e^{-\frac{\gamma}{\lambda} t}$  as  $n \rightarrow \infty$  (*dotted line*). That is, there is an upper bound on how fast traders will sell their position, regardless of how many traders are in the race. Note that the shape of the curve depends on  $\gamma$

Insert  
Figure 1  
here

and on  $\lambda$  only through the ratio  $\frac{\gamma}{\lambda}$ . However, the scale of the  $\gamma$  and  $\lambda$  parameters does otherwise matter in relation to  $U_t$ . For instance, the expected long-term fractional loss in value of the asset is proportional to  $\gamma/U_t$ .

Insert  
Figure 2  
here

Figure 2 plots the corresponding price process (for a constant  $U_t = 50$ ). For a single trader, the price changes linearly over the trading period. By trading at a constant rate, the single trader is able to “walk down the demand curve” and not incur a loss in surplus due to excessive short-term price pressure from the trading intensity (*straight solid line*). For a large number of traders, the price function over  $t \in [0, T]$  quickly approaches a constant value. The information regarding the trader’s target position in the asset quickly becomes incorporated in the asset price (*dotted line*). There is a surplus loss to the strategic traders as trading pressure depresses prices quickly. We quantify these surplus changes in the next subsection.

**Case 2 (Distressed Trader and Predatory Trader).** We now set up and analyze the two-player predatory stage game, which will form the basis for the games in Section II. Consider that there exists one distressed and one opportunistic trader. Each trader chooses a trading schedule ( $Y_t^d$  and  $Y_t^p$ ) over the period  $[0, T]$  to maximize his own expected value, assuming the other trader will do likewise. From Result 1, the unique equilibrium smooth trading policies are

$$\begin{aligned} Y_t^d &= a e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} + b e^{\frac{\gamma}{\lambda}t} \\ Y_t^p &= a e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} - b e^{\frac{\gamma}{\lambda}t}, \end{aligned} \tag{8}$$

where

$$a = \frac{\gamma}{6\lambda} \left(1 - e^{-\frac{1}{3}\frac{\gamma}{\lambda}T}\right)^{-1} \Delta x, \quad b = \frac{\gamma}{2\lambda} \left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1} \Delta x. \tag{9}$$

Insert  
Figure 3  
here

The shape of the trading strategy depends on the parameters of the market. Figure 3 gives an

example, with  $\Delta x = 1$ ,  $T = 1$ ,  $\gamma = 10$  and  $\lambda = 1$ . The strategy involves the opportunistic trader initially racing the distressed trader to the market in an exponential fashion, and then fading the distressed trader towards the end of the period, also exponentially. If the first trader needs to sell, that is  $\Delta x < 0$ , the predatory trader sells short at beginning and buys back in later periods to cover his position. If the distressed trader is required to buy a block of the asset, the opposite strategy by the predator ensues. In general, we see that the presence of the predator will lead the distressed trader to increase his trading volume at the beginning and at the end of the trading period.

### *C. Surplus Effects*

Based on the trading dynamics in Section I B, we quantify the surplus changes that occur when traders race to market and when predatory trading occurs. The surplus values that we derive will be used in the following sections.

First, consider the expected value for a single trader with monopoly power. Given the price (1) and the optimal trading rate  $Y_t = \frac{\Delta x}{T}$ , the expected value for the single trader is given by

$$V_1 = -U_0 \Delta x - \left( \frac{\gamma}{2} + \frac{\lambda}{T} \right) \Delta x^2. \quad (10)$$

This is the trader's first best when there are no other competing traders informed of the trader's trading requirement  $\Delta x$ . The costs due to short-term trading pressure are minimized. When other players trade strategically at the same time, the value that the trader can derive is strictly lower than  $V_1$ . We will also see that when multiple traders compete in a sell-off or if there is predatory trading, the aggregate surplus available to all traders is decreased.

Define  $V_n$  as the total expected value for the strategic traders when  $n$  traders play this game

and define  $\Delta V_n$  as the change in total surplus that occurs compared to the expected value when all participants trade at a constant rate ( $V_1$ ). The following result provides expressions for  $V_n$  and  $\Delta V_n$ , and shows that the loss in surplus is increasing with the number of traders. It will lay some groundwork for the surplus results for the case where there is a distressed and a predatory trader (that is, for  $n = 2$ ), and is also of interest on its own for the monotonicities.

**RESULT 2** (*Expected Total Surplus and Loss for Multiple Traders*) *The total expected value for  $n$  traders with a combined trading target  $\Delta x$  is*

$$V_n = -U_0 \Delta x - \frac{\gamma}{2} \left( 1 + \frac{n-1}{n+1} \cdot \frac{e^{\frac{n-1}{n+1} \frac{\gamma T}{\lambda}} + 1}{e^{\frac{n-1}{n+1} \frac{\gamma T}{\lambda}} - 1} \right) \Delta x^2. \quad (11)$$

*The expected loss in total surplus from competition is*

$$\Delta V_n = V_1 - V_n = \gamma \left( \frac{1}{2} \cdot \frac{n-1}{n+1} \cdot \frac{e^{\frac{n-1}{n+1} \frac{\gamma T}{\lambda}} + 1}{e^{\frac{n-1}{n+1} \frac{\gamma T}{\lambda}} - 1} - \frac{1}{\lambda T} \right) \Delta x^2. \quad (12)$$

$\Delta V_n$  is positive, monotonic increasing in  $\gamma$ ,  $T$  and  $n$ , and monotonic decreasing in  $\lambda$ .

**Proof.** See Appendix A. ■

Now we apply Result 2 to the two-trader case and derive a surplus result that we will use in Section II. We define  $V_2$  as the total expected value for the strategic traders when two traders play this game, and we define  $V_d$  and  $V_p$  as the expected values to the distressed trader and to the opportunistic trader (as defined in Section I B). Likewise, we define  $\Delta V_2$  as the change in surplus that occurs compared to the expected value  $V_1$  that is obtained when the participants trade at a constant rate.



**RESULT 3** (*Expected Total Surplus and Loss for Two Traders*) The total expected value for the distressed trader and the predatory trader is

$$V_2 = V_d + V_p = -U_0 \Delta x - \frac{\gamma}{3} \cdot \frac{2e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1} \Delta x^2. \quad (13)$$

The expected value is divided as

$$\begin{aligned} V_d &= -U_0 \Delta x - \frac{\gamma}{6} \frac{5e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1}{e^{\frac{2}{3}\frac{\gamma}{\lambda}T} - 1} \Delta x^2, \\ V_p &= \frac{\gamma}{6} \cdot \frac{e^{\frac{2}{3}\frac{\gamma}{\lambda}T} - 1}{e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1} \Delta x^2. \end{aligned} \quad (14)$$

The expected loss due to predation for the distressed trader is

$$\Delta V_d = V_1 - V_d = \gamma \left( \frac{1}{6} \cdot \frac{2e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{2}{3}\frac{\gamma}{\lambda}T} + e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 2}{e^{\frac{2}{3}\frac{\gamma}{\lambda}T} - 1} - \frac{1}{\frac{\gamma}{\lambda}T} \right) \Delta x^2, \quad (15)$$

and the expected loss from predation in total surplus for the strategic traders is

$$\Delta V_2 = V_1 - V_2 = \gamma \left( \frac{1}{6} \cdot \frac{e^{\frac{1}{3}\frac{\gamma}{\lambda}T} + 1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T} - 1} - \frac{1}{\frac{\gamma}{\lambda}T} \right) \Delta x^2. \quad (16)$$

$V_p$  is monotonically increasing in  $\gamma$  and in  $T$ , and monotonically decreasing in  $\lambda$ .

The ratio of gains to the predator to the losses to the distressed trader,  $\frac{V_p}{\Delta V_d}$ , is monotonically decreasing in  $\gamma$  and in  $T$ , monotonically increasing in  $\lambda$ , and is bounded by  $\frac{4}{5} > \frac{V_p}{\Delta V_d} > \frac{1}{2}$ . Note that it follows from Result 2 that  $\Delta V_2$  is positive, monotonic increasing in  $\gamma$  and  $T$ , and monotonic decreasing in  $\lambda$ . From the monotonicity of  $V_p$  it also follows that  $\Delta V_d = \Delta V_2 + V_p$  is monotonic.

**Proof.** See Appendix A. ■

From the solutions for the rate of trading, we can see that a larger  $\frac{\gamma}{\lambda}$  ratio (more permanent price impact and less temporary price impact) creates conditions for more aggressive predation, in the sense that trading will be relatively more concentrated at the beginning and at the end of the period. Racing is faster, and fading occurs closer to the end of the trading period.

Since  $\frac{V_p}{\Delta V_d}$  is bounded in the interval  $[\frac{1}{2}, \frac{4}{5}]$ , the losses to the distressed trader are strictly higher than the gains by the predator<sup>24</sup>. Even though the monotonicity of  $V_p$  implies that market conditions that lead to more aggressive predation (larger  $\gamma$  or lower  $\lambda$  or both) will lead to more gains from predation, since  $\frac{V_p}{\Delta V_d}$  decreases in  $\gamma$  and increases in  $\lambda$ , the losses to the distressed trader grow faster than the gains to the predator. In this one-shot stage game, this represents a significant surplus loss to the traders as a whole.<sup>25</sup> In a dynamic setting, which we model in the next section, if both traders have a possible liquidity need in each period, there exists a potential for Pareto improvement if the traders can cooperate. As we will see, the ratio  $\frac{\gamma}{\lambda}$  is the key determinant of whether cooperation is possible.

Finally, for some insight into the magnitude of the available Pareto improvement, consider the case when  $\lambda \ll \gamma T$ . These are the conditions under which predation is most aggressive, that is when racing and fading are fastest as a consequence of the low transaction costs. Taking the limit  $\lambda \rightarrow 0$  (which, by change of units, is immediately seen to be equivalent to  $T \rightarrow \infty$ ), L'Hôpital's rule yields for the overall surplus loss, the distressed trader's losses from predation, and the predator's gains:

$$\begin{aligned}\Delta V_2 &\rightarrow \frac{\gamma}{6} \Delta x^2, \\ \Delta V_d &\rightarrow \frac{\gamma}{3} \Delta x^2, \\ V_p &\rightarrow \frac{\gamma}{6} \Delta x^2.\end{aligned}\tag{17}$$

In the limit, under market conditions that favor the most aggressive predation, the predator gains ( $V_p$ ) half of what the distressed trader loses ( $\Delta V_d$ ). This is the lower bound for  $\frac{V_p}{\Delta V_d}$ .

## II. Cooperation and Liquidity

To illustrate the incentives for cooperation in financial markets and the implications for market liquidity, we consider a dynamic game in which there are two strategic traders, as well as a large number of long-term investors. Each player faces a common discount factor  $\delta$ , and the common asset price determinants  $U_t$ ,  $\gamma$ , and  $\lambda$ . At the beginning of each stage, nature moves first and assigns a type to each of the traders. With probability  $p_i$ ,  $i = 1, 2$ , each trader must liquidate a large position of size  $\Delta x$ . With probability  $1 - p_i$ , they may act as a predator if their competitor needs liquidity. In each round, the traders have perfect information about each other's type.<sup>26</sup>

We assume that the distress probabilities  $p_1$  and  $p_2$  are mutually independent. In each time period one of the following four events occurs: neither of the two players is distressed, with probability  $(1 - p_1)(1 - p_2)$ ; the second player is distressed but the first is not, with probability  $(1 - p_1)p_2$ ; the first player is distressed but the second is not, with probability  $p_1(1 - p_2)$ ; both players are distressed, with probability  $p_1p_2$ . The four probabilities add to one. Cooperation is possible when either there exists one predator and one distressed trader (with probability  $p_1 + p_2 - 2p_1p_2$ ), or when both players are distressed (with probability  $p_1p_2$ ). If only one of the players is distressed and needs to liquidate a position, cooperation involves the other refraining from engaging in predatory trading. If both traders are distressed, cooperation involves both traders selling at a constant rate and refraining from racing each other to the market for their own gain.

Cooperation provides the players with the ability to quickly sell large blocks of shares for the price that would be obtained by selling them progressively over time. That is, while cooperation is

sustained, the distressed trader is allowed to ‘walk down the demand curve’, rather than having the information regarding the trading target quickly incorporated into the asset price, ahead of most of his trading. In this sense, that large blocks of shares can be moved for a better price, the market will appear more liquid. It will also avoid the volatility and potential instability and disruption from the large trading volume peaks associated with the racing and fading.

In what follows, we consider two versions of the model. As a building block, we develop a benchmark model in which the magnitude of the shock  $\Delta x$  is constant. The two traders participate in an infinitely-repeated game and the punishment strategy that they use is a grim-trigger strategy (Friedman 1971). We determine which markets are prone to breakdowns in cooperation and analyze the effect of distress probabilities on the ability to support cooperation. We show that the ability to cooperate in markets has market structure implications.

Subsequently, we build upon the benchmark model and allow  $\Delta x$  to be a stochastic random variable. The traders choose optimal strategies by taking expectations over  $\Delta x$  and the distress probabilities that they face. In the equilibrium of this dynamic game, the traders implicitly agree not to punish each other for predated when the stakes are high. That is, they use a punishment strategy along the lines of Rotemberg and Saloner (1986). Episodically, cooperation breaks down, leading to episodic illiquidity, which is short-lived. Finally, we consider the effect of multi-market contact and the contagion of illiquidity in the stochastic model.

#### *A. Fixed Liquidity Needs and Cooperation*

Consider that the two strategic traders play an infinitely-repeated game in which the magnitude of the liquidity shock  $\Delta x$  is constant. The punishment strategy considered is a trigger strategy in the spirit of Friedman (1971) and Dutta and Madhavan (1997). That is, for cooperation to occur

in equilibrium for a given discount factor  $\delta$ , the expected value of a perpetuity of cooperation must exceed that of a one-time deviation plus a perpetuity of non-cooperation. By the Folk theorem, a convex set of subgame-perfect Nash equilibria may exist in which intermediate levels of cooperation occur. For clarity of exposition, we focus on the extremal equilibrium that allows for the maximal cooperation.

The goal here to predict when traders will abandon a cooperative effort, thereby leading to a reduction of the apparent liquidity in the market. We perform a ‘comparative statics’ analysis by comparing the discount factor  $\delta$  required for cooperation to be possible under different scenarios of the other problem parameters. Given any particular punishment scheme, such as the more complicated penal codes in Abreu (1988), such a critical  $\delta$  can be derived. For this analysis, we do not allow the players to change punishment schemes to achieve cooperation. We focus on trigger strategies because they lead to the same economic results, while maintaining clarity of the model.

Table I  
about  
here

The following result describes the extremal equilibrium of our repeated game with fixed liquidity needs ( $\Delta x$ ) in each period, using a trigger strategy. Refer to Table I for the expected values derived in Section I.

**RESULT 4** (*Repeated Game with Two Symmetric Traders*) Define the expected values as in Section I. When a trigger strategy (punishment strategy) is used, the discount factor required to support collusion is

$$\delta \geq \delta_{\min} = \max \{ \delta_1, \delta_2 \}, \tag{18}$$

where the  $\delta_i$  are the lowest discount factors for which each player does not have an incentive to predate given the opportunity to do so (and given that the same is true for the other player), which

are

$$\begin{aligned}\delta_1 &= \frac{V_p}{p_1(1-p_2)\Delta V_d + 2p_1p_2\Delta V_2 + (1-(1-p_1)p_2)V_p}, \\ \delta_2 &= \frac{V_p}{(1-p_1)p_2\Delta V_d + 2p_1p_2\Delta V_2 + (1-p_1(1-p_2))V_p}.\end{aligned}\tag{19}$$

The  $\delta_1$  and  $\delta_2$  bounds on the discount factor (and therefore  $\delta_{\min}$ ) are monotonically increasing in  $\lambda$  and in  $T$ , and monotonically decreasing in  $\gamma$ .

**Proof.** See Appendix A. ■

Result 4 predicts that cooperation is more likely for assets with a higher permanent price impact of trading, since the minimum discount factor  $\delta$  for which cooperation can be supported is monotonically decreasing in  $\gamma$ . In Section I C, we showed that the surplus loss to predatory trading is monotonically increasing in  $\gamma$ . Since with higher  $\gamma$  there exists a higher Pareto improvement available, it becomes more desirable for the traders to maintain cooperation. In contrast, in markets with a high temporary impact of trading  $\lambda$ , we would expect a lower level of cooperation. Since the Pareto improvement available monotonically decreases in  $\lambda$ , the level of cooperation should be lower for these markets.

Given Result 4, we can focus on the ratio  $\frac{\gamma}{\lambda}$  to predict whether there will exist more or less aggressive cooperation. For large  $\frac{\gamma}{\lambda}$ , we expect cooperation to dominate predation. For small  $\frac{\gamma}{\lambda}$ , we expect predation to be more prevalent. The ratio  $\frac{\gamma}{\lambda}$  impacts the liquidity available in asset markets. In Section II B, we will show that markets with a large  $\frac{\gamma}{\lambda}$  have relatively smooth liquidity most of the time, but have the most marked spikes of illiquidity. These results arise because the monotonicities in Result 4 hold in the stochastic model.

From Result 4, we analyze how  $\delta$  relates to the probabilities of distress. That is, how the

probabilities of each trader having future liquidity needs affects the traders' ability to cooperate. We will see that it is easier to support cooperation when the probabilities of distress are higher and more symmetric.

Insert  
Figure 4  
here

Consider the example in Figure 4. The figure plots the minimum  $\delta$  for which cooperation is feasible, as a function of  $\gamma$ . The other parameters are  $\Delta x = 1$ ,  $U_0 = 10$ ,  $T = 10$ , and  $\lambda = 1$ . The base case is the solid-line, in which  $p_1 = 0.5$  and  $p_2 = 0.5$ . For all values of  $\gamma$ , the  $\delta$  required for cooperation is less than 1, which means that if the strategic traders are sufficiently patient they can cooperate in any market. Now consider the case where  $p_1 = 0.1$  and  $p_2 = 0.1$ . It is still possible to cooperate in any market, but the required  $\delta$  is higher. As the probability of distress decreases, a player which is not distressed and has the opportunity to predate will likely have to wait longer until the next event where he might in turn benefit from cooperation from the other player. For the value of future benefits from cooperation to be sufficient for the player not to have an incentive to predate, the future needs to be less discounted (*i.e.*, higher  $\delta$ ). Finally, consider the case where  $p_1 = 0.5$  and  $p_2 = 0.3$ . In this case, it is not possible for the traders to cooperate in markets with low  $\gamma$ . Since we must have  $\delta \in [0, 1]$ , we can find a bound on  $\gamma/\lambda$ , below which cooperation should never be observed. Symmetry in distress probabilities between the traders is an important factor for cooperation. In markets with a low  $\gamma/\lambda$ , even a small degree of asymmetry will be enough to cause the traders to abandon a cooperative relationship.

These conclusions are supported analytically as follows. Without loss of generality, assume  $\delta_1 = \delta_{\min}$  (that is,  $p_1 < p_2$ ). Equation (19) can be rewritten as

$$\delta_1 = \frac{V_p}{p_2 [p_1 \Delta V_2 - V_p] + p_1 \Delta V_d + V_p}. \quad (20)$$

From Equation 20 we can see that  $\delta_1$  is monotonically increasing in  $p_2$  (since we can establish that  $V_p \geq \Delta V_2$  from  $V_p/\Delta V_d > 1/2$  in Result 3 and  $\Delta V_d = \Delta V_2 + V_p$ ). A larger probability of distress for the ‘larger’ trader makes cooperation by the ‘smaller’ trader more difficult (cooperation is possible only under a narrower range of market conditions). We can also rewrite Equation (19) as

$$\delta_1 = \frac{V_p}{p_1 [(1 - p_2)\Delta V_d + 2p_2\Delta V_2 + p_2V_p] + (1 - p_2)V_p}, \quad (21)$$

from which we see that  $\delta_1$  is monotonic decreasing in  $p_1$ . A larger probability of distress for ‘smaller’ trader makes cooperation easier (possible under a wider range of market conditions).

Using Result 4 it is possible to place bounds on how symmetric the distress probabilities must be in order to support cooperation. Consider, for example, that the traders are infinitely patient ( $\delta = 1$ ) and, without loss of generality, that  $p_1 < p_2$ . To support cooperation it must be that

$$p_1 \geq p_2 \frac{V_p}{\Delta V_d} - p_1 p_2 \frac{\Delta V_2}{\Delta V_d}. \quad (22)$$

Evaluating Equation (22) under extreme market conditions ( $\gamma/\lambda \rightarrow 0$  and  $\gamma/\lambda \rightarrow \infty$ ) allows us to derive the relative values of the distress probabilities for which cooperation is possible.<sup>27</sup> Taking the limits above, Equation (22) becomes

$$p_1 \geq \frac{4}{5}p_2 - \frac{1}{5}p_1 p_2 \quad \text{and} \quad p_1 \geq \frac{1}{2}p_2 - \frac{1}{2}p_1 p_2. \quad (23)$$

Another useful relation is from the case where distress events are infrequent ( $p_1, p_2 \ll 1$ ). The probably of both players being simultaneously distressed is then negligible. The size of the smaller



player relative to larger player is then bounded by

$$\frac{p_1}{p_2} \geq \frac{V_p}{\Delta V_d}. \quad (24)$$

In the limit cases above ( $\gamma/\lambda \rightarrow 0$  and  $\gamma/\lambda \rightarrow \infty$ ) this is

$$\frac{p_1}{p_2} \geq \frac{4}{5} \quad \text{and} \quad \frac{p_1}{p_2} \geq \frac{1}{2}. \quad (25)$$

As discussed above, the traders' distress probabilities need to be sufficiently symmetric for cooperation to be possible. Equation (25) provides some insight on the required level of symmetry.

The requirement for symmetry in distress probabilities has implications for market structure. Strategic traders often trade on behalf of external clients who use these markets. For example, proprietary trading desks trade for both their clients and on their own account. If the probability of needing to trade large blocks in a short time period is linked to the market share of external clients that a trader serves, the model predicts that active traders may have an incentive to share the market with their competition. If a duopoly exists (when a monopoly is not possible), it may be to the benefit of a large trader to allow a smaller trader to grow in size so that a Pareto superior outcome for the strategic traders can be achieved. This incentive to share the market will need to be weighed against the benefits of having more business for oneself. However, should the surplus from cooperation be lost, the value of being an 'insider' will be eroded, and with it the ability to extract rents from external clients.

Consider the following scenario in which there exists two strategic traders and a representative outsider who seeks to trade a block of an asset. The strategic traders have two alternatives when an

outsider needs to trade. They may initiate a predatory strategy, race and fade the external player to the market, and earn a profit by affecting the price of the asset. Alternatively, the outsider may become a client of the traders so that the traders may exact rents for use of their services (these rents may arise in the form of a bid-ask spread). The fact that there exists a cooperative outcome in this market between the insiders provides a means by which a relatively stable, albeit widened, bid-ask spread may exist, and we do not necessarily observe price volatility when a non-member needs liquidity. The amount of the surplus available between the traders and the client is  $\Delta V_d$ , since the outsider is indifferent between receiving  $V_d$ , and paying  $\Delta V_d$  in order to receive  $V_1$  when using the services of the strategic traders.<sup>28</sup>

Insert  
Figure 5  
here

The external client uses each trader with probabilities  $p_1$  and  $p_2$ . Equation (22) implies that the relative market shares of the traders should be reasonably symmetric to support cooperation. Consider the example in Figure 5 in which the minimum  $\delta$  necessary to support cooperation is plotted as a function of  $p_1$ . Two ranges of scenarios are illustrated:  $p_1 = p_2$  (*bold line*) and  $p_1 + p_2 = 0.5$  (*dotted line*). When the traders' distress probabilities (market shares) are symmetric, cooperation is always possible, as long as traders are sufficiently patient. However, when the market shares are asymmetric and  $p_1 < 0.18$  (36% market share) or  $p_1 > 0.32$  (64% market share), cooperation is not possible. If one trader has a larger than 64% share, he may find it to his own interest to allow his opponent to gain market share so that their ongoing Pareto superior relationship may continue. This may provide an explanation for the observation in practice of deviations in the bid-ask spread without resulting in price wars.

Insert  
Figure 6  
here

Figure 6 illustrates, for three different values of  $\delta$ , the  $(p_1, p_2)$  pairs that can support cooperation. The values of  $\delta$  plotted are 1, 0.9 and 0.8. The other parameters are  $\Delta x = 1$ ,  $U_0 = 10$ ,  $T = 10$ ,  $\gamma = 1$ , and  $\lambda = 1$ . The shaded region corresponds to the  $(p_1, p_2)$  pairs for which cooperation

is possible if both traders use a discount factor  $\delta = 0.8$ . Note that the boundaries of the sets are not straight lines due to the bilinear terms  $p_1 p_2$ , but are nearly so for small values of  $p_1$  and  $p_2$ . For  $\delta = 1$ , and for small probabilities, the set boundaries go to the origin with slope  $V_p/\Delta V_d$  and  $\Delta V_d/V_p$ . As the sum of the two probabilities becomes smaller, traders are required to be of more similar sizes for a cooperative outcome to be feasible. Note that the bold and dashed lines correspond to the cases plotted in Figure 5. Considering cases with smaller overall frequency of events  $p_1 + p_2$  corresponds to moving the dashed line to the lower-left. Figure 5 and Figure 6 both illustrate the result that market shares need to be sufficiently symmetric for cooperation to be possible.

In the next section, we consider these relationships when the liquidity event ( $\Delta x$ ) is stochastic across time. We will also discuss the effect of multimarket contact and contagion of illiquidity.

## *B. Episodic Illiquidity and Contagion*

### *B.1. Shocks of Random Magnitude*

In Section IIA, we evaluated the requirements for cooperation given that  $\Delta x$  is a fixed amount of the asset. In that formulation, if cooperation is possible (based on the market parameters and  $\delta$ ), the traders never deviate. To characterize episodic illiquidity,  $\Delta x$  is better modeled as a random variable. In the event of a large  $\Delta x$ , it is more profitable for the traders to deviate for a one-time gain. However, instead of initiating a grim-trigger strategy, there are more profitable strategies available to the cartel.

The large traders implicitly agree to restrain from predateding when the magnitude of the shock is below some threshold  $\Delta x^*$  and, conversely, not to punish other players in future periods for predateding when the shock is above that threshold. That is, when a player has trading requirement

that exceeds  $\Delta x^*$ , the other player will predate, but cooperation is resumed in subsequent periods. This equilibrium behavior results in episodically increased volatility.<sup>29</sup>

Another way to describe this equilibrium is that each trader agrees to restrain from predating on the other, but only as long as they ‘behave responsibly’ in their risk management. This creates a natural restriction on the exposure that each trader can take without a disproportionate increase in the risk of their portfolio.

The value of  $\Delta x^*$  which is optimal for the cartel (in the sense of leading to the highest expected value for its members) can be computed for any distribution of the trading requirement for each player. In general,  $\Delta x^*$  can only be characterized implicitly.

**RESULT 5** (*Shocks of Random Magnitude*) Consider trading requirements for each of two players which are shocks of random magnitude  $\Delta x$  distributed i.i.d. according to the density  $f(y)$ , which we assume to be

(i) symmetric,  $f(y) = f(-y)$ ,

(ii) with unbounded support,  $f(y) > 0, \forall y \in \mathbb{R}$ ,

(iii) and with finite variance,  $\int_{-\infty}^{\infty} y^2 f(y) dy < \infty$ .

A strategy with episodic predation with threshold  $\Delta x^*$  is feasible with any  $\Delta x^*$  that satisfies

$$2C \int_0^{\Delta x^*} y^2 f(y) dy \geq K(\Delta x^*)^2. \quad (26)$$

The supremum of  $\Delta x^*$  such that the inequality is satisfied exists, and we designate it by  $\overline{\Delta x}$ . The following strategy profile constitutes a sub-game perfect Nash equilibrium. At time  $t = 0$ , we predate if  $|\Delta x| > \Delta x^*$ , and otherwise cooperate. At time  $t \neq 0$ ,

1. If the history of play  $h^{t-1}$  is such that for every period in which  $|\Delta x| < \Delta x^*$  there was no predation, then

(a) If  $|\Delta x| > \Delta x^*$ , predate this period.

(b) If  $|\Delta x| < \Delta x^*$ , cooperate.

2. If  $h^{t-1}$  is such that for  $|\Delta x| < \Delta x^*$ , there was predation, then predate.

The constants above are

$$C = \frac{\delta}{1-\delta} \left[ p_1(1-p_2)K_d + 2p_1p_2K_2 - (1-p_1)p_2K \right] \quad (27)$$

and  $K = V_p/\Delta x^2$ , with  $K_d = \Delta V_d/\Delta x^2$  and  $K_2 = \Delta V_2/\Delta x^2$  (that is,  $K$ ,  $K_d$  and  $K_2$  are the factors multiplying  $\Delta x^2$  in the expected values  $V_p$ ,  $\Delta V_d$  and  $\Delta V_2$ ; note that  $K$  in (26) is the factor in the expected gain to the predator).

**Proof.** See Appendix A. ■

In what follows, the parameter  $C/K$  has the interpretation of representing the benefits from cooperation relative to the costs of cooperation. For each of the players  $i$ , the numerator  $C$  is increasing in the discount rate ( $\delta$ ), increasing in  $p_i$ , and decreasing in  $p_{j \neq i}$ . The ratio  $C/K$  is also increasing in  $K_d$  (the welfare loss if player  $i$  is distressed and player  $p_{j \neq i}$  is not), increasing in  $K_1$  (the total welfare loss if both are distressed and do not cooperate), and is decreasing in  $K$  (the profit from predation).  $C/K$  is then a measure of the relative strengths of the incentives to cooperate and predate.

Insert  
Figure 7  
here

To gain intuition about how the parameter  $C/K$  affects market liquidity, consider that the shock  $\Delta x$  is normally distributed. In Figure 7, the left-hand-side of the inequality in Equation 26

is plotted with the solid-line and the right-hand-side is plotted with the dashed-line. For  $\Delta x^*$  in an interval  $[\underline{\Delta x}, \overline{\Delta x}]$ , it is possible to sustain the subgame-perfect Nash equilibrium. For  $\Delta x^* > \overline{\Delta x}$ , the value gained for deviation is too high, and cooperation cannot be maintained. The supremum  $\overline{\Delta x}$  defines the most profitable strategy for the cartel. (Note that  $\overline{\Delta x}$  might be zero, in which case traders never cooperate.)

The nature of the solutions is essentially independent of the scale parameter of the distribution. Consider a family of distributions  $f_a(y) = af(ay)$ . If we determine solutions in terms of  $\Delta x^*/a$ , the set of feasible thresholds is independent of the asset parameters. The inequality in Equation 26 is equivalent to

$$2\frac{C}{K} \int_0^{\frac{\Delta x^*}{a}} y^2 f_a(y) dy \geq \left(\frac{\Delta x^*}{a}\right)^2, \quad (28)$$

so that, after the corresponding scaling, the solutions to the inequality are constant with scaling of the distribution. For example, if we consider the zero-mean normal distribution

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}, \quad (29)$$

we can rewrite the inequality as

$$\frac{K}{C} \left(\frac{\Delta x^*}{\sigma}\right)^2 \leq 2 \int_0^{\frac{\Delta x^*}{\sigma}} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy. \quad (30)$$

Insert Therefore, we can parameterize any such problem using only  $C/K$  and  $\Delta x^*/\sigma$ .

Figure 8 Figure 8 plots  $\underline{\Delta x}/\sigma$  and  $\overline{\Delta x}/\sigma$  as a function of  $C/K$ . The solid-line represents  $\overline{\Delta x}/\sigma$  and the dotted-line represents  $\underline{\Delta x}/\sigma$ . For any value of  $C/K$ , the vertical segment between the lines is the set of  $\Delta x^*$  (in standard deviations) such that cooperation is possible. Note that there exists a

critical value for  $C/K$  (represented by the small circle), below which it is impossible to support cooperation. Any threshold we might consider would always be too high, in that the immediate gains from deviation provide an incentive to deviate, and too small, in that the expected future rewards from cooperation during those shocks that do fall below the threshold do not provide sufficient incentive to cooperate.

We can now determine the minimum or critical  $C/K$  for which there is a non-zero  $\overline{\Delta x}$ , that is for which an episodic predation strategy is feasible. Consider again the normal distribution and the inequality in Equation (30). Taking the derivative of the difference between the two sides of the inequality with respect to  $\Delta x^*$  and equating to zero leads to

$$\frac{K}{C} = \frac{1}{\sqrt{2\pi}} \frac{\Delta x^*}{\sigma} e^{-\frac{1}{2}\left(\frac{\Delta x^*}{\sigma}\right)^2}. \quad (31)$$

This characterizes the points in Figure 7 where the solid and dashed lines have the same derivative. Using this at the supremum (*i.e.*, with equality holding in (30)), we obtain the case where the lines touch at a single point rather than having two intersections (other than zero). After a change of variable in the integral, we obtain

$$\int_0^{\frac{\Delta x^*}{\sigma}} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \frac{1}{2} \left(\frac{\Delta x^*}{\sigma}\right)^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\Delta x^*}{\sigma}\right)^2}, \quad (32)$$

which is straightforward to solve numerically for  $\Delta x^*/\sigma$ . Since  $C/K$  only depends on  $\Delta x^*$  through  $\Delta x^*/\sigma$ , the minimum  $C/K$  ratio for which there is a feasible strategy of the episodic predation type does not depend on the scale parameter of the distribution. For the normal distribution, the minimum  $C/K$  for which there is a non-zero  $\overline{\Delta x}$ , that is for which an episodic predation strategy is

feasible, is  $C/K = 4.6729$  for any  $\sigma$ . This imposes restrictions on the parameters. For assets with parameters such that  $C/K$  is less than this critical value, no episodic predation strategy is feasible. The threshold associated with this  $C/K$  ratio is  $\underline{\Delta x} = \overline{\Delta x} = 1.3688\sigma$ .

In Section II A, we showed that the ability for traders to cooperate when the liquidity needs are fixed is increasing in  $\frac{\gamma}{\lambda}$ . By Result 3, it is easy to show that  $C/K$  is also monotonically increasing in  $\gamma$  and monotonically decreasing in  $\lambda$ . This implies that markets with high  $\frac{\gamma}{\lambda}$  will have a higher  $C/K$ . Based on this, the following predictions are natural. For assets with a high level of asymmetric information (high  $\gamma$ ) that are widely traded in large volume (low  $\lambda$ ), we would expect there to be liquidity that is stable most of the time (apparent liquidity), but disappears episodically. An example of this type of security would be a growth stock (perhaps a tech stock) with diffuse ownership. In contrast, a low  $\frac{\gamma}{\lambda}$  asset may have liquidity levels that fluctuate more regularly, but will not exhibit marked illiquidity in an episodic way. An example of a low  $\frac{\gamma}{\lambda}$  asset would be a thinly traded AA-rated corporate bond. To our knowledge, securities have not been grouped into these categories before and studied for their permanent and temporary illiquidity. This would be necessary to test the empirical implications of our model.

### *B.2. Contagion Across Markets*

Suppose that the members of the oligopoly can cooperate in more than one market. As an example, consider institutional traders who dominate mortgage markets and are also strategic traders in other fixed income markets. If a liquidity event is large enough to disturb cooperation in one market, it may also affect cooperation in the others. Bernheim and Whinston (1990) find that if markets are not identical, multimarket contact supports cooperation. In our case, and since assets are not perfectly correlated, multimarket contact makes it easier to maintain cooperation.



In this section, we discuss the effects of multimarket contact on episodic illiquidity across markets.

Consider traders that participate in  $n$  markets, where the trading requirement in each market is a stochastic random variable. We define a liquidity event to be such that all trading targets for each of the  $n$  assets have the same sign (*i.e.*, liquidity shocks occur in the same direction in all markets).<sup>30</sup> The trading targets are modeled as jointly normal and conditionally independent given that they are either all positive or all negative. We also use the simplifying assumption that trading targets in all the assets have the same variance. The density in the positive orthant ( $y$  such that  $y_i \geq 0$ , all  $i$ ) and in the negative orthants ( $y$  such that  $y_i \leq 0$ , all  $i$ ) is

$$f(y) = \frac{2^{n-1}}{\sigma^n (2\pi)^{n/2}} e^{-\frac{y^T y}{\sigma^2}}, \quad (33)$$

and zero elsewhere.

The shape of the optimal region for cooperation is spherical. This is the region in which the incentive to predate, which is proportional to  $\sum_{i=1}^n \Delta x_i^2$ , is constant. The inequality for  $n$  assets involves an integral in  $n$  dimension which, using the radial symmetry of the normal distribution can be written as

$$2C \int_0^r S_n y^{n+1} f(y) dy \geq Kr^2, \quad (34)$$

where  $r$  is the radius of the cooperation region, and

$$S_n = \frac{1}{2^n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \quad (35)$$

is the area of the intersection of the sphere of unit radius in  $n$  dimension with the positive orthant.

It can easily be verified that, as for the one-asset case, the nature of the solutions is essentially

Insert independent of the scale parameter of the distribution (say the standard deviation).

Figure 9 here Figure 9 is the multimarket version of Figure 8 in that it plots  $\underline{\Delta x}$  and  $\overline{\Delta x}$  for episodic predation over  $n$  markets,  $n = 1, 2, \dots, 8$ . The minimum value of  $C/K$  that is required to support cooperation decreases as the number of markets increases. Adding markets can make cooperation possible where it would otherwise not be possible. This is because adding markets increases the relative value of cooperation, given that punishment is effected over all markets. (Consider, for instance,  $C/K = 4.0$ . With these parameters, traders are unable to cooperate over one market, but are able to do so over 2 or more markets.) Also note that the supremum of  $r$  increases with  $n$ . The probability that an episode of predation will occur is in fact seen to decrease with  $n$ . We expect episodes of predation to be more significant, since they now affect  $n$  markets, but less frequent with contagion strategies. The minimum values of  $C/K$  for cooperation over multiple markets,  $n = 1, 2, \dots, 20$ , are listed in

Table II Table II.

about here

### III. Conclusion

We have presented a model where the breakdown of cooperation in financial markets leads to episodic illiquidity. This model is based on an equilibrium strategy in which traders cooperate most of the time through repeated interaction, providing ‘apparent liquidity’ to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity. Our model provides an explanation for why episodic liquidity breakdowns do not occur more often, and predicts that this apparent liquidity is more easily sustained in asset markets with a high permanent price impact of trading and a low temporary impact of trading.

We solve a competitive trading game for strategic traders (predatory stage game), which is

formulated as a continuous-time dynamic programming problem using an asset pricing equation which accounts for transaction costs from price impact of trading. According to this model, traders ‘race’ to market, selling quickly in the beginning of the period. Also in equilibrium, predators initially race distressed traders to market, but eventually ‘fade’ them and buy back. The presence of predators in the market leads to a surplus loss to the strategic traders.

Cooperation in the market is modeled by embedding this predatory stage game in a dynamic game with infinite horizon. Cooperation allows for the trading of large blocks of the asset at more favorable prices, so that the surplus loss due to predatory trading can be avoided. This leads to predictions about the types of markets in which cooperation is more likely.

We believe that our model presents a plausible argument for the level of predation or cooperation in financial markets being a determinant of available liquidity, and a contributor to the episodic nature of illiquidity.

## ENDNOTES

<sup>1</sup> See Acharya and Pedersen (2005) for evidence of a flight to quality during these episodes.

<sup>2</sup> Predatory trading has been defined by Brunnermeier and Pedersen (2005) as trading that induces and/or exploits another investor's need to change their position. It is important to distinguish predatory trading from front-running. Front-running is an illegal activity in which a specialist, acting as an agent of an investor, trades on his own account in the same direction as his client before he fulfills his client's order. In this way, the specialist profits but violates his legal obligation as an agent of the investor. Predatory activity occurs in the absence of such a legal obligation.

<sup>3</sup> On February 2, 2005 the Wall Street Journal reported that this predatory trading plan was referred to as "Dr. Evil" by traders working at Citicorp.

<sup>4</sup> This is an exogenous event that changes long-run relationships between brokers and the specialist.

<sup>5</sup> Other articles in this literature include Berhardt, Dvoracek, Hughson, and Werner (2005), Desgranges and Foucault (2005), Reiss and Werner (2004), Ramadorai (2003), Hansch, Naik, and Viswanathan (1999), Massa and Simonov (2003).

<sup>6</sup> Motivation for partitioning price effects based on permanent and temporary components of liquidity is given in the body of the paper, but is based on work by Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1990), Cheng and Madhavan (1997), Huang and Stoll (1997), Sadka (2005).

<sup>7</sup> In the formulation by Brunnermeier and Pedersen, no transaction costs are incurred in the equilibrium solution and all gains by the predators are exactly offset by losses by distressed traders, so that there would be no feasible Pareto improvement in a repeated game. There are other substantial differences between the models. Brunnermeier and Pedersen impose exogenous holding limits  $[-\bar{x}, \bar{x}]$  for traders, whereas we do not make this restriction. Our model involves a stochastic price process, while in Brunnermeier and Pedersen the asset pricing relationship is deterministic. Finally note that Brunnermeier and Pedersen’s model predicts “price-overshooting”, whereas our model does not. However, this is a consequence of, in our stage game model, all traders having an identical time horizon. If we relax this as in Brunnermeier and Pedersen to allow predators a longer horizon, price overshooting is also observed in our model.

<sup>8</sup> Attari, Mello, and Ruckes (2005) also describe predatory trading behavior with a two-period model. They show that predators may even lend to others that are “financially fragile” because they can obtain higher profits by trading against them for a longer period of time. Our model is more general in that it is in an infinite-horizon, multi-period framework, with each period a continuous-time game.

<sup>9</sup> Our mechanism for contagion is different from that of Brunnermeier and Pedersen (2004). The contagion in Brunnermeier and Pedersen (2004) is caused by a wealth effect. As prices in the market drop, additional traders are induced into a state of distress and a market-wide sell-off is observed. In contrast, contagion in our model occurs when traders abandon cooperation in a model of repeated interaction.

<sup>10</sup> Another possible test of our model would be a natural experiment. For example, comparing the liquidity pattern in the foreign exchange market before and after electronic automation might

reveal the effects of relationships and cooperation on liquidity.

<sup>11</sup> Directly linking trading pressure and price distinguishes our model from Brunnermeier and Pedersen (2005), where transaction costs are modeled via an exogenous parameter  $A$  (the maximum trading rate at which transaction costs are avoided), which does not directly affect prices in the market. This results in there being no surplus to be gained from cooperation.

<sup>12</sup> Further motivation for our use of this pricing relationship are the empirical and theoretical studies that link trading pressure and asset prices (Keim and Madhavan 1996, Kaul, Mehrotra, and Morck 2000, Holthausen, Leftwich, and Mayers 1990, Chan and Lakonishok 1995, Bertsimas and Lo 1998, Fedyk 2001, DeMarzo and Urošević 2000, Almgren and Chriss 2000, Almgren and Chriss 1999, Huberman and Stanzl 2004a).

<sup>13</sup> In a model with no discounting, Huberman and Stanzl (2004a) show that the presence of a drift term is inconsistent with no arbitrage. This result can be extended to the case with discounting. For no arbitrage to hold, the difference between the drift coefficient and the continuous-time discount factor must be zero. For the multi-period game which we will later discuss, the assumption is that  $T$  is relatively small, that is the distress and predation events develop over short periods of time, and the discounting within each period is not significant. The period-to-period discount factor is then also close to one. Since each period is short, the multi-stage game will consist of many short periods, where the probabilities of a player being distressed in any one of those periods are small, so that the period-to-period discount factor is significant to the problem.

<sup>14</sup> Huang and Stoll (1997) show large variation in the ratio  $\frac{\gamma}{\lambda}$  in Table 2 in their paper. For the stocks considered, the ratio  $\frac{\gamma}{\lambda}$  varies from 0.02 to 0.22, which seems especially significant given the sample of large liquid stocks that they consider.

<sup>15</sup> Duffie, Garleanu, and Pedersen (2005) also study the factors which affect liquidity in non-anonymous markets.

<sup>16</sup> A particularly clear example of this is the mortgage market developed by Salomon Brothers. Once this market was established and profitable, many of Salomon's mortgage traders were hired by other investment banks to run their mortgage desks. As a consequence, the trading habits of all of the desks were especially well-known to each other.

<sup>17</sup> A variation on this model would be to allow the opportunistic traders to trade over a longer time period than the distressed traders. The solution for such a model is similar. However, the predatory trader will now choose what position to have by the end of the distressed seller's deadline. The choice of this position is made by maximizing the expected value from trading over the distressed seller's period, plus the expected value from selling the position at the end of that period at a constant rate over the additional time. (See also Footnote 7.)

<sup>18</sup> Our admissibility restriction is similar in spirit to Back and Baruch (2004). It restricts trader  $i$ 's strategy given the price process, hence it depends on the strategies of other traders. As noted, it is easy to show that the restriction  $Y_t^i \in L^2$  suffices to ensure admissibility, so that there are standard restrictions that ensure admissibility.

<sup>19</sup> The single trader version of this problem is related to Huberman and Stanzl (2004b), Huberman and Stanzl (2004a), Almgren and Chriss (1999), Almgren and Chriss (2000), Bertsimas and Lo (1998).

<sup>20</sup>In Appendix B we analyze closed-loop strategies, where players take into account the other players' response functions and are able to change their trading schedules part-way through the game. From an asymptotic approximation (and numerical experiments), we have found that the

solution to the closed-loop differential game is qualitatively similar to that of the open-loop differential game, except that the welfare loss in closed loop is higher. Hence, the incentives for cooperation are stronger in the closed-loop solution, which would make the players more likely to cooperate in the repeated-game analysis of Section II.

<sup>21</sup>Our approach to existence is similar to Huberman and Stanzl (2004a), who consider a related problem in continuous time for a single trader. As in their paper, we restrict ourselves to the class of smooth differentiable strategies. An alternative approach is to ensure smoothness by defining the solution to the continuous-time problem as the limit of the solutions to a sequence of discrete-time problems, which is the approach we use in the analysis of the closed-loop version of the problem in Appendix B.

<sup>22</sup>See Theorem 6.12 in Basar and Olsder (1999) and related discussion.

<sup>23</sup>However, the open loop strategy is not ‘strongly time consistent’ or subgame perfect. The closed-loop solution discussed in Appendix B is strongly time-consistent.

<sup>24</sup>Note that these bounds are tight, in that they can be approximated arbitrarily close under some combination of valid parameters.

<sup>25</sup> In our stage game, we do not allow for ex-post renegotiation. Surplus losses are common in many models in non-cooperative game theory (*i.e.*, Prisoner’s Dilemma and Centipede Game) and motivate cooperation in repeated play.

<sup>26</sup>In models of implicit collusion under imperfect information (Green and Porter 1984, Abreu, Pearce, and Stacchetti 1986), players never deviate in equilibrium, but enter punishment phases because of exogenous price changes. Our game is one of complete information, where deviations



cannot arise from such exogenous factors. This allows for a simpler model which captures the key issues of interest, namely how apparent liquidity arises from incentives to cooperate.

<sup>27</sup> The limits  $\frac{\gamma}{\lambda} \rightarrow 0$  and  $\frac{\gamma}{\lambda} \rightarrow \infty$  are defined as follows. The limit  $\frac{\gamma}{\lambda} \rightarrow 0$  is achieved by letting  $\gamma \rightarrow 0$ , or  $\lambda \rightarrow \infty$ , or both. The limit  $\frac{\gamma}{\lambda} \rightarrow \infty$  is achieved analogously.

<sup>28</sup>To determine the division of this surplus between the insiders and the external player, it is possible to use a generalized Nash bargaining solution in which the insiders receive fraction  $\tau$  of the surplus and the client receives fraction  $1 - \tau$ . The example that we consider (Figure 5), is equivalent to assuming  $\tau = 1$ . This is without any significant loss of generality, since relaxing this assumption lead to the same comparative statics.

<sup>29</sup> Episodic illiquidity also occurs during extreme financial distress. During extreme distress, a member of the oligopoly becomes a finite concern. Because the horizon of this game is finite, the players work out their strategy profiles by backwards induction and cooperation disappears.

<sup>30</sup> Implicitly, we are assuming that there is a common cause driving the liquidity event. That is, the liquidity needs for each trader are positively correlated across markets. We can generalize the model to account for arbitrary correlation. However, it seems less plausible that a trader would simultaneously have both large positive and large negative liquidity needs.

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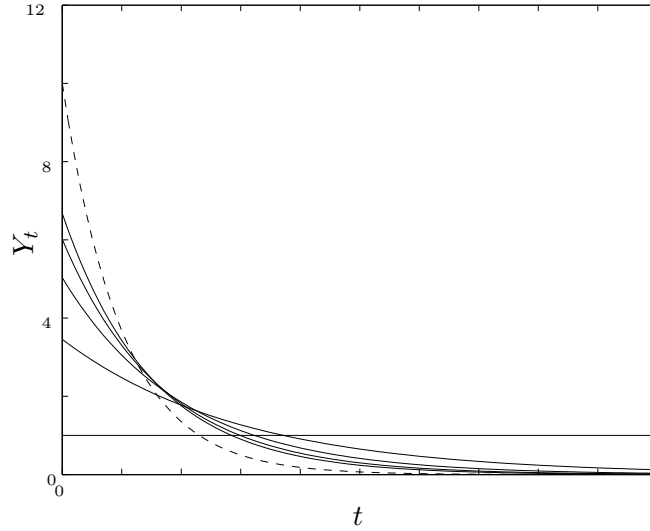
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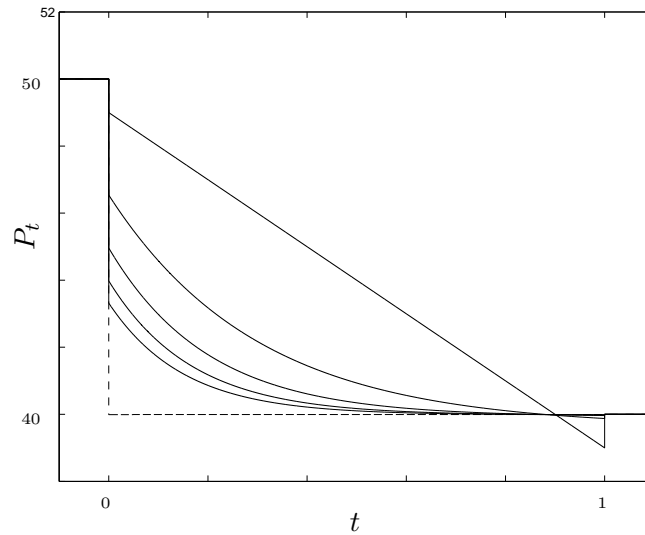
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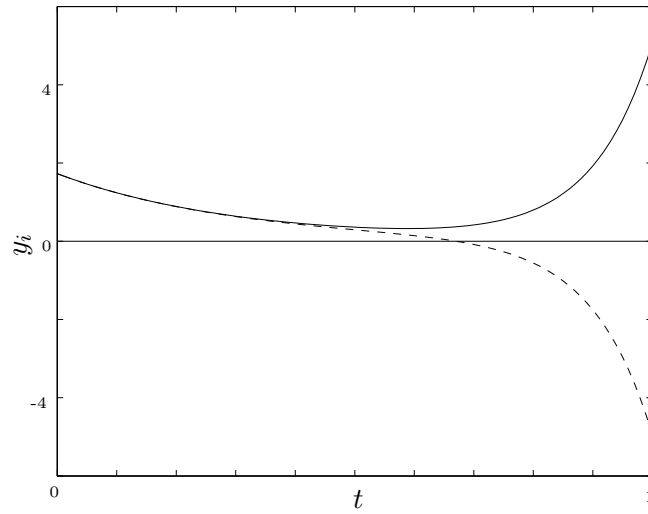
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**Figure 1.** Trading rate ( $Y_t$ ) for multiple traders with identical targets (solid for  $n = 1, 2, 3, 4, 5$ , dashed for  $n = \infty$ ) during a market sell-off. Competition among traders leads to a ‘race to trade’. Traders sell at a decreasing exponential rate. For  $n = 1$  we obtain the constant selling rate (*solid horizontal line*). If there are more traders, everybody will trade earlier. Note that there is an upper bound on how fast traders will sell their position ( $Y_t \rightarrow e^{-\frac{\gamma}{\lambda}t}$  as  $n \rightarrow \infty$ ), regardless of how many traders are in the race (*dotted line*). The parameters for this example are  $\Delta x = 1$ ,  $T = 1$ ,  $\gamma = 10$ ,  $\lambda = 1$ .

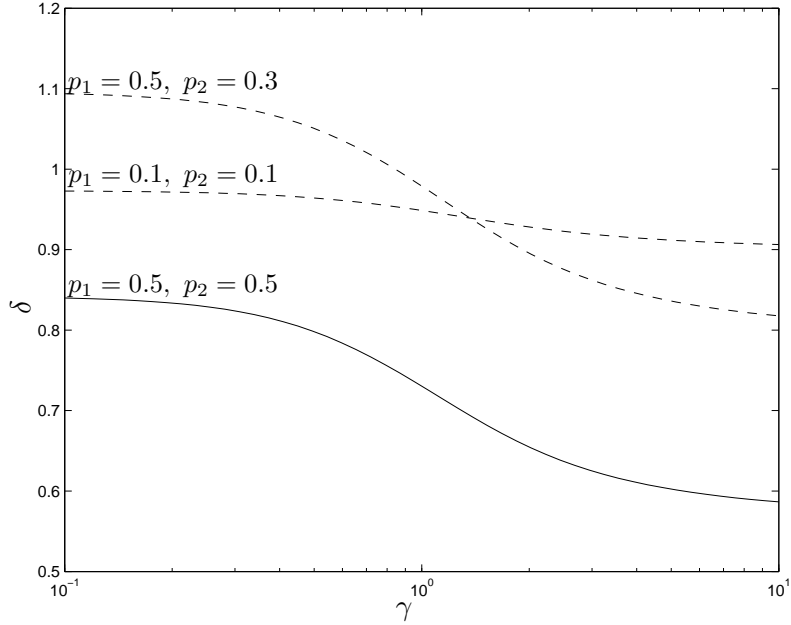


**Figure 2.** Expected price for multiple traders with identical targets (solid for  $n = 1, 2, 3, 4, 5$ , dashed for  $n = \infty$ ). When  $n = 1$ , trading occurs at a constant rate (*straight solid line*). When there are a large number of traders present ( $n$  large), trading pressure depresses prices quickly (*dotted line*). The parameters for this example are  $\Delta x = -1$ ,  $T = 1$ ,  $\gamma = 10$ ,  $\lambda = 1$ , with constant  $U_t = 50$ .

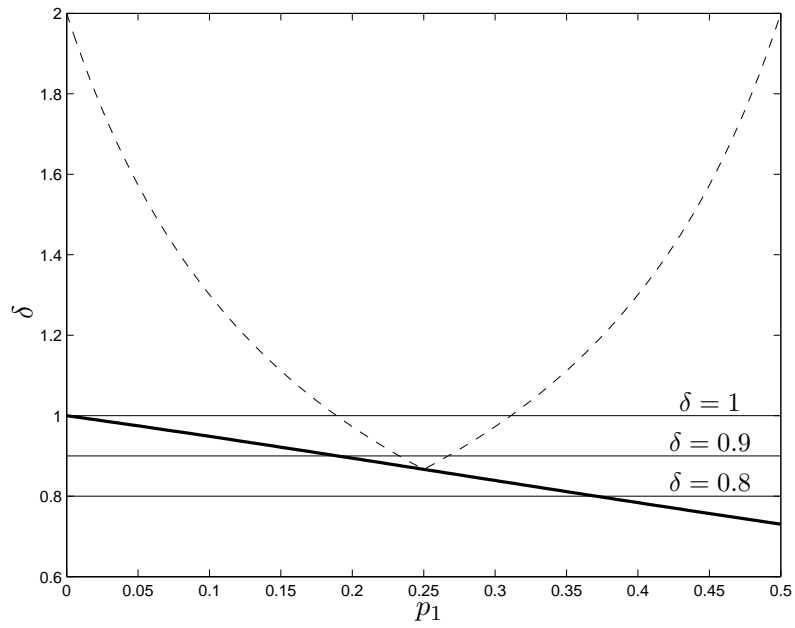


**Figure 3.** One trader with position target (solid) and one ‘opportunistic’ trader (dashed). The opportunistic trader initially races the distressed trader to the market in a sell-off. Then, toward the end of the period, the opportunistic trader reverses their position and fades the distressed trader. The parameters for this example are  $\Delta x = 1, T = 1, \gamma = 10, \lambda = 0.1$ .

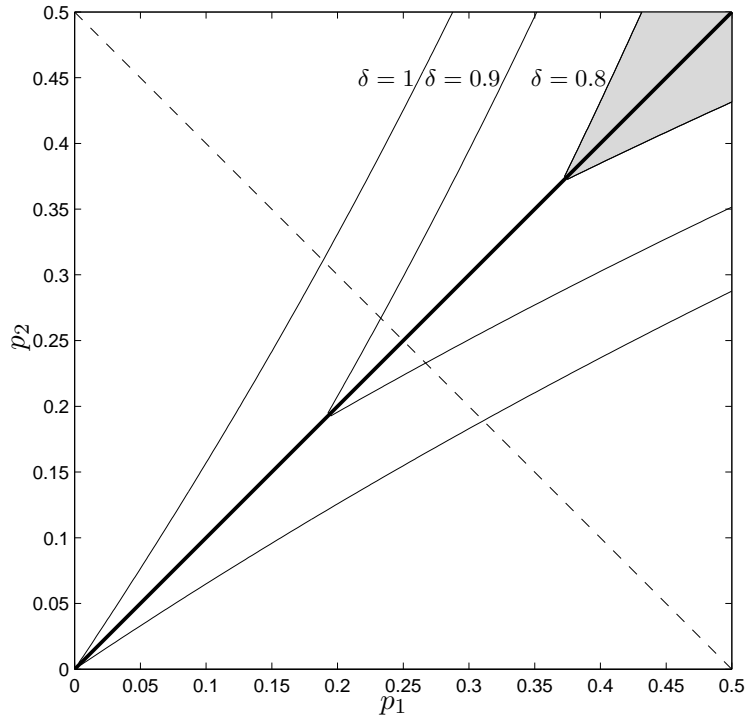




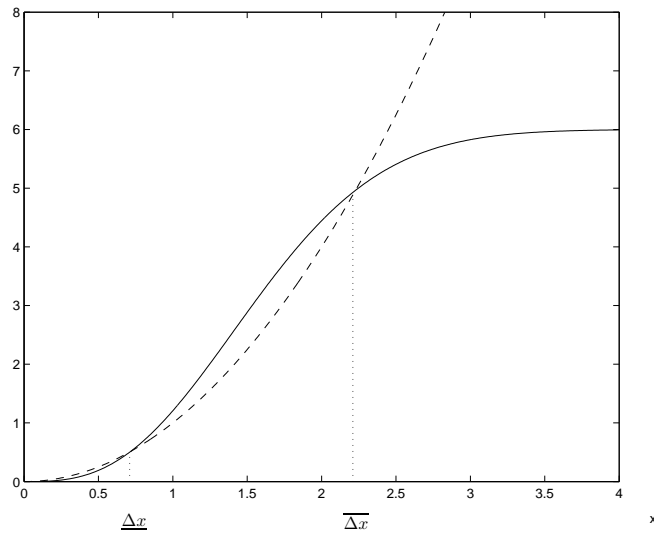
**Figure 4.** Minimum  $\delta$  for which cooperation is feasible as a function of  $\gamma$ , with logarithmic scale on the  $\gamma$ -axis. Three different cases of the distress probabilities of each trader are plotted (the probabilities are assumed independent). It is always possible to support cooperation when  $p_1 = p_2 = 0.5$  and when  $p_1 = p_2 = 0.1$ , but it is harder when the distress probabilities are lower. When the permanent price impact parameter  $\gamma$  is low it is impossible support cooperation when the distress probabilities are asymmetric ( $p_1 = 0.5$  and  $p_2 = 0.3$ ). The other parameters are  $\Delta x = 1$ ,  $U_0 = 10$ ,  $T = 10$ , and  $\lambda = 1$ .



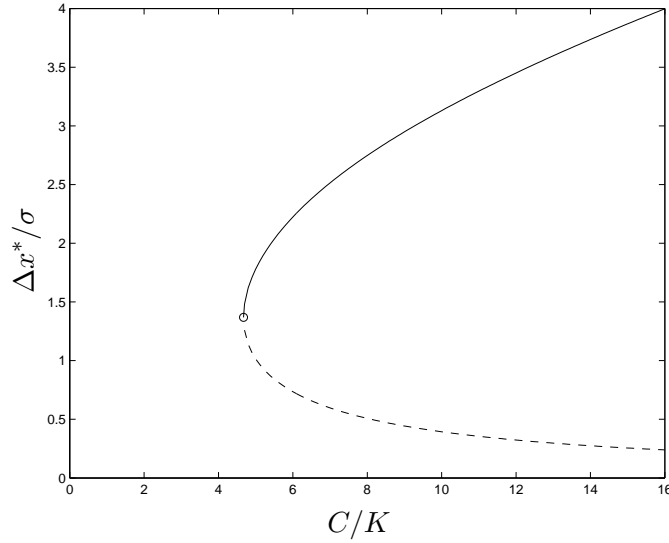
**Figure 5.** Minimum  $\delta$  as a function of distress probabilities, the cases of  $p_1 = p_2$  (bold), and  $p_2 = 0.5 - p_1$  (dashed). Parameters are  $\Delta x = 1$ ,  $U_0 = 10$ ,  $T = 10$ ,  $\gamma = 1$ , and  $\lambda = 1$ . The horizontal reference lines correspond to the iso- $\delta$  lines in Figure 6.



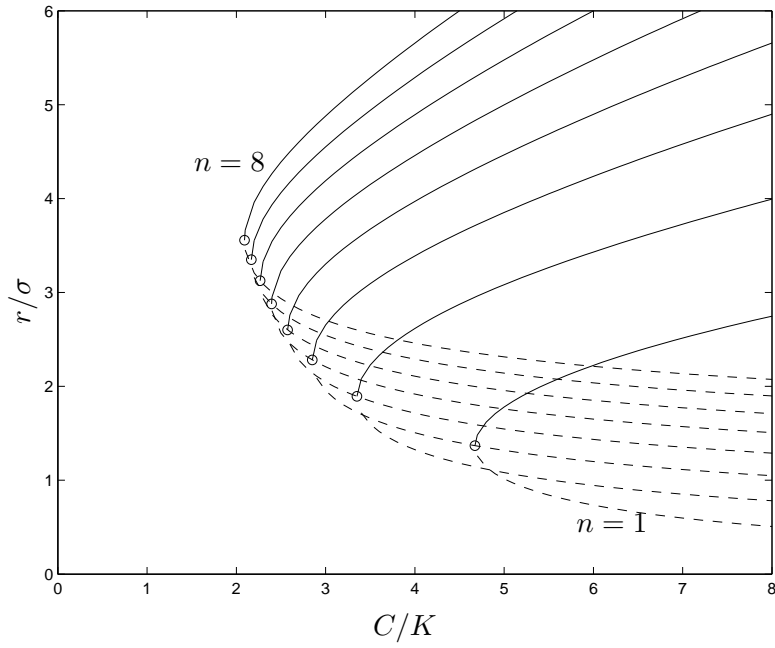
**Figure 6.** Values of  $p_1$  and  $p_2$  for which cooperation can be sustained, given  $\delta$  (plotted for the cases  $\delta = 1$ ,  $\delta = 0.9$ , and  $\delta = 0.8$ ). For instance, for  $\delta = 0.8$  the cooperative equilibrium exists if  $p_1$  and  $p_2$  fall in the shaded region. The bold and dashed lines indicate the values of  $p_1$  and  $p_2$  that are plotted in Figure 5.



**Figure 7.** Left- (solid) and right-hand-side (dashed) of Equation (26). The curves intersect at zero,  $\underline{\Delta x}$ , and  $\overline{\Delta x}$ .



**Figure 8.** Normally distributed shocks, upper and lower bounds for  $\Delta x^*/\sigma$  as a function of the asset parameters. The ratio  $C/K$  measures the benefits from cooperation relative to the costs of cooperating in financial markets. The open-circle at  $C/K = 4.6729$  is the minimum  $C/K$  for which there exists a non-zero  $\overline{\Delta x}$ , that is for which an episodic predation strategy is feasible.



**Figure 9.** Lowest and highest  $r$  such that the strategy (episodic predation with contagion over  $n$  assets,  $n = 1, 2, \dots, 8$ ) is an equilibrium, plotted as a function of the asset parameters, common to all assets. The shocks are independent and normally distributed conditional on shocks being either all positive or all negative. The ratio  $C/K$  measures the benefits from cooperation relative to the costs of cooperating in financial markets. The open-circle for each curve is the minimum  $C/K$  for which there exists a non-zero  $\overline{\Delta x}$ , that is for which an episodic predation strategy is feasible. Note that the minimum  $C/K$  decreases as the number of markets  $n$  increase.

**Table I**  
**Expected Values from Strategic Trading**

Listed are the expected values that traders gain or lose when a sell-off occurs in the market. If only one trader is present, they sell a block of shares  $\Delta x$  in the time interval  $[0, T]$  and gain the value  $V_1$ . When a predator is present they will earn less and the loss they incur is  $\Delta V_d = V_1 - V_d$ , where  $V_d$  is the value they gain when they trade against the predator. The predator makes a profit of  $V_p$  and the aggregate surplus to the predator and distressed trader is  $V_2 = V_d + V_p$ . We show that the change in overall surplus,  $\Delta V_2$ , is always negative. The expressions in the right-hand column of the table are derived in the paper.  $U_0$  is the expected value of future dividends,  $\gamma$  measures the permanent impact of trading, and  $\lambda$  measures the temporary impact of trading.

Surplus to distressed trader, no predation	$V_1$	$-U_0 \Delta x - \left(\frac{1}{2} + \frac{1}{\lambda T}\right) \gamma \Delta x^2$
Surplus to distressed trader during predation	$V_d$	$-U_0 \Delta x - \frac{1}{6} \frac{5e^{\frac{2}{3}\lambda T} + e^{\frac{2}{3}\lambda T} + e^{\frac{1}{3}\lambda T} - 1}{e^{\lambda T} - 1} \gamma \Delta x^2$
Change in surplus to distressed trader ( $V_1 - V_d$ )	$\Delta V_d$	$\left(\frac{1}{6} \cdot \frac{2e^{\frac{2}{3}\lambda T} + e^{\frac{2}{3}\lambda T} + e^{\frac{1}{3}\lambda T} + 2}{e^{\lambda T} - 1} - \frac{1}{\lambda T}\right) \gamma \Delta x^2$
Surplus to predator	$V_p$	$\frac{1}{6} \cdot \frac{e^{\frac{2}{3}\lambda T} - 1}{e^{\frac{2}{3}\lambda T} + e^{\frac{1}{3}\lambda T} + 1} \gamma \Delta x^2$
Total surplus to predator and distressed trader	$V_2$	$-U_0 \Delta x - \frac{1}{3} \cdot \frac{2e^{\frac{1}{3}\lambda T} - 1}{e^{\frac{1}{3}\lambda T} - 1} \gamma \Delta x^2$
Change in overall surplus with predatory trading	$\Delta V_2$	$\left(\frac{1}{6} \cdot \frac{e^{\frac{1}{3}\lambda T} + 1}{e^{\frac{1}{3}\lambda T} - 1} - \frac{1}{\lambda T}\right) \gamma \Delta x^2$

**Table II**  
**Multimarket contact across  $n$ -markets.**

The ratio  $C/K$  measures the benefits from cooperation relative to the costs of cooperating in financial markets. For  $n = 1, 2, \dots, 20$ , we calculate the minimum values of  $C/K$  for which cooperation is possible over multiple markets. As the number of markets increases ( $n$  increases), the relative benefit needed to support cooperation decreases.

$n$	$C/K$	$n$	$C/K$	$n$	$C/K$	$n$	$C/K$
1	4.6729	6	2.2664	11	1.9299	16	1.7740
2	3.3509	7	2.1687	12	1.8907	17	1.7517
3	2.8507	8	2.0913	13	1.8563	18	1.7314
4	2.5747	9	2.0280	14	1.8259	19	1.7127
5	2.3950	10	1.9751	15	1.7986	20	1.6954



**Table III**  
**Expected Values from Strategic Trading: Comparing Open-Loop and Closed-Loop Strategies**

Listed are the expected values that traders gain or lose when a sell-off occurs in the market. Considering the limit conditions when  $T \rightarrow \infty$  or  $\lambda \rightarrow 0$ , we calculate the values to the strategic traders when they use open-loop and closed-loop strategies.  $U_0$  is the expected value of future dividends and  $\gamma$  measures the permanent impact of trading. The values to the traders are qualitatively similar, though the aggregate surplus loss to the traders is worse in the closed-loop case.

		Open-Loop	Closed-Loop
Surplus to distressed trader, no predation	$V_1$	$-U_0 \Delta x - \frac{1}{2} \gamma \Delta x^2$	$-U_0 \Delta x - \frac{1}{2} \gamma \Delta x^2$
Surplus to distressed trader during predation	$V_d$	$-U_0 \Delta x - \frac{5}{6} \gamma \Delta x^2$	$-U_0 \Delta x - \frac{7}{8} \gamma \Delta x^2$
Change in surplus to distressed trader ( $V_1 - V_d$ )	$\Delta V_d$	$\frac{1}{3} \gamma \Delta x^2$	$\frac{3}{8} \gamma \Delta x^2$
Surplus to predator	$V_p$	$\frac{1}{6} \gamma \Delta x^2$	$\frac{1}{8} \gamma \Delta x^2$
Total surplus to predator and distressed trader	$V_2$	$-U_0 \Delta x - \frac{2}{3} \gamma \Delta x^2$	$-U_0 \Delta x - \frac{3}{4} \gamma \Delta x^2$
Change in overall surplus with predatory trading	$\Delta V_2$	$\frac{1}{6} \gamma \Delta x^2$	$\frac{1}{4} \gamma \Delta x^2$

## Appendix A

### Derivations

#### Proof of Result 1 (General Solution)

Given that each player's objective is linear in  $U_t$ , and that the strategies are open-loop, we can bring the expectation inside the integral (Fubini's theorem applies since the trading rate  $Y_t^i$  is admissible) and consider the equivalent problem with a deterministic asset pricing equation, in which  $U_t$  is replaced by a constant  $u = U_0$ ,

$$P_t = u + \gamma \sum_{j=1}^n X_t^j + \lambda \sum_{j=1}^n Y_t^j. \quad (\text{A.1})$$

This is a standard deterministic control problem with objective concave in  $X_t^i$  and  $Y_t^i$  and constraint linear in the control  $Y_t^i$ . If the solutions to the necessary conditions are continuous, sufficiency follows.

With the multiplier function  $Z_t^i$  associated with the constraint  $dX_t^i = Y_t^i dt$ , necessary optimality conditions for the problem faced by trader  $i$  are (Basar and Olsder 1999)

$$\begin{aligned} u + \gamma \sum_{j=1}^n X_t^j + \lambda \sum_{j=1}^n Y_t^j + \lambda Y_t^i + Z_t^i &= 0 \\ dZ_t^i &= -\gamma Y_t^i dt. \end{aligned} \quad (\text{A.2})$$

The second equation ensures that the multiplier function is continuous (as required for sufficiency).

Differentiating the first equation with respect to  $t$ , and substituting the second,

$$\gamma \sum_{j=1}^n Y_t^j dt + \lambda \sum_{j=1}^n dY_t^j + \lambda dY_t^i - \gamma Y_t^i dt = 0. \quad (\text{A.3})$$

The  $n$  such equations for each trader can be collected together as

$$\lambda(I + \mathbf{1}\mathbf{1}^T)dY_t = \gamma(I - \mathbf{1}\mathbf{1}^T)Y_t dt, \quad (\text{A.4})$$

where  $I$  is the  $n \times n$  identity matrix,  $\mathbf{1}$  is the  $n$ -vector with all entries equal to one, and  $\mathbf{1}\mathbf{1}^T$  is an  $n \times n$  matrix with all elements equal to one. From the formula for the inverse of the rank-one update of a matrix (Kailath 1980), the inverse of  $I + \mathbf{1}\mathbf{1}^T$  is  $I - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T$ , which we use to write the linear dynamic system in the form

$$dY_t = \frac{\gamma}{\lambda} A Y_t dt, \quad \text{where } A = I - \frac{2}{n+1}\mathbf{1}\mathbf{1}^T. \quad (\text{A.5})$$

Since  $A\mathbf{1} = \mathbf{1} - \frac{2}{n+1}n\mathbf{1} = -\frac{n-1}{n+1}\mathbf{1}$ , the vector of ones is an eigenvector of the matrix  $A$ , with associated eigenvalue  $-\frac{n-1}{n+1}$ . Likewise, vectors in the null-space of  $\mathbf{1}$  are eigenvectors of  $A$ , with eigenvalue 1: for  $v$  orthogonal to the vector of ones, that is satisfying  $\mathbf{1}^T v = 0$ , we find that  $Av = v$ . The dimension of this sub-space, and multiplicity of the eigenvalue 1, is  $n - 1$ . Since the matrix  $A$  has a full set of  $n$  independent eigenvectors, all Jordan blocks are of size 1 and solutions to the system of linear differential equations are as stated in (5). This characterizes any continuous policy (with continuous dual functional) which is an extremal of the problem. Since a continuous extremal exists, from the concavity of the objective in the state and control and linearity of the constraint, this is the unique extremal of the problem (in the admissible class). The  $n$  trading target constraints and  $\mathbf{1}^T b = 0$  uniquely determine the  $n$  free parameters in the solution (integrate the  $Y_t^i$ , equate to  $\Delta x_i$ , and solve for  $a$  and  $b$ ).

■

We next show a Lemma and Corollary, which will be of use in proving Result 2.

**LEMMA A.1** *The function  $f : \mathbf{R}^+ \mapsto \mathbf{R}$*

$$f(y) = \frac{1 + e^{-y}}{1 - e^{-y}} - \frac{2}{y} \quad (\text{A.6})$$

*is positive increasing.*

**Proof.** We first show  $\lim_{y \rightarrow 0} f(y) = 0$ . Applying l'Hôpital's rule, we find

$$\lim_{y \rightarrow 0} f(y) = \lim_{y \rightarrow 0} \frac{-ye^{-y} + 1 + e^{-y} - 2e^{-y}}{1 - e^{-y} + ye^{-y}} = \lim_{y \rightarrow 0} \frac{ye^{-y}}{2e^{-y} - ye^{-y}} = 0. \quad (\text{A.7})$$

We now show  $f'(y) > 0$ .

$$f'(y) = \frac{-2e^{-y}}{(1 - e^{-y})^2} + \frac{2}{y^2} = \frac{2}{(1 - e^{-y})^2 y^2} (-y^2 e^{-y} + 1 + e^{-2y} - 2e^{-y}). \quad (\text{A.8})$$

For  $y > 0$ , the denominator is positive. We show that the numerator is also positive,  $g(y) = -y^2 e^{-y} + 1 + e^{-2y} - 2e^{-y} > 0$ , from  $g(0) = 0$  and  $g'(y) > 0$ :

$$g'(y) = -2ye^{-y} + y^2 e^{-y} - 2e^{-2y} + 2e^{-y} = e^{-y} (-2y + y^2 - 2e^{-y} + 2). \quad (\text{A.9})$$

Likewise, we show  $h(y) = -2y + y^2 - 2e^{-y} + 2 > 0$ , from  $h(0) = 0$  and  $h'(y) > 0$ :

$$h'(y) = -2 + 2y + 2e^{-y}, \quad (\text{A.10})$$

which is positive if  $e^{-y} > 1 - y$ , which is true for any  $y \neq 0$  (from the intercept and derivative at

zero and from the convexity of the exponential). ■

**COROLLARY A.1** *The function  $g : \mathbf{R}^+ \mapsto \mathbf{R}$*

$$g(y) = y \frac{1 + e^{-y}}{1 - e^{-y}} \tag{A.11}$$

*is positive increasing.*

**Proof.** Write

$$g(y) = yf(y) + 2, \tag{A.12}$$

where  $f$  is as in the previous Lemma. The product of two positive increasing functions is positive increasing. ■

**Proof of Result 2 (Expected Total Surplus and Loss for Multiple Traders)**

The expected surplus is obtained by integration of

$$-P_t \sum_{i=1}^n Y_t^i = -P_t n a e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} t} \tag{A.13}$$

over  $t \in [0, T]$ , followed by algebraic simplification. The proofs of the monotonicities are direct applications of the Lemma above or of its Corollary, using  $y = T$ ,  $y = \gamma$ ,  $y = \frac{1}{\lambda}$ , and  $y = \frac{n-1}{n+1}$  (with  $n$  relaxed to be in  $\mathbf{R}$ ). For  $\lambda$  and  $n$ , we also need the fact that the composition of two monotonic functions is monotonic. ■

**Proof of Result 3 (Expected Total Surplus and Loss for Two Traders)**

By Equation (8),  $Y = Y_t^d + Y_t^p = 2ae^{-\frac{1}{3} \frac{\gamma}{\lambda} t}$ . By integration of  $-P_t Y$  over  $t \in [0, T]$ , followed by

algebraic simplification, the results in Equations (13) and (16) are derived. The monotonicities are verified by differentiation of Equation (16). We define  $V_- = V_d - V_p$  and  $Y_- = Y_t^d - Y_t^p = 2be^{\frac{\gamma}{\lambda}t}$ . Integrate  $-P_t Y_-$  over  $t \in [0, T]$  and simplify to obtain

$$V_- = V_d - V_p = -u \Delta x - \gamma \frac{e^{\frac{\gamma}{\lambda}T}}{e^{\frac{\gamma}{\lambda}T} - 1} \Delta x^2. \quad (\text{A.14})$$

$V_d$  and  $V_p$  are obtained by simplification of  $(V_2 + V_-)/2$  and  $(V_2 - V_-)/2$ .

The proof for the monotonicity of  $\frac{V_p}{\Delta V_d}$  is along the same lines as for Result 2 (with lengthier algebra). The bounds on  $\frac{V_p}{\Delta V_d}$  are the limits at 0 and  $+\infty$ , obtained by applying l'Hôpital's rule as needed. ■

#### **Proof of Result 4 (Repeated Game with Two Symmetric Traders)**

For trader 1, the gains from cooperation must exceed those of one-time deviation and infinite non-cooperation or

$$\begin{aligned} \frac{\delta_1}{1-\delta_1} \left[ p_1(1-p_2)V_1(\Delta x) + \frac{1}{2}p_1p_2V_1(2\Delta x) \right] \geq \\ V_p(\Delta x) + \frac{\delta_1}{1-\delta_1} \left[ (1-p_1)p_2V_p(\Delta x) + p_1(1-p_2)V_d(\Delta x) + \frac{1}{2}p_1p_2V_2(2\Delta x) \right]. \end{aligned} \quad (\text{A.15})$$

Since  $\Delta V_2$  is quadratic in  $\Delta x$ , we have that  $\frac{1}{2} \Delta V_2(2\Delta x) = 2 \Delta V_2(\Delta x)$  (we then omit the argument when it is  $\Delta x$ ). The first equation in (19) follows by solving for  $\delta_1$ . The second equation is derived similarly for trader 2. For both traders to cooperate, it must be that  $\delta \geq \max\{\delta_1, \delta_2\}$ . The monotonicities can be proved algebraically, along the same lines as for Result 2. ■

#### **Proof of Result 5 (Shocks of Random Magnitude)**

Equation (A.15) can be written in this context as Equation (26), where  $f$  is the density of  $\Delta x$

and  $C$  and  $K$  are as defined. For values of  $C$  sufficiently large or values of  $K$  sufficiently small, Equation (26) will be satisfied and there will exist a  $\Delta x^*$  such that cooperation is possible. As long as  $|\Delta x| < \Delta x^*$ , the traders will cooperate since the value of cooperating exceeds that of a one-time deviation and subsequent grim-trigger play. If  $|\Delta x| \geq \Delta x^*$ , the traders will predate and resume cooperation in the next period if possible. If Equation 26 is not satisfied, then cooperation is not possible and the traders will always predate. Thus, there exists a subgame perfect Nash equilibrium as described. The left-hand side of Equation 26 is bounded since  $f$  has finite variance and the right-hand-side is unbounded. Hence the supremum of  $\Delta x^*$  is bounded. For existence of  $\overline{\Delta x}$ , note that zero is a solution. ■

## Appendix B

### Closed-Loop Solution

In our analysis of the stage game, the strategies considered are deterministic. They are *open-loop*, in the sense that traders choose their strategies at time  $t = 0$ . It is assumed that traders would not respond to other traders' deviations from their optimal strategies. We would have obtained the same solution had we defined the  $Y_t^i$  to be  $\mathcal{F}_t$ -adapted (where  $\mathcal{F}_t$  is the filtration of the  $\sigma$ -algebras generated by  $B_t$ ). Note however that this is in the strict sense that traders are aware of the underlying process  $U_t$  that defines price fundamentals, but not of the actual price  $P_t$  and of the other players' trading rates  $Y_t$ . (The argument for the solution to this variation on the open-loop problem to be the same as in Result 1 is based on the value functions' linearity in  $U_t$ , as is done below in this section.)

We now consider *closed-loop* strategies, in that traders know or can infer the other traders' rate of trading, and respond accordingly. Under such strategies,  $Y_t^i$  is adapted to all the information existing at time  $t$ , and players can revise their trading decisions at any time based on such information. This means that earlier decisions must account for other players' response function at later times. This inability to commit ahead of time to not deviate from a given strategy over the entire  $[0, T]$  period leads to more aggressive strategies than in the open-loop case, including faster racing.

Closed-loop strategies are substantially more difficult to analyze than open-loop, and we are not able to provide a closed-form solution. We do provide a description of the equilibrium optimal strategies in terms of a value function with two scalar parameter which satisfy a triangular system of nonlinear differential equations. Our numerical simulations have shown the open- and closed-loop solutions not to be substantially different. We provide some analytical justification for this



observation. In particular, from a fixed-point analysis of the Riccati equations, we provide closed-form expressions for the expected value for each of the players when  $T$  is large.

We derive the closed-loop result for the deterministic case and then show that it holds for the stochastic pricing equation. We consider strategies where the trading rate is constant over time increments of length  $\Delta t$ . The reward for trader  $i$  over each time increment is

$$r_i(u, y, \Delta t) = \int_0^{\Delta t} - (u + \gamma \mathbf{1}^T y \tau + \lambda \mathbf{1}^T y) y_i d\tau \quad (\text{B.1})$$

$$= - \left( u + \left( \frac{\gamma}{2} \Delta t + \lambda \right) \mathbf{1}^T y \right) y_i \Delta t. \quad (\text{B.2})$$

We formulate the problem as a dynamic game, with an  $n + 1$ -dimensional state, composed of  $u(t) \in \mathbf{R}$  and  $\phi(t) \in \mathbf{R}^n$ . The first component,  $u(t)$ , is the expected price at time  $t$ , including the permanent price impact of previous trades (in previous notation,  $u(t) = U_t + \gamma \sum_{i=1}^n X_t^i$ ). The  $n$  components of  $\phi(t)$  are the remaining trading targets for each trader, that is, the amount they still need to trade by  $T$  (in previous notation,  $\phi_i(t) = x_{T_i} - X_t^i$ ). The state transition over a period of length  $\Delta t$  with each player trading at a constant rate  $y_i$  is

$$u'(u, y, \Delta t) = u + \gamma \mathbf{1}^T y \Delta t, \quad (\text{B.3})$$

$$\phi'(\phi, y, \Delta t) = \phi - y \Delta t. \quad (\text{B.4})$$

The value functions for each of the  $n$  traders must simultaneously satisfy

$$V^i(u, \phi, t) = \max_{y_i} r_i(u, y, \Delta t) + V^i(u'(u, y, \Delta t), \phi'(\phi, y, \Delta t), t + \Delta t), \quad (\text{B.5})$$

$$i = 1, \dots, n.$$

We will show that the value functions can be represented in the form

$$V^i(u, \phi, t) = -u\phi_i - \alpha(t)\phi_i\mathbf{1}^T\phi + \beta(t)(\mathbf{1}^T\phi)^2, \quad (\text{B.6})$$

and derive Riccati equations for  $\alpha(t), \beta(t) \in \mathbf{R}$ . We take the limit  $\Delta t \rightarrow 0$  to find the differential equations for the continuous-time case.

Substituting (B.6), (B.2), (B.3), and (B.4) in (B.5) and letting  $\Delta t \rightarrow 0$ , we obtain

$$\begin{aligned} -\frac{d\alpha(t)}{dt}\phi_i\mathbf{1}^T\phi + \frac{d\beta(t)}{dt}(\mathbf{1}^T\phi)^2 &= \\ &= \max_{y_i} -\lambda y_i\mathbf{1}^T y - \gamma\phi_i\mathbf{1}^T y + \alpha\phi_i\mathbf{1}^T y + \alpha y_i\mathbf{1}^T\phi - 2\beta\mathbf{1}^T y\mathbf{1}^T\phi. \end{aligned} \quad (\text{B.7})$$

Differentiating with respect to  $y_i$  and equating to zero, we obtain the optimality condition

$$\lambda(y_i + \mathbf{1}^T y) = (\alpha - \gamma)\phi_i + (\alpha - 2\beta)\mathbf{1}^T\phi, \quad (\text{B.8})$$

which we collect over  $i = 1, \dots, n$  as

$$\lambda(I + \mathbf{1}\mathbf{1}^T)y = ((\alpha - \gamma)I + (\alpha - 2\beta)\mathbf{1}\mathbf{1}^T)\phi. \quad (\text{B.9})$$

Multiplying on the left by  $\frac{1}{\lambda}(I - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T)$ , we solve for the equilibrium trading rates as a function of the state,

$$y = \frac{1}{\lambda} \left( (\alpha - \gamma)I + \frac{1}{n+1}(-2\beta + \gamma)\mathbf{1}\mathbf{1}^T \right) \phi, \quad (\text{B.10})$$

which can equivalently be written individually as

$$y_i = \frac{1}{\lambda} \left( (\alpha - \gamma)\phi_i + \frac{1}{n+1}(-2\beta + \gamma)\mathbf{1}^T\phi \right), \quad i = 1, \dots, n. \quad (\text{B.11})$$

The sum of the trading rates is found to be

$$\mathbf{1}^T y = \frac{1}{\lambda} \left( \alpha - 2\frac{n}{n+1}\beta - \frac{1}{n+1}\gamma \right) \mathbf{1}^T \phi. \quad (\text{B.12})$$

Substituting (B.11) and (B.12) in (B.7), we verify that the structure of the value function as postulated in (B.6) is in fact preserved over time (more strongly, it can be verified that the structure is preserved before letting  $\Delta t \rightarrow 0$ ). Collecting terms and simplifying, we obtain the following triangular representation for the Riccati equations,

$$\frac{d\alpha}{dt} = \frac{1}{\lambda}\alpha(\alpha - \gamma), \quad (\text{B.13})$$

$$\frac{d\beta}{dt} = \frac{1}{\lambda} \left( 2\alpha\beta - \frac{1}{(n+1)^2}(2n\beta + \gamma)^2 \right). \quad (\text{B.14})$$

So far we have assumed a deterministic  $u$ . The objectives for the deterministic case are linear in  $u$ , the corresponding optimal policies do not depend on  $u$ , and the value functions were found to be linear in  $u$ . Using these value functions for the stochastic case, the  $\partial^2 V^i / \partial u^2$  terms in the HJB equations are zero, and all other terms are as in the HJB equations for the deterministic case. That is, if given value functions and optimal policies satisfy the HJB equations for the deterministic case, they also satisfy the HJB equations for the stochastic case.

Conditions for verification of the HJB equation are met (Karatzas and Shreve 1988) as the value function is differentiable in  $t$  and smooth in the state.

While the system of nonlinear equations (B.13) and (B.14) is difficult to solve in closed form, a number of its properties can be studied. As  $T \rightarrow \infty$ , and since the expected values are bounded due to the overall convexity of the problem and the triangular structure precludes oscillatory behavior,  $\alpha(0)$  and  $\beta(0)$  must converge to fixed points of the differential equations. The fixed points for the first equation are 0 and  $\gamma$ , of which only  $\alpha = \gamma$  is stable (note that (B.13) is convex-quadratic in  $\alpha$ , and we are considering integration backwards in time). With  $\alpha = \gamma$ , we solve for the fixed points of  $\beta$ , which are found to be  $\frac{1}{2}\gamma$  and  $\frac{1}{2n^2}\gamma$ , of which only  $\beta = \frac{1}{2n^2}\gamma$  is stable (note that (B.14) is concave-quadratic in  $\beta$ ).

In the case  $n = 1$ , we have  $\phi_i(0)\mathbf{1}^T\phi(0) = (\mathbf{1}^T\phi(0))^2 = \Delta x^2$ , so that

$$V = -U_0\Delta x - \gamma\Delta x^2 + \frac{1}{2}\gamma\Delta x^2 = -U\Delta x - \frac{1}{2}\gamma\Delta x^2. \quad (\text{B.15})$$

As expected, we recover the same value as for the open-loop case (which corresponds to a constant trading rate).

Consider now the case  $n = 2$ , with the distressed trader needing to trade  $\phi_1(0) = \Delta x$ , and the predatory trader's target being  $\phi_2(0) = 0$ . For the distressed trader,  $\phi_1(0)\mathbf{1}^T\phi(0) = (\mathbf{1}^T\phi(0))^2 = \Delta x^2$ . For the predatory trader,  $\phi_2(0)\mathbf{1}^T\phi(0) = 0$  and  $(\mathbf{1}^T\phi(0))^2 = \Delta x^2$ . Table III summarizes the same expected values as in Table I, but for  $T$  large ( $T \rightarrow \infty$  or, equivalently by change of units,  $\lambda$  small, *i.e.*,  $\lambda \rightarrow 0$ ). Overall, the expected values are similar. The loss to the distressed trader is somewhat larger in the closed-loop case, the gain to predatory trader somewhat smaller, and the overall welfare loss somewhat larger. The ratio of gains to the predatory trader per losses to the distressed trader decreases from 1/2 in the open-loop case to 1/3 in the closed-loop case (*cf.* lower bound in Result 3). Cooperation between traders will therefore be more likely in the closed-loop

Table III case.

about  
here

For  $n$  traders, and still under the assumption of a long trading horizon, we can derive the shape of the racing under closed-loop strategies. During the racing stage, that is for small  $t$ ,  $\alpha(t)$  and  $\beta(t)$  are approximately constant (under the assumption of large  $T$  or small  $\lambda$ ). Using the stable fixed-point values  $\alpha = \gamma$  and  $\beta = \frac{1}{2n^2}\gamma$  in (B.11), we obtain

$$y_i = \frac{\gamma}{\lambda} \cdot \frac{n-1}{n} \cdot \frac{\mathbf{1}^T \phi}{n}, \quad i = 1, \dots, n. \quad (\text{B.16})$$

Since  $y_i = -d\phi_i/dt$ , we conclude that, with closed-loop strategies, the racing is of the form

$$y_i = a e^{-\frac{\gamma}{\lambda} \frac{n-1}{n} t}, \quad (\text{B.17})$$

with the constant  $a \in \mathbf{R}$  a function of the average trading target. This is slightly faster than what we found for the open-loop case, which was  $y_i = a e^{-\frac{\gamma}{\lambda} \frac{n-1}{n+1} t}$ .