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Convergence concepts on quasi-probability space

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Abstract. Quasi-probability is a kind of fuzzy measure, its properties will be further discussed in this paper. New convergence concepts for quasi-random variables are then introduced and the relationships among the convergence concepts are investigated. All obtained results are natural extensions of the classical convergence theory to the case where the measure tool is fuzzy.

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1. INTRODUCTION

Convergence concepts are basic and important concepts in classical measure theory [15], [19]. The additivity property of classical measure is in some application contexts too restrictive and, consequently, unrealistic [1], [10], [13]. Therefore, many mathematicians tried to explore some kinds of non-additive measures. Zadeh [20] proposed the concept of possibility measure in 1999; Liu Baoding [11] founded an uncertainty theory in 2007; Wang Zhenyuan [17] discussed quasi-probability measure in 1992, and so on. Quasi-probability is a fuzzy (non-additive) measure and, it is a generalization of probability measure and Sugeno measure. Quasi-probability was widely applied by some scholars [3], [8], [9], [17], [21]. When the measure tool is non-additive, the convergence concepts are very different from additive case [2], [5], [14], [16], [18]. Some mathematicians have explored them such as Gianluca [4], B. Hazarika [6, 7], Liu Baoding [12], [13], Wang Zhenyuan [17], Zhang Zhiming [22], and so forth. For the sake of investigating quasi-probability theory deeper, we will propose in the present paper some new convergence concepts on quasi-probability space, and discuss the relationships among the convergence concepts. Our work helps to build important theoretical foundations for the development of quasi-probability theory.

2. The Definition and Properties of Quasi-Probability Measure

In this paper, let X be a nonempty set and $(X, \mathcal{F})$ be a measurable space. Here $\mathcal{F}$ is a $\sigma-$algebra of X.

Definition 2.1 ([17])**.** Let $\alpha \in (0, +\infty]$, an extended real function is called a T-function iff $\theta : [0, a] \to [0, +\infty]$ is continuous, strictly increasing, and such that $\theta(0) = 0,\ \theta^{-1}(\{\infty\}) = \varnothing$ or $\{\infty\}$, according to a being finite or not.

Definition 2.2. Let $\alpha \in (0, +\infty]$, an extended real function $\theta : [0, a] \to [0, +\infty]$ is called a regular function, if θ is continuous, strictly increasing, and $\theta(0) = 0, \theta(1) = 1$.

Obviously, if θ is a regular function, then θ^{-1} is also a regular function.

Definition 2.3 ([17])**.** μ is called quasi-additive iff there exists a T-function θ , whose domain of definition contains the range of μ , such that the set function $\theta \circ \mu$ defined on $\mathcal{F}$ by $(\theta \circ \mu)(E) = \theta\ [\mu(E)]\ \ (\forall E \in \mathcal{F})$, is additive; μ is called a quasi-measure iff there exists a T-function θ such that $\theta \circ \mu$ is a classical measure on $\mathcal{F}$. The T-function θ is called the proper T-function of μ.

Definition 2.4. Let μ be a quasi-measure on $\mathcal{F}$, if θ is a regular T-function of μ, and $\mu(X) = 1$, then μ is called a quasi-probability. The triplet $(X, \mathcal{F}, \mu)$ is called a quasi-probability space.

Example 2.5. Let μ be a probability measure. From definition 2.3, we know that μ is a quasi-probability with $\theta(x) = x$ as its T-function.

Example 2.6 ([17])**.** Suppose that $X = \{1, 2, \cdots, n\}$, $\rho(X)$ is the power set of X. If

$$\mu(E) = (\frac{|E|}{n})^2,$$

where $|E|$ is the number of those points that belong to E, then μ is a quasi-probability with $\theta(x) = \sqrt{x},\quad x \in [0, 1]$ as its T-function.

Theorem 2.7. *If μ is a quasi-probability, then $\mu(\varnothing) = 0$.*

Proof. μ is a quasi-probability, there exists a T-function θ such that $\theta \circ \mu$ is a classical measure, $(\theta \circ \mu)(\varnothing) = 0$, i.e., $\theta[\mu(\varnothing)] = 0$, it follows from definition 2.2 that $\mu(\varnothing) = 0$. □

Theorem 2.8. *Let μ be a quasi-probability on $\mathcal{F}$. If $A, B \in \mathcal{F}$, and $A \subset B$, then we have $\mu(A) < \mu(B)$.*

Proof. Since $A \subset B$, and there exists a T-function θ such that $\theta \circ \mu$ is a classical measure, we have $(\theta \circ \mu)(A) < (\theta \circ \mu)(B)$. θ is continuous, strictly increasing, it is clear that $\mu(A) < \mu(B)$. □

Theorem 2.9. *Let μ be a quasi-probability on $\mathcal{F}$, then there exists a regular T-function θ such that $\theta \circ \mu$ is a probability on $\mathcal{F}$.*

Proof. μ is a quasi-probability, it follows from definition 2.4 that there exists a regular T-function θ, such that $\theta \circ \mu$ is a classical measure. $\forall A \in \mathcal{F}, \quad (\theta \circ \mu)(A) = \theta(\mu(A)) \geq 0$, and $(\theta \circ \mu)(X) = \theta(\mu(X)) = \theta(1) = 1$. It implies that $\theta \circ \mu$ is a probability measure. □

Theorem 2.10. *If μ be a quasi-probability, then μ is continuous.*

Proof. According to definition 2.4 and theorem 2.9, the continuity of μ is obvious. □

Definition 2.11. Let $(X, \mathcal{F}, \mu)$ be a quasi- probability space and $\xi = \xi(\omega), \omega \in \mathcal{F}$, be a real set function on $\mathcal{F}$. For any given real number x, if $\{\omega |\ \xi(\omega) \leq x\} \in \mathcal{F}$, then ξ is called a quasi-random variable, denoted by q-random variable.

Definition 2.12. The distribution function of q-random variable ξ is defined by

$$F_\mu(\mathrm{x}) = \mu\{\omega \in \mathcal{F} |\ \xi(\omega) \leq \mathrm{x}\}.$$

Let ξ and η be two q-random variables. $\forall x, y \in R$, if

$$\mu(\xi \leq x, \eta \leq y) = \theta^{-1}[(\theta \circ \mu)(\xi \leq x) \cdot (\theta \circ \mu)(\eta \leq y)],$$

then ξ and η are independent q-random variables [8].

The q-random variables $\xi_1,\ \xi_2,\ ...,\ \xi_n$ ... are said to be identically distribution iff

$$\mu\{\xi_i \in B\} = \mu\{\xi_j \in B\},\ i, j = 1, 2, ...,$$

for any Borel set B of $\mathcal{R}$.

3. Convergence Theorems of q-Random Variables Sequence

In the section, two new convergence concepts are proposed, and then the relationships between the two convergence concepts are investigated.

Definition 3.1. Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ is a sequence of q-random variables. If there exists a q-random variable ξ, such that

$$\mu\{\lim_{n\to\infty} \xi_n = \xi\} = 1,$$

then we say that $\{\xi_n\}$ converges with quasi-probability 1 to ξ. Denoted by

$$\lim_{n\to\infty} \xi_n = \xi \quad (\mu - a.s.).$$

Definition 3.2. Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ is a sequence of q-random variables. If there exists a q-random variable ξ, such that $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{|\ \xi_n - \xi\ | \geq \varepsilon\} = 0,$$

namely,

$$\lim_{n\to\infty} \mu\{|\ \xi_n - \xi\ | < \varepsilon\} = 1,$$

then we say that $\{\xi_n\}$ converges in quasi-probability to ξ. Denoted by

$$\lim_{n\to\infty} \xi_n = \xi \qquad (\mu).$$

Lemma 3.3. *Suppose that* $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ *is a sequence of q-random variables,* ξ *is a q-random variable. Then the following propositions are equivalent.*

(1) $\lim_{n\to\infty} \xi_n = \xi \qquad (\mu - a.s.)$;

(2) $\mu\{\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} [|\xi_k - \xi| < \frac{1}{m}]\} = 1$, *and* $\mu\{\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} [|\xi_k - \xi| \geq \frac{1}{m}]\} = 0$;

(3) $\forall \varepsilon > 0,\ \mu\{\bigcup_{n=1}^{\infty} \bigcap_{k\geq n} [|\xi_k - \xi| < \varepsilon]\} = 1,$ *and* $\mu\{\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon\} = 0.$

(4) $\forall \varepsilon > 0,\ \lim_{n\to\infty} \mu\{\bigcap_{k\geq n} [|\xi_k - \xi| < \varepsilon]\} = 1,$ *and* $\lim_{n\to\infty} \mu\{\bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon]\} = 0.$

Proof. The equivalences among propositions (1) to (3) have been proved in [21]. Now we prove that (3) is equivalent to (4). therefor, we only need to pay attention to the following facts: $\forall n$,

$$\bigcap_{k\geq n} [|\xi_k - \xi| < \varepsilon] \subset \bigcap_{k\geq n+1} [|\xi_k - \xi| < \varepsilon], \quad \text{or} \quad \bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon] \supset \bigcup_{k\geq n+1} [|\xi_k - \xi| \geq \varepsilon],$$

it follows from the continuity of quasi-probability measure that

$$\lim_{n\to\infty} \mu\{\bigcap_{k\geq n} [|\xi_k - \xi| < \varepsilon]\} = \mu\{\bigcup_{n=1}^{\infty} \bigcap_{k\geq n} [|\xi_k - \xi| < \varepsilon]\},$$

or

$$\lim_{n\to\infty} \mu\{\bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon]\} = \mu\{\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon]\}.$$

So propositions (3) is equivalent to propositions (4). □

Lemma 3.4 ([21])**.** *Suppose that* X *is a nonempty set,* $\rho(X)$ *is the power set of* X*. Let* $A_k \in \rho(X), c_k = \mu\{A_k\},\ k = 1, 2, ...,$

(1)*If* $\sum_{k=1}^{\infty} \theta(c_k) < \infty,$ *then* $\mu\{\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\} = 0.$

(2) If $\sum_{k=1}^{\infty} \theta(c_k) = \infty$*, and* $A_k, k = 1, 2, ...,$ *are independent, then*

$$\mu\{\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\} = 1.$$

Where θ *is the proper T-function of* μ*.*

Theorem 3.5. *Suppose that* $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ *are independent q-random variables defined on the quasi-probability space* $(X, \mathcal{F}, \mu)$*. Then* $\{\xi_n\}$ *converges with quasi-probability 1 to 0 if and only if* $\forall c \in (0, \infty)$*, there exists a T-function* θ*, such that*

$$\sum_{k=1}^{\infty} (\theta \circ \mu)(|\xi_k| \geq c) < \infty.$$

Proof. By virtue of lemma 3.3,

$$\lim_{n\to\infty} \xi_n = 0 \qquad (\mu - a.s.)$$

if and only if $\forall c > 0$,

$$\mu\{\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} [|\xi_k| \geq c]\} = 0.$$

According to lemma 3.4, we know that theorem 3.5 holds. □

Theorem 3.6. *Suppose that $\{\xi_n\}$ is a sequence of q-random variables, ξ is a q-random variable. If $\{\xi_n\}$ converges with quasi-probability 1 to ξ, then $\{\xi_n\}$ converges in quasi-probability to ξ.*

Proof. $\forall n$,

$$\{|\xi_n - \xi| \geq \varepsilon\} \subset \bigcup_{k\geq n} \{|\xi_k - \xi| \geq \varepsilon\},$$

therefore

$$0 \leq \mu\{|\xi_n - \xi| \geq \varepsilon\} \leq \mu\{\bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon]\}.$$

$\{\xi_n\}$ converges with quasi-probability 1 to ξ, it follows from lemma 3.3 that

$$\lim_{n\to\infty} \mu\{\bigcup_{k\geq n} [|\xi_k - \xi| \geq \varepsilon]\} = 0.$$

Hence we have

$$\lim_{n\to\infty} \mu\{| \xi_n - \xi |\geq \varepsilon\} = 0, \quad \text{namely} \quad \lim_{n\to\infty} \xi_n = \xi \qquad (\mu),$$

it means that $\{\xi_n\}$ converges in quasi-probability to ξ. □

Example 3.7. Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are independent q-random variables defined on the quasi-probability space $(X, \mathcal{F}, \mu)$, θ is the proper T-function of μ. If

$$\mu\{\xi_n = \frac{1}{n}\} = 1 - \frac{1}{n}, \quad \mu\{\xi_n = n+1\} = \frac{1}{n}, \quad n = 1, 2, \cdots,$$

then $\{\xi_n\}$ converges in quasi-probability to 0. However, $\{\xi_n\}$ does not converge with quasi-probability 1 to 0.

In fact, $\forall \varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{|\xi_n| \geq \varepsilon\} = \lim_{n\to\infty} \mu\{\xi_n = n+1\} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

which implies that $\{\xi_n\}$ converges in quasi-probability to 0. But if we denote that

$$A_n = \{\xi_n = n+1\}, \quad c_n = \mu\{A_n\} = \frac{1}{n},$$

then

$$\sum_{n=1}^{\infty} \theta(c_n) = \infty.$$

By virtue of lemma 3.3 and lemma 3.4, we know that $\{\xi_n\}$ does not converge with quasi-probability 1 to 0.

Theorem 3.6 shows that convergence with quasi-probability 1 implies convergence in quasi-probability. Example 3.7 shows that convergence in quasi-probability does not imply convergence with quasi-probability 1. But for independent q-random series, convergence with quasi-probability 1 is equivalent to convergence in quasi-probability.

Theorem 3.8. *If $\{\xi_n\}$ is a sequence of independent q-random variables, then $\sum_{n=1}^{\infty}\xi_n$ converges with quasi-probability 1 if and only if $\sum_{n=1}^{\infty}\xi_n$ converges in quasi-probability.*

Proof. It is sufficient to prove that $\sum_{n=1}^{\infty}\xi_n$ converges in quasi-probability implies convergence with quasi-probability 1. Denoted by $S_n=\sum_{k=1}^{n}\xi_k$. Since $\sum_{n=1}^{\infty}\xi_n$ converges in quasi-probability, $\forall\varepsilon>0$, there exists a positive integer n_ε such that

$$\forall k\geq 1,\quad n\geq n_\varepsilon,\quad \mu\{|S_{n+k}-S_n|>\varepsilon\}<\varepsilon.$$

On the other hand, $\sum_{n=1}^{\infty}\xi_n$ converges with quasi-probability 1 if and only if there exists a non-negative decreasing sequence ε_m, $\lim_{m\to\infty}\varepsilon_m=0$, such that

$$\lim_{m\to\infty}\overline{\lim_{n\to\infty}}\lim_{k\to\infty}\mu\{\max_{1\leq v\leq k}|S_{n+v}-S_n|\geq 2\varepsilon_m\}=0.$$

It is easy to verify that

$$\bigcup_{v=1}^{k}\{\max_{1\leq j\leq v-1}|S_{n+j}-S_n|\leq 2\varepsilon,\quad |S_{n+v}-S_n|>2\varepsilon,\quad |S_{n+k}-S_{n+v}|\leq\varepsilon\}$$
$$\subset\{|S_{n+k}-S_n|>\varepsilon\},$$

furthermore,

$$\{\max_{1\leq j\leq v-1}|S_{n+j}-S_n|\leq 2\varepsilon\},\quad \{|S_{n+v}-S_n|>2\varepsilon\},\ \text{and}\ \{|S_{n+k}-S_{n+v}|\leq\varepsilon\}$$

are disjoint. Since μ is a quasi-probability, there exists a T-function θ such that $\theta\circ\mu$ is a probability. It follows from the independence between

$$\{\max_{1\leq j\leq v-1}|S_{n+j}-S_n|\leq 2\varepsilon,\quad |S_{n+v}-S_n|>2\varepsilon\}$$

and

$$\{|S_{n+k}-S_{n+v}|\leq\varepsilon\}$$

that

$$(\theta\circ\mu)\{\max_{1\leq v\leq k}|S_{n+v}-S_n|>2\varepsilon\}\min_{1\leq v\leq k}(\theta\circ\mu)\{|S_{n+k}-S_{n+v}|\leq\varepsilon\}$$

$$\leq\sum_{v=1}^{k}(\theta\circ\mu)\{\max_{1\leq j\leq v-1}|S_{n+j}-S_n|\leq 2\varepsilon,\ |S_{n+v}-S_n|>2\varepsilon\}(\theta\circ\mu)\{|S_{n+k}-S_{n+v}|\leq\varepsilon\}$$

$$=(\theta\circ\mu)\{\bigcup_{v=1}^{k}\{\max_{1\leq j\leq v-1}|S_{n+j}-S_n|\leq 2\varepsilon,\quad |S_{n+v}-S_n|>2\varepsilon,\quad |S_{n+k}-S_{n+v}|\leq\varepsilon\}\}$$

$$\leq(\theta\circ\mu)\{|S_{n+k}-S_n|>\varepsilon\}.$$

Because $\forall k\geq 1$, $n\geq n_\varepsilon$, $(\theta\circ\mu)\{|S_{n+k}-S_n|>\varepsilon\}<\theta(\varepsilon)$,

$$(\theta\circ\mu)\{\max_{1\leq v\leq k}|S_{n+v}-S_n|>2\varepsilon\}<\frac{\theta(\varepsilon)}{1-\theta(\varepsilon)},\quad \forall k\geq 1,\quad n\geq n_\varepsilon.$$

And

$$(\theta\circ\mu)\{\bigcup_k\{|S_{n+k}-S_n|>2\varepsilon\}\}=(\theta\circ\mu)\{\bigcup_k\{\max_{1\leq v\leq k}|S_{n+v}-S_n|>2\varepsilon\}\}$$

$$=\lim_{k\to\infty}(\theta\circ\mu)\{\max_{1\leq v\leq k}|S_{n+v}-S_n|>2\varepsilon\}<\frac{\theta(\varepsilon)}{1-\theta(\varepsilon)}.$$

It implies that

$$\mu\{\bigcup_k\{|S_{n+k}-S_n|>2\varepsilon\}\}<\theta^{-1}(\frac{\theta(\varepsilon)}{1-\theta(\varepsilon)}).$$

Thus for any positive integer m,

$$\lim_{n\to\infty}\sup\mu\{\bigcup_k\{|S_{n+k}-S_n|>2\varepsilon_m\}\}<\theta^{-1}(\frac{\theta(\varepsilon_m)}{1-\theta(\varepsilon_m)}).$$

Since θ and θ^{-1} are continuous, we have

$$\lim_{m\to\infty}\theta^{-1}(\frac{\theta(\varepsilon_m)}{1-\theta(\varepsilon_m)})=0$$

which proves

$$\lim_{m\to\infty}\overline{\lim_{n\to\infty}}\lim_{k\to\infty}\mu\{\max_{1\leq v\leq k}|S_{n+v}-S_n|\geq 2\varepsilon_m\}=0.$$

This means $\sum_{n=1}^{\infty}\xi_n$ converges with quasi-probability 1.

Now the theorem is proved. □

Example 3.9. Suppose that $\xi_1,\ \xi_2,\ ...,\ \xi_n,\ ...$ are independent q-random variables defined on the quasi-probability space $(X,\mathcal{F},\mu)$, and

$$\mu\{\xi_n=-1\}=\mu\{\xi_n=1\}=\frac{1}{2},\quad n=1,2,\cdots,$$

then series

$$\sum_n\frac{\xi_n}{n^{\alpha}},\quad \alpha\in(\frac{1}{2},1]$$

not only converges in quasi-probability but also converges with quasi-probability 1.

4. Conclusions

This paper proposed two new convergence concepts for quasi-random variables. Firstly, the properties of quasi-probability measure were further discussed. Then the concepts of convergence with quasi-probability 1 and convergence in quasi-probability were introduced. Finally, the relationships between the two convergence concepts were discussed. The investigations helped to build important theoretical foundations for the systematic and comprehensive development of quasi-probability theory.

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