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IDENTIFICATION OF RIEMANNIAN FOLIATIONS ON THE TANGENT BUNDLE VIA SODE STRUCTURE

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ABSTRACT. The geometry of a system of second order differential equations is the geometry of a semispray, which is a globally defined vector field on TM. The metric compatibility of a given semispray is of special importance. In this paper, the metric associated with the semispray S is applied in order to study some types of foliations on the tangent bundle which are compatible with SODE structure. Indeed, sufficient conditions for the metric associated with the semispray S are obtained to extend to a bundle-like metric for the lifted foliation on TM. Thus, the lifted foliation converts to a Riemanian foliation on the tangent space which is adapted to the SODE structure. Particularly, the metric compatibility property of the semispray S is applied in order to induce SODE structure on transversals. Finally, some equivalent conditions are presented for the transversals to be totally geodesic.

1. **Introduction**

Differential geometry of the total space of a manifold's tangent bundle has its roots in various problems like Differential Equations, Calculus of Variations, Mechanics, Theoretical Physics and Biology. Nowadays, it

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is a distinct domain of differential geometry and has important applications in the theory of physical fields and special problems from mathematical biology [1]. This significance, has led to the creation of new concepts and geometric structures, which are specific to TM, such as systems of Second Order Differential Equations (SODE), metric structures, semisprays and nonlinear connections. Actually, investigating these concepts can be regarded as a powerful device for the study of the geometric properties of the tangent bundle.

The geometric theory of a dynamical system is described by a system of second order differential equations, as the geometry can be derived from a special vector field that exists on the tangent bundle of a manifold, see [6, 12, 21, 16, 14]. This vector field is a semispray which is a globally defined vector field on TM. If a semispray is given, then one can associate to it different geometric objects like nonlinear and N-linear connections. Basing on these entities, the differential geometry of the pair $(TM; S)$ can be developed. Such a geometric study given a geometric study of SODE, which appears in the theory of dynamical or mechanical systems.

In 1959, B.Reinhart introduced a particular type of foliations called Riemannian foliations [18]. The existence of a bundle-like metric on a foliated manifold, leads to the creation of Riemannian foliations. This class of foliations are of special importance, because they are the natural setting for generalizing Riemannian geometry to foliated manifolds (refer to [2, 8, 13] for more information). But their analogy to Finsler and Lagrange metrics is not studied extensively yet. In [11], Miernowski and Mozgawa investigated Finslerian foliations. Popescu et al., [15, 17] studied some classes of foliations on a Lagrangian manifold which were related to the Lagrange metric structure.

The main purpose of this paper is to characterize Riemannian foliations on the tangent bundle which are compatible with SODE structure. Indeed, a metric which is compatible with the lifted foliation is constructed by imposing some conditions on the metric which is associated with the semispray S. Thus, this metric extends to a bundle-like metric for the lifted foliation on TM and is called a metric compatible with the lifted foliation. So, the lifted foliation converts to a Riemanian foliation on the tangent space. Also, by inducing the nonlinear connection corresponding to the semispray S on the transverse bundle, a characterization of foliations with totally geodesic transversals is presented.

The structure of the present paper, is as follows: In Section 2 preliminaries are presented. Section 3 is devoted to the investigation of metric compatibility of semisprays. In Section 4 the metric associated with a semispray is applied in order to construct a metric which is compatible with the lifted foliation. An example is also presented which satisfies all the metric compatibility conditions with the lifted foliation. In Section 5 the nonlinear connection which is associated with the semispray S is induced on the transverse bundle. Finally, in Section 6 the Riemannian foliations are defined which are compatible with SODE structure and, at last a characterization of foliations with totally geodesic transversals is presented.

2. Preliminaries

Let M be a real n-dimensional smooth manifold and TM be its tangent bundle with natural projection $\pi : TM \longrightarrow M$ as a submersion. The notion of semispray on the total space TM is related to the second order ordinary differential equation (SODE) on the base manifold M,

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$$

These equations on TM can be written as:

$$\frac{dy^i}{dt} + 2G^i(x, y) = 0, \quad y^i = \frac{dx^i}{dt} \tag{2.1}$$

If $(U, \varphi = x^i)$ and $(V, \psi = \widetilde{x}^i)$ are coordinate charts on M with $U \cap V \neq \emptyset$, and $rank(\frac{\partial \widetilde{x}^i}{\partial x^i}) = n$, the corresponding change of coordinates on TM will be $(\pi^{-1}(U), \varphi = (x^i, y^i))$, $(\pi^{-1}(V), \psi = (\widetilde{x}^i, \widetilde{y}^i))$, in which

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \qquad \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \quad rank(\frac{\partial \widetilde{x}^i}{\partial x^i}) = n$$

with respect to this change of coordinates on TM, the functions $G^i(x, y)$ transform according to:

$$2\widetilde{G}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} 2G^j - \frac{\partial y^i}{\partial x^j} y^j \tag{2.2}$$

On the other hand, the equation (2.1) is the integral curve of the vector field

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

By (2.2), it can be seen that S is a global vector field on TM. It is called a **semispray** on TM and G^i are called the **coefficients** of S.

The semispray S is called **homogeneous of degree** 2 if G^i are homogeneous functions of degree 2, in this case S is called a **spray**.
If the base manifold M is paracompact, then there always exist semisprays on TM. It is well-known that in this case there exists a nonlinear connection on TM. If S is a semispray with the coefficients $G^i(x,y)$, then the functions

$$G^i_j(x,y) = \frac{\partial G^i}{\partial y^j}$$

will be the coefficients of the nonlinear connection N.
It is known that a nonlinear connection N determines a horizontal distribution which is complementary to the vertical distribution. So:

$$T_uTM = H_uTM \oplus V_uTM, \quad \forall u \in TM \tag{2.3}$$

An adapted local basis to the direct decomposition (2.3) is in the form of $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_u$, where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^i_j \frac{\partial}{\partial y^i}$$

The adapted dual basis $\{dx^i, \delta y^i\}$ of the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ has the 1-forms δy^i as follows:

$$\delta y^i = dy^i + G^i_j dx^j.$$

Let $\mathcal{F}$ be a foliation of dimension p and codimension $q = n - p$ on the manifold M. Let D be the tangent distribution to $\mathcal{F}$. The vector fields on M, which are tangent to the leaves of $\mathcal{F}$, are denoted by $\mathcal{X}(\mathcal{F})$. A smooth function f on M is said to be **basic**, if for every $X \in \mathcal{X}(\mathcal{F})$, the derivative Xf of f along X is identically zero. According to [13], the following can be assumed:

Proposition 2.1. *Let $f \in \Omega^0_{bas}(M)$ (the ring of smooth functions). The following properties are equivalent:*
(1) f is basic.
(2) f is constant on each leaf.
(3) In every simple distinguished open set, equipped with distinguished local coordinates $(x^1, \ldots, x^p, x^{p+1}, \ldots, x^{p+q})$, f is a function only of the variables $x^{p+1}, \ldots, x^{p+q}$.

A vector field $X \in \mathcal{X}(M)$ is called **foliate** if, for all $Y \in \mathcal{X}(\mathcal{F})$, the bracket $[X,Y]$ also belongs to $\mathcal{X}(\mathcal{F})$. [Note that in the literature, foliate vector fields are also called basic, base-like, foliated and, projectable]. The set $L(M,F)$ of foliate vector fields is a Lie sub-algebra of $\mathcal{X}(M)$. According to [13] the following proposition can be stated:

Proposition 2.2. *Let $X \in \mathcal{X}(M)$. The following properties are equivalent:*
(1) X is foliate.
(2) If $(\varphi_t)_{t<\varepsilon}$ is the local one- parameter group associated with X, on the neighborhood of an arbitrary point of M, then for all t, the local diffeomorphism φ_t leaves the distribution D invariant.
(3) In every simple distinguished open set, equipped with distinguished local coordinates $(x^1,\dots,x^p,x^{p+1},\dots,x^{p+q})$, the last q components of X, depends only on the variables $x^{p+1},\dots,x^{p+q}$.

A metric g is **bundle-like** for the foliation $\mathcal{F}$ if for any open set U of M and for all vector fields Y and Z on U, which are foliate and perpendicular to the leaves, the function $g(Y,Z)$ is basic on U. When for a given foliation there exists a Riemannian metric g on M which is bundle-like for $\mathcal{F}$, it can be said that $\mathcal{F}$ is a **Riemannian foliation** on (M,g).

3. Metrizability of semisprays

The question of metric compatibility has been investigated in several aspects. Indeed a semispray is called metrizable if the paths of the semispray are just the geodesics of some metric space. The problem of compatibility between a system of second order differential equations and a metric structure on tangent bundle, has been studied by many authors [5, 6, 7, 10, 20] and it is known as one of the Helmholtz conditions from the inverse problem of Lagrangian mechanic [9, 19].
Let S be a semispray which is locally represented as :

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

and let $N = (G^i_j = \frac{\partial G^i}{\partial y^j})$ be the nonlinear connection associated with S. By a generalized Lagrange metric or shortly a **GL-metric** on TM, it is meant a metric $g^v = g_{ij}\delta y^i \otimes \delta y^j$ on VTM, where $\delta y^i = dy^i + G^i_j dx^i$. Hence, this metric can be extended on TM as follows: $g = g_{ij}\delta y^i \otimes \delta y^j + g_{ij}dx^i \otimes dx^j$ where $g^h = g_{ij}dx^i \otimes dx^j$ is a metric on the horizontal distribution HTM. This metric i.e. $g = g^v + g^h$ is called the **Sasaki metric**. The geometry of $(M, g_{ij}(x))$ is called the geometry of a generalized Lagrange space which was studied by R.Miron in [12] and also in [6]. Actually, what is meant by **metric compatibility** is the compatibility of a GL-metric g^v with the given semispray.

Definition 3.1. The **covariant derivative** associated with the semispray $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is defined as

$$\nabla^* : \Gamma(VTM) \longrightarrow \Gamma(VTM)$$

$$\nabla^*(X^i \frac{\partial}{\partial y^i}) := (S(X^i) + G^i_j X^j)\frac{\partial}{\partial y^i}.$$

We see that, for all $f \in C^\infty(TM)$ and $X, Y \in \Gamma(VTM)$, the following properties are satisfied by ∇^*:
$(a) : \nabla^*(\frac{\partial}{\partial y^i}) = G^j_i \frac{\partial}{\partial y^j}$
$(b) : \nabla^*(X + Y) = \nabla^* X + \nabla^* Y$
$(c) : \nabla^* f X = S(f)X + f\nabla^* X$

Note that this covariant derivative is extendable on each d-tensor of degree (p, q). Particularly, for each GL-metric g, the following can be defined:

$$\nabla^* g(X, Y) = S(g(X, Y)) - g(\nabla^* X, Y) - g(X, \nabla^* Y) \tag{3.1}$$

J.Bucataru in [5], stated the following definition for a metric compatibility of a given semispray S:

Definition 3.2. The semispray S is called metric with respect to the metric g if $\nabla^* g = 0$. In this case, g is called the **metric associated with the semispray** S.

Considering (3.1), this condition is locally expressed as

$$S(g_{ij}) = g_{ik}G^k_j + g_{kj}G^k_i. \tag{3.2}$$

As an example, the metric compatibility condition, i.e., the relation (3.2), is verified in both Riemannian and Finsler cases as follows:

Riemannian metrics: Suppose M is an n-dimensional manifold and $g = g(x)$ is a Riemannian metric on M. Then (3.2) becomes

$$y^l \frac{\partial g_{ij}}{\partial x^l} = g_{ik}G^k_j + g_{kj}G^k_i. \tag{3.3}$$

The Levi-Civita connection of g is a linear connection on M. A symmetric linear connection with coefficients $({\Gamma_i}^k{}_j)$ yields a semispray S with

$$G^k = \frac{1}{2}{\Gamma_i}^k{}_j \, y^i y^j$$

The canonical nonlinear connection of this semispray has the coefficients:

$$G_i^k = {\Gamma_i}^k{}_l \, y^l \tag{3.4}$$

By inserting (3.4) in (3.3), the following formula can be obtained:

$$\frac{\partial g_{ij}}{\partial x^l} = g_{ik}{\Gamma_i}^k{}_l + g_{kj}{\Gamma_l}^k{}_j$$

by which one can get the Christoffel symbols. So, the metric compatibility condition (3.2) was verified in the Riemannian case.

Finsler metrics: For a Finsler space $\mathbf{F}^n$, the variational problem of the energy function F^2 determines a system of second order differential equations [6],

$$E_i(F^2) := \frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\frac{\partial F^2}{\partial y^i} = 0\ , \quad y^i = \frac{dx^i}{dt}.$$

As stated before, such systems of second order differential equations determines a semispray. For a Finsler space the semispray is homogeneous so it is a spray, and its integral curves are solutions of the Euler-Lagrange equations. This spray is given by:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}.$$

where the local coefficients $G^i(x,y)$ are given by the following formula:

$$2G^i(x,y) = \frac{1}{2}g^{ij}(x,y)[\frac{\partial^2 F^2}{\partial x^k \partial y^j}(x,y)y^k - \frac{\partial F^2}{\partial x^j}(x,y)].$$

Then the coefficients G_j^i of the nonlinear connection of the spray S are given by

$$\begin{aligned} G_j^i = \frac{\partial G^i}{\partial y^j} &= \frac{1}{4}\frac{\partial g^{ip}}{\partial y^j}(\frac{\partial^2 F^2}{\partial x^m \partial y^p}y^m - \frac{\partial F^2}{\partial x^p}) \\ &+ \frac{1}{4}g^{ip}(\frac{\partial g_{jp}}{\partial x^m}y^m - \frac{\partial^2 F^2}{\partial x^p \partial y^j}) + \frac{1}{4}g^{ip}\frac{\partial^2 F^2}{\partial x^j \partial y^p}. \end{aligned}$$

Lemma 3.3. *Let $\mathbf{F}^n$ be a Finsler space and g_{ij} be the metric tensor of $\mathbf{F}^n$. Then the spray S is metrizable, i.e., in local coordinates, the following relation is assumed:*

$$S(g_{ij}) = g_{im}G_j^m + g_{jm}G_i^m$$

Proof. Refer to [6]. □

4. **Metrics Compatible with the Lifted Foliation**

Let $\mathcal{F}$ be a foliation of dimension p and codimention $q = n - p$ on the manifold M. Then, consider the local coordinates $(x^i) = (x^a, x^\alpha)$ where $a, b, \ldots \in \{1, \ldots, p\}$ and $\alpha, \beta, \ldots \in \{p+1, \ldots, p+q = n\}$. Now, suppose that L_t is a leaf of $\mathcal{F}$ and $\{(U, \varphi) : (x^1, \ldots, x^p, x^{p+1}, \ldots, x^{p+q})\}$ is a foliated chart on the n- foliated manifold $(M, \mathcal{F})$. This means that each plaque P_c^t of $\mathcal{F}$ in U is described by the equations of the form

$$x^{p+1} = c^{p+1}, \ldots, x^{p+q} = c^{p+q}.$$

where $c = (c^{p+1}, \ldots, c^{p+q})$ is a point of $\mathbb{R}^q$. Hence, $\{\frac{\partial}{\partial x^a}\}, a \in \{1, \ldots, p\}$ are vector fields on U which are tangent to each n-dimensional submanifold P_c^t of U. Let $\{(\widetilde{U}, \widetilde{\varphi}) : (\tilde{x}^1, \ldots, \tilde{x}^p, \tilde{x}^{p+1}, \ldots, \tilde{x}^{p+q})\}$ be another foliated chart in a way that $U \cap \widetilde{U} \neq \emptyset$. Assume that P_c^t and $P_{\tilde{c}}^t$ are two plaques in U and $\widetilde{U}$ respectively, in a way that $P_c^t \cap P_{\tilde{c}}^t \neq \emptyset$. As P_c^t and $P_{\tilde{c}}^t$ are the domains of some local charts on the leaf L_t which is a p-dimensional submanifold of M, on $P_c^t \cap P_{\tilde{c}}^t$.

$$\frac{\partial}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b} \quad , \quad a, b, \ldots \in \{1, \ldots, p\}. \tag{4.1}$$

Since $U \cap \widetilde{U}$ is covered by the intersections of plaques of $\mathcal{F}$, it can be deduced that (4.1) is satisfied on the whole $U \cap \widetilde{U}$. On $U \cap \widetilde{U}$ the following is generally assumed:

$$\frac{\partial}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b} + \frac{\partial \tilde{x}^\alpha}{\partial x^a} \frac{\partial}{\partial \tilde{x}^\alpha}.$$

So, according to (4.1), it can be inferred that:

$$\frac{\partial \tilde{x}^\alpha}{\partial x^a} = 0 \quad , \quad \forall\ \alpha \in \{p+1, \ldots, p+q\}\ ,\ a \in \{1, \ldots, p\}.$$

Hence, the coordinate transformations on the n-foliated manifold $(M, \mathcal{F})$ have the following special form:

$$(a)\tilde{x}^a = \tilde{x}^a(x^b, x^\beta) \quad , \quad (b) : \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta).$$

As $\{\frac{\partial}{\partial x^a}\},\ a \in \{1, \ldots, p\}$, are tangent to leaves of $\mathcal{F}$. $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^\alpha}\}$ is called an $\mathcal{F}$-**natural frame field** on $(M, \mathcal{F})$ (refer to [2] for more details). Then, the transformations of $\mathcal{F}$- natural frame fields on $(M, \mathcal{F})$ are given by the relation (4.1) and

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial \tilde{x}^a}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^a} + \frac{\partial \tilde{x}^\beta}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\beta}.$$

Theorem 4.1. *A distribution* $D(\mathcal{F}) = span\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}\}$ *defines canonically a foliation* $\mathcal{F}^*$ *on* TM*, called the natural lift of* $\mathcal{F}$ *to the tangent space* TM*.*

Proof. For any $x \in M$ there exists a local foliated chart $\{(U, \varphi) : (x^a, x^\alpha)\}$ on M in a way that all the submanifolds of U given by $x^\alpha = c^\alpha,\ \alpha \in \{p+1, \dots, p+q\}$, are integral manifolds of D. A chart can be induced: $(\bar{U}, x^a, y^b, x^\alpha, y^\beta)$ on TM where (x^α, y^β) are the transverse coordinates. Let $(\widetilde{U}, \tilde{x}^a, \tilde{y}^b, \tilde{x}^\alpha, \tilde{y}^\beta)$ be another coordinate system on TM. Then, the theorem follows directly from the transformation rule:

$$\tilde{x}^a = \tilde{x}^a(x^b, x^\beta) \quad , \quad \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta)$$
$$\tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} y^b + \frac{\partial \tilde{x}^a}{\partial x^\beta} y^\beta \quad , \quad \tilde{y}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} y^\beta .$$

Taking into account the above mentioned coordinate transformations, two foliations can be deduced as the natural lift of $\mathcal{F}$ to the tangent space TM. These foliations are locally spanned by: $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^a}\}$ and $\{\frac{\partial}{\partial y^a}\}$. □

The semispray $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, determines a nonlinear connection N with local coefficients $G^i_j = \frac{\partial G^i}{\partial y^j}$. N is called the **nonlinear connection of the semispray** S. The nonlinear connection N has the local components as follows:

$$(G^i_j) = \begin{bmatrix} G^a_b & G^\alpha_b \\ G^a_\beta & G^\alpha_\beta \end{bmatrix}$$

Each of the local components $G^a_b, G^\alpha_b, G^a_\beta, G^\alpha_\beta$ has $x^a, x^\alpha, y^b, y^\beta$ as variables.

The nonlinear connection N defines a local base of its horizontal vector fields given by:

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G^b_a \frac{\partial}{\partial y^b} - G^\beta_a \frac{\partial}{\partial y^\beta}$$
$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - G^b_\alpha \frac{\partial}{\partial y^b} - G^\beta_\alpha \frac{\partial}{\partial y^\beta} \qquad (4.2)$$

In [15] Popescu et al. defined the notion of the Lagrangian adapted to the lifted foliation. As follows, the results of that paper are generalized by constructing a metric which is compatible with the lifted foliation via SODE structure.

Definition 4.2. Let $\mathcal{F}$ be a foliation of codimension q on M and $\mathcal{F}^*$ be the natural lift of $\mathcal{F}$ to TM. Let S be a semispray which is locally represented as $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$. The metric g is **compatible with** $(\mathcal{F}^*, S)$ if the following conditions are satisfied:
(1) $S(g_{ij}) = g_{ik}G^k_j + g_{kj}G^k_i$.
(2) $g_{b\beta} = g(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^\beta}) = 0$.
(3) The local functions $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ are basic functions, i.e., they do not depend on the tangent variables (x^a, y^a).

Remark 4.3. Let $T(\mathcal{F}^*)$ be the tangent distribution to the foliation $\mathcal{F}^*$ and VTM be the vertical distribution. The integrable distribution $V\mathcal{F}^*$ on TM is defined as follows

$$V\mathcal{F}^* = T(\mathcal{F}^*) \cap VTM$$

Note that similar to [15], in the above definition condition (2) means that $V\mathcal{F}^*$ is perpendicular to the distribution spanned by the vertical foliated vector fields of the foliation which are locally generated by the vector fields $\{\frac{\partial}{\partial y^\alpha}\}$, $\alpha \in \{p+1,\dots,p+q\}$.

The above definition will be clarified by presenting an example as follows.

Example 4.4. Let M be an $(n = p+q)$-dimensional manifold. Let N and N' be two smooth manifolds and $M = N \times N'$ be their product manifolds. Then M carries two complementary foliations of $\mathcal{F}$ and $\mathcal{F}'$ by copies of N and N' respectively. Let D and D' be their tangent distributions with projection morphisms $\mathbb{P}$ and $\mathbb{P}'$. If ∇ and ∇' are linear connections on N and N', respectively, then a linear connection $\widetilde{\nabla}$ on M can be defined as follows: For any point $x^* = (x, x')$ of M consider the coordinate systems $(x^1,\dots,x^p;U)$ and $(x^{p+1},\dots,x^{p+q};U')$ about $x \in N$ and $x' \in N'$, respectively. Then $(x^1,\dots,x^p,x^{p+1},\dots,x^{p+q};U\times U')$ is a coordinate system about x^*. Suppose $\{\Gamma_b{}^a{}_c\}$, $a,b,c \in \{1,\dots,p\}$ and $\{\Gamma'_\beta{}^\alpha{}_\gamma\}$, $\alpha,\beta,\gamma \in \{p+1,\dots,p+q\}$ are the local coefficients of ∇ and ∇' with respect to the coordinate systems $(x^a;U)$ and $(x^\alpha;U')$, respectively. Then, the local components of $\widetilde{\nabla}$ can be defined with respect to the coordinate system $(x^a, x^\alpha; U\times U')$ as follows:

$$(a)\ \Gamma^*{}_b{}^a{}_c = \Gamma_b{}^a{}_c\ , \quad a,b,c \in \{1,\dots,p\}$$
$$(b)\ \Gamma^*{}_\beta{}^\alpha{}_\gamma = \Gamma'_\beta{}^\alpha{}_\gamma\ , \quad \alpha,\beta,\gamma \in \{p+1,\dots,p+q\}$$

$$(c)\ {\Gamma^*}_{j\ k}^{\ i} = 0\ , \quad \text{for all other triples.}$$

So, $\widetilde{\nabla}$ can be expressed as follows:

$$\widetilde{\nabla}_X Y = (\nabla_{\mathbb{P}X}\mathbb{P}Y, \nabla'_{\mathbb{P}'X}\mathbb{P}'Y)\ , \quad \forall \ \ X, Y \in \Gamma(TM).$$

Let S and S' be the semisprays of N and N' with local coefficients G^a and G'^{α} and assume that $g = [g_{ab}(x^c)]$ and $g' = [g'_{\alpha\beta}(x^\gamma)]$ are the metrics associated with S and S' on N and N' respectively, therefore:

$$S = y^a \frac{\partial}{\partial x^a} - 2G^a(x, y)\frac{\partial}{\partial y^a} \quad , \quad a, b, c \in \{1, \ldots, p\}$$

$$S' = y^\alpha \frac{\partial}{\partial x^\alpha} - 2G'^\alpha(x, y)\frac{\partial}{\partial y^\alpha} \quad , \quad \alpha, \beta, \gamma \in \{p+1, \ldots, p+q\}$$

where $G^a = \frac{1}{2}\Gamma_{b\ c}^{\ a}\, y^a y^b$ and $G'^\alpha = \frac{1}{2}\Gamma'^{\ \alpha}_{\beta\ \gamma}\, y^\beta y^\gamma$. Hence $\widetilde{S}$, the semispray of M is defined as the sum of S and S', i.e. $\widetilde{S} = S + S'$. Then, the metric $\widetilde{g}$ associated with the semispray $\widetilde{S}$ on $M = N \times N'$ can be defined as follows:

$$\widetilde{g}(X, Y) = g(\mathbb{P}X, \mathbb{P}Y) + g'(\mathbb{P}'X, \mathbb{P}'Y)\ , \quad \forall\ X, Y \in \Gamma(TM).$$

From the above relation, the matrix of local components of $\widetilde{g}$ can be inferred and expressed as follows:

$$[\widetilde{g}_{ij}(x^k)] = \begin{bmatrix} g_{ab}(x^c) & 0 \\ 0 & g'_{\alpha\beta}(x^\gamma) \end{bmatrix}$$

The metric $\widetilde{g}$ which has been constructed above, satisfies all the conditions of definition (4.2), hence it can be regarded as an interesting example of a metric which is compatible with the lifted foliation and the semispray $\widetilde{S}$.

Remark 4.5. Let (N_1, F_1) and (N_2, F_2) be two Finsler manifolds with Finsler metrics F_1 and F_2 respectively, and $f : N_1 \times N_2 \longrightarrow \mathbb{R}^+$ be a smooth function. On the product manifold $M = N_1 \times N_2$ the metric

$$F(y^a, v^\alpha) = \sqrt{F_1^2(y^a) + f^2 F_2^2(v^\alpha)}$$

is considered. For all $(y^a, v^\alpha) \in TN_1^0 \times TN_2^0$ and $a \in \{1,\ldots, p\}$, $\alpha \in \{p+1,\ldots, p+q\}$.

The manifold $M = N_1 \times N_2$ endowed with this metric is called the **twisted product** of the manifolds N_1 and N_2 and is denoted by $N_1 \times_f N_2$. The function f will be called the **twisted function**. The Hessian of F with respect to the vector variables is of the form

$$\begin{bmatrix} A & 0 \\ 0 & f^2 B \end{bmatrix}$$

where A and B are the Hessians of F_1 and F_2 respectively. Then the functions

$$(i): \ \mu_{ab}(x,y) = \frac{1}{2}\frac{\partial^2 F_1^2(x,y)}{\partial y^a \partial y^b}, \qquad (ii): \ \eta_{\alpha\beta}(u,v) = \frac{1}{2}\frac{\partial^2 F_2^2(u,v)}{\partial v^\alpha \partial v^\beta}$$

define a Finsler tensor of type (0,2) on TN_1^0 and TN_2^0, respectively.
Now, let $\mathbb{F} = (N_1 \times_f N_2, F)$ be a twisted product Finsler manifold, and let $\mathbf{x} \in M$ and $\mathbf{y} \in T_x M$, where $\mathbf{x} = (x, u)$, $\mathbf{y} = (y, v)$, $M = N_1 \times N_2$ and $T_{\mathbf{x}} M = T_x N_1 \oplus T_u N_2$. Then, using (4.7), (4.8), it can be inferred that:

$$(g_{ij}(x,u,y,v)) = (\frac{1}{2}\frac{\partial^2 F^2(x,u,y,v)}{\partial \mathbf{y}^i \partial \mathbf{y}^j}) = \begin{bmatrix} \mu_{ab}(x,y) & 0 \\ 0 & f^2 \eta_{\alpha\beta}(u,v) \end{bmatrix}$$

where $\mathbf{y}^i = (y^a, v^\alpha)$ and $\mathbf{y}^j = (y^b, v^\beta)$ and $a, b, \cdots \in \{1,\ldots,p\}$, $\alpha, \beta, \cdots \in \{1,\ldots,q\}$ and $i, j, \cdots \in \{1,\ldots,p+q\}$.
If the twisted function f is a basic function, then according to Lemma (3) the example can be extended to the twisted product of Finsler manifolds in order to support the presented idea.

Definition 4.6. Let S be a semispray which is locally represented as $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$. Then, the semispray S is foliated if:

$$(i): G_b^\alpha = \frac{\partial G^\alpha}{\partial y^b} = 0 \quad , \quad (ii): \frac{\partial G^\alpha}{\partial x^b} = 0$$

or

$$(i): G_\alpha^b = \frac{\partial G^b}{\partial y^\alpha} = 0 \quad , \quad (ii): \frac{\partial G^b}{\partial x^\alpha} = 0$$

Theorem 4.7. *Let $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ be a foliated semispray and $N = (G_j^i)$ be the nonlinear connection of S. Then, a p - distribution $H\mathcal{F}^*$ exists on TM which is complementary to $V\mathcal{F}^*$ in $T(\mathcal{F}^*)$, i.e.:*

$$T(\mathcal{F}^*) = V\mathcal{F}^* \oplus H\mathcal{F}^*.$$

where

$$H\mathcal{F}^* = T(\mathcal{F}^*) \cap HTM$$

$$V\mathcal{F}^* = T(\mathcal{F}^*) \cap VTM$$

Proof. Since the semispray S is foliated, $G_a^\beta = 0$. So from relation (4.2), it can be inferred that

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G_a^b \frac{\partial}{\partial y^b}$$

which means that $H\mathcal{F}^*$ is locally spanned by the vector fields $\{\frac{\delta}{\delta x^a}\}$, $a \in \{1,\dots,p\}$. Since $\{\frac{\partial}{\partial y^a}\}$ is a local base in $\Gamma(V\mathcal{F}^*)$. The proof comes to the end. □

Corollary 4.8. *According to the above Theorem, it can be deduced that if the semispray S is foliated then $V\mathcal{F}^*$ and $H\mathcal{F}^*$ induce the nonlinear connection (G_b^a) , $a, b \in \{1,\dots,p\}$, on the leaves of the foliation.*

Lemma 4.9. *Let $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ be a foliated semispray. Then $\{\frac{\delta}{\delta x^\alpha}\}$ are local foliated vector fields.*

Proof. Since the semispray S is foliated, $G_\alpha^b = 0$. So, from (4.2), it can be deduced that:

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - G_\alpha^\beta \frac{\partial}{\partial y^\beta}$$

Also, due to definition (4.6), the following relations can be obtained:

$$\frac{\partial G_\alpha^\beta}{\partial x^a} = \frac{\partial^2 G^\beta}{\partial x^a \partial y^\alpha} = 0$$

$$\frac{\partial G_\alpha^\beta}{\partial y^a} = \frac{\partial^2 G^\beta}{\partial y^a \partial y^\alpha} = 0$$

Thus G_α^β does not depend on the tangent variables (x^a, y^a), so by definition and due to propositions (1) and (2), the proof is complete. □

5. Induced non-linear connection on the transverse bundle

Let $\mathcal{F}$ be a foliation of dimension p and co-dimension $q = n - p$ on the manifold M. Consider the local coordinates $(x^i) = (x^a, x^\alpha)$ where $a, b,\ldots \in \{1,\ldots,p\}$ and $\alpha, \beta,\ldots \in \{p+1,\ldots,p+q = n\}$. Let $Q = TM/T(\mathcal{F})$ be the transverse bundle of $\mathcal{F}$ and let $\tilde{M}$ be a transverse submanifold (transversal) of M. By definition $\tilde{M}$ is a q dimensional immersed submanifold of the n-dimensional manifold M. Let $i : \tilde{M} \longrightarrow$

M be an immersion and $i(u) = (x^1(u),\dots,x^n(u))$, where $u = (u^1,\dots,u^q)$ and x^i, $i \in \{1,\dots,n\}$ are smooth functions (C^∞). So

$$\begin{array}{rcl} i_* : T\tilde{M} & \longrightarrow & TM \\ (x^\alpha, y^\alpha) & \longmapsto & (x^i(u), y^i(u,v)) \end{array}$$

and

$$(a)\ y^i(u,v) = B^i_\alpha y^\alpha, \quad (b)\ B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$$

$$(c)\ B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\ , \quad (d)\ B^i_{\alpha 0} = B^i_{\alpha\beta} v^\beta.$$

Let S be a semispray which is locally represented as $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ and let $N = (G^i_j = \frac{\partial G^i}{\partial y^j})$ be the nonlinear connection associated with S. Assume that S is a metrizable semispray, due to Bucataru's definition, i.e. there exists a metric g in a way that the relation (3.2) holds.

Remark 5.1. Note that throughout this section, by g we mean, the metric which is obtained from the metric compatibility of the semispray S.

The metric $(g_{ij}(x))$ on TM induces a Riemannian metric $\tilde{g}$ on $T\tilde{M}$ such that

$$\tilde{g}_{\alpha\beta}(u) = g_{ij}(x(u)) B^{ij}_{\alpha\beta}$$

where $B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta$. The natural local frame fields and coframe fields on M' and $\tilde{M}'$ are related by the following relations

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{\alpha 0}\frac{\partial}{\partial y^i} \quad , \quad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i},$$

$$dx^i = B^i_\alpha du^\alpha \quad , \quad dy^i = B^i_{\alpha 0} du^\alpha + B^i_\alpha dv^\alpha.$$

Let G be the Sasakian metric on M', where

$$G = g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j$$

for $\delta y^i = dy^i + G^i_j dx^j$. With this metric, the following can be defined:

$$N^\beta_\alpha = \tilde{g}^{\beta\gamma} G(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\gamma})$$

It can be proved that $N = (N_\alpha^\beta)$ is a non-linear connection on $\tilde{M}'$, cf.[4]. (N_α^β) is called the **induced nonlinear connection**. $(\tilde{M}, N_\alpha^\beta)$ is called the **SODE submanifold** of M.
Now, the following is defined:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta}$$

The horizontal vector bundle is denoted by $H\tilde{M}' = (\frac{\delta}{\delta u^\alpha})$.

Proposition 5.2. *The local frame field* $\frac{\delta}{\delta u^\alpha}$ *of the induced nonlinear connection can be written as*

$$\frac{\delta}{\delta u^\alpha} = B_\alpha^i \frac{\delta}{\delta x^i} + H_\alpha^a B_a$$

where $H_\alpha^a = B_i^a(B_{\alpha 0}^i + B_\alpha^j G_j^i)$.

Proof. cf. [4]. □

Now, let $(M, \mathcal{F})$ be a foliated manifold. Then, with respect to the metric g, the following decomposition of TM can be stated:

$$TM = T(\mathcal{F}) \oplus T(\mathcal{F})^\perp$$

6. Riemannian Foliations Compatible with SODE Structure

Let g be the metric compatible with $(\mathcal{F}^*, S)$ (definition 4.2). By applying g a metric g^* on TM can be defined as follows:

$$g_{IJ}^*(x,y) = \begin{bmatrix} g_{ij}(x,y) & 0 \\ 0 & g_{ij}(x,y) \end{bmatrix} \quad I, J \in \{1, \dots, 2n\} \ , \ i, j \in \{1, \dots, n\}.$$

This means that with respect to the frame field $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ which is locally defined on TM, the following can be stated:

$$g^*(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g^*(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij} \ , \tag{6.1}$$

$$g^*(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0. \tag{6.2}$$

Theorem 6.1. *Let* $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *be a foliated semispray and let* g *be a metric which is compatible with* $(\mathcal{F}^*, S)$. *Then* g^* *is a bundle-like metric and* $\mathcal{F}^*$ *is a Riemannian foliation on* TM.

Proof. As it is shown in Lemma (5), $\{\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial y^{\beta}}\},\ \alpha, \beta \in \{p+1, \ldots, p+q\}$, is a local base of foliated vector fields for the foliation $\mathcal{F}^*$. Furthermore:

$$g^*(\frac{\delta}{\delta x^{\alpha}}, \frac{\delta}{\delta x^{\beta}}) = g^*(\frac{\partial}{\partial y^{\alpha}}, \frac{\partial}{\partial y^{\beta}}) = g_{\alpha\beta}\ , \tag{6.3}$$

$$g^*(\frac{\delta}{\delta x^{\alpha}}, \frac{\partial}{\partial y^{\beta}}) = 0. \tag{6.4}$$

since $g_{\alpha\beta}$ is a basic function, by definition it can be inferred that g^* is a bundle-like metric for the foliation $\mathcal{F}^*$. Hence, $\mathcal{F}^*$ is a Riemannian foliation on TM. □

$(\mathcal{F}^*, g^*)$ is called the **compatible Riemannian foliation with SODE structure** on TM. By applying the metric g^*, the following decomposition can be obtained:

$$TT(M) = T(\mathcal{F}^*) \oplus T(\mathcal{F}^*)^{\perp}$$

By $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ the projection morphism can be denoted on $T(\mathcal{F}^*)$ and $T(\mathcal{F}^*)^{\perp}$ respectively. According to Theorem (1.5.1) of [2], there exists a unique linear connection ∇ (resp. $\nabla^{\perp}$) with respect to the above decomposition. ∇ and $\nabla^{\perp}$ are called the **intrinsic connections** on $T(\mathcal{F}^*)$ and $T(\mathcal{F}^*)^{\perp}$, respectively. Next, the Levi-Civita connection $\widetilde{\nabla}$ on (TM, g^*) is considered, then, according to [2], the following can be presented:

$$(a) : \nabla_X \mathbb{P}Y = \mathbb{P}\widetilde{\nabla}_{\mathbb{P}X}\mathbb{P}Y + \mathbb{P}[\widetilde{\mathbb{P}}X, \mathbb{P}Y]$$
$$(b) : \nabla^{\perp}_X \widetilde{\mathbb{P}}Y = \widetilde{\mathbb{P}}\widetilde{\nabla}_{\widetilde{\mathbb{P}}X}\widetilde{\mathbb{P}}Y + \widetilde{\mathbb{P}}[\mathbb{P}X, \widetilde{\mathbb{P}}Y].$$

We call $h : \Gamma(T(\mathcal{F}^*)) \times \Gamma(T(\mathcal{F}^*)) \to \Gamma(T(\mathcal{F}^*)^{\perp})$ and $\widetilde{h} : \Gamma(T(\mathcal{F}^*)^{\perp}) \times \Gamma(T(\mathcal{F}^*)^{\perp}) \to \Gamma(T(\mathcal{F}^*))$, given by:

$$h(\mathbb{P}X, \mathbb{P}Y) = \widetilde{\mathbb{P}}\widetilde{\nabla}_{\mathbb{P}X}\mathbb{P}Y$$

$$\widetilde{h}(\widetilde{\mathbb{P}}X, \widetilde{\mathbb{P}}Y) = \mathbb{P}\widetilde{\nabla}_{\widetilde{\mathbb{P}}X}\widetilde{\mathbb{P}}Y \tag{6.5}$$

the **second fundamental forms** of $T(\mathcal{F}^*)$ and $T(\mathcal{F}^*)^{\perp}$ respectively.

Here, a linear connection known as Vránceanu connection is introduced. The connection plays a fundamental role in foliated manifolds with bundle-like metrics [2, 3]. Vránceanu connection, $\widehat{\nabla}$ on $(TM, g^*, \mathcal{F}^*)$ is defined as follows:

$$\widehat{\nabla}_X Y = \nabla_X \mathbb{P}Y + \nabla^{\perp}_X \widetilde{\mathbb{P}}Y \tag{6.6}$$

where ∇ and $\nabla^{\perp}$ are the intrinsic connections on $T(\mathcal{F}^*)$ and $T(\mathcal{F}^*)^{\perp}$ respectively, or by:

$$\widehat{\nabla}_X Y = \mathbb{P}\widetilde{\nabla}_{\mathbb{P}X}\mathbb{P}Y + \widetilde{\mathbb{P}}\widetilde{\nabla}_{\widetilde{\mathbb{P}}X}\widetilde{\mathbb{P}}Y + \mathbb{P}[\widetilde{\mathbb{P}}X, \widetilde{\mathbb{P}}Y] + \widetilde{\mathbb{P}}[\mathbb{P}X, \widetilde{\mathbb{P}}Y]$$

for any $X, Y \in \Gamma(TTM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection.

According to [3], Proposition 1.1, the following theorem can be stated:

Theorem 6.2. *Let $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ be a foliated semispray. Assume that $\widehat{\nabla}$ is the Vránceanu connection on $(TM, g^*, \mathcal{F}^*)$, then:*

$$(\widehat{\nabla}_{\mathbb{P}X}\ g^*)(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) = 0, \qquad \forall \ \ X, Y, Z \in \Gamma(TTM).$$

This means that the induced metric g^* on $T(\mathcal{F}^*)^{\perp}$ is parallel with respect to the Vránceanu connection $\widehat{\nabla}$.

Proof. Since the semispray S is foliated due to Theorem (6.1), it can be deduced that the metric g^* on TM is bundle-like for the foliation $\mathcal{F}^*$. So the induced Riemannian metric on $T(\mathcal{F}^*)^{\perp}$ (denoted by the same symbol g^*) is parallel with respect to the intrinsic connection $\nabla^{\perp}$, thus for all $X, Y, Z \in \Gamma(TTM)$, we have

$$\begin{aligned}(\nabla^{\perp}_X g^*)(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) &= X(g^*(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z)) - g^*(\nabla^{\perp}_X\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) \\ &\quad -g^*(\widetilde{\mathbb{P}}Y, \nabla^{\perp}_X\widetilde{\mathbb{P}}Z) \\ &= 0.\end{aligned} \tag{6.7}$$

now by using (6.6) and (6.7) the proof is complete. □

Lemma 6.3. *Let $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ be a foliated semispray. Consider $(TM, g^*, \mathcal{F}^*)$ (as in theorem (6.1)), then the second fundamental form $\widetilde{h}$ of $T(\mathcal{F}^*)$ is given by*

$$\widetilde{h}(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z) = \frac{1}{2}\mathbb{P}[\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z] \ , \quad \forall \ X, Y \in \Gamma(TTM). \tag{6.8}$$

Proof. Since g^* is a bundle-like metric for the foliation $\mathcal{F}^*$ (Theorem (6.1)), the Levi-Civita connection $\widetilde{\nabla}$ on (TM, g^*) satisfies the following equality:

$$2g^*(\widetilde{\nabla}_{\widetilde{\mathbb{P}}Y}\widetilde{\mathbb{P}}Z, \mathbb{P}X) = g^*([\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z], \mathbb{P}X) \ , \quad \forall X, Y, Z \in \Gamma(TTM) \tag{6.9}$$

Next, by using (6.5) and (6.9) the following can be obtained

$$g^*(\widetilde{h}(\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z), \mathbb{P}X) = g^*(\mathbb{P}\nabla_{\widetilde{\mathbb{P}}Y}\widetilde{\mathbb{P}}Z, \mathbb{P}X) = \frac{1}{2}g^*(\mathbb{P}[\widetilde{\mathbb{P}}Y, \widetilde{\mathbb{P}}Z], \mathbb{P}X)$$

which proves the relation (6.8). □

Theorem 6.4. *Let* $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *be a foliated semispray. Consider* $(TM, g^*, \mathcal{F}^*)$ *as stated in Theorem* (6.1)*, then* $T(\mathcal{F}^*)^\perp$ *is an integrable distribution if and only if the second fundamental form* $\widetilde{h}$ *of* $T(\mathcal{F}^*)^\perp$ *vanishes identically on* M.

Proof. Taking into account the relation (6.8) the proof is gotten. □

Corollary 6.5. *Let* $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *be a foliated semispray. Then* $T(\mathcal{F}^*)^\perp$ *is a totally geodesically distribution i.e. any leaf of* $T(\mathcal{F}^*)^\perp$ *is totally geodesic immersed in* $(TM, g^*, \mathcal{F}^*)$ *if and only if* $T(\mathcal{F}^*)^\perp$ *is an integrable distribution.*

Taking into account Theorem (6.4) and due to Theorem (1.5.9) of [2] the following theorem can be presented:

Theorem 6.6. *Let* $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *be a foliated semispray. Consider* $(TM, g^*, \mathcal{F}^*)$ *(as in Theorem* (6.1)*, then* $T(\mathcal{F}^*)^\perp$ *is an integrable distribution if and only if* g^* *is parallel with respect to the intrinsic connection* $\nabla^\perp$ *on* $T(F^*)^\perp$.

As the final result by combining Theorems (5.1), (6.4) and (6.6) the following theorem can be stated. Indeed, a characterization for the foliations with totally geodesic transversals is obtained.

Theorem 6.7. *Let* $(M, \mathcal{F})$ *be a foliated manifold and* $\mathcal{F}^*$ *be the natural lift of* $\mathcal{F}$ *to* TM*. Let* $S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *be a foliated semispray and* g *be a metric compatible with* $(\mathcal{F}^*, S)$*. Consider* $(TM, \mathcal{F}^*, g^*)$*, where* g^* *is the metric defined by* (6.1) *and* (6.2)*. Assume that* $T(\mathcal{F})^\perp$ *(the transverse bundle of* $\mathcal{F}$*) is integrable. Then the following assertions are equivalent:*

(1) g^* *is parallel with respect to the intrinsic connection* $\nabla^\perp$ *on* $T(F^*)^\perp$.
(2) $H^a_\alpha = B^a_i(B^i_{\alpha 0} + B^j_\alpha G^i_j) = 0$.
(3) $T(F^*)^\perp$ *is an integrable distribution.*
(4) $T(F)^\perp$ *is totally geodesic.*

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References

[1] P. L. Antonelli and R .H. Bradbury, Volterra Hamilton Models in the Ecology and Evolution of Colonial Organisms, World Scientific Press, Singapore, 1994.
[2] A. Bejancu and H. R. Farran, Foliations and Geometric Structures, Springer, Dordrecht, 2006.
[3] A. Bejancu and H. R. Farran, Vranceanu Connections and Foliations with Bundle-Like Metrics, *Proc. Indian Acad. Sci. Math. Sci.* **118** (2008), no. 1, 99–113.
[4] A. Bejancu and H. R. Farran, Geometry of Pseudo-Finsler Submanifolds, Kluwer Academic Publishers, Dordrecht, 2000.
[5] I. Bucataru, Metric nonlinear connections, *Differntial Geom. Appl.* **25** (2007), no. 3, 335–343.
[6] I. Bucataru and R. Miron, Finsler-Lagrange Geometry. Applications to dynamical systems, Editura Academiei Române, Bucharest, 2007.
[7] M. Crasmareanu, Metrizable systems of autonomous second order differential equations, *Carpathian J. Math.* **25** (2009), no. 2, 163–176.
[8] D. Gromoll and G. Walschap, Metric Foliations and Curvature, *Progress in Mathematics*, Birkhäuser Verlag, Basel, 2009.
[9] O. Krupková, Variational metric structures, *Publ. Math. Debrecen* **62** (2003), no. 3-4, 461–498.
[10] B. Lackey, A model of trophodynamics, *Nonlinear Anal.* **35** (1999), no. 1, 37–57.
[11] A. Miernowski and W. Mozgawa, Lift of Finsler foliation to its normal bundle, *Differential Geom. Appl.* **24** (2006), no. 2, 209–214.
[12] R. Miron and M. Anastasiei, The Geometry of Lagrange Spaces, *Theory and Applications*, Kluwer Academic Publishers Group, Dordrecht, 1994.
[13] P. Molino, Riemannian Foliations, *Progress in Mathematics* **73**, Birkhäuser Boston, Inc., Boston, 1988.
[14] Z. Muzsnay, The Euler-Lagrange PDE and Finsler metrizability, *Houston J. Math.* **32** (2006), no. 1, 79–98.
[15] M. Popescu and P. Popescu, Projectable Non-Linear Connections and Foliations, *Proc. Summer School on Differential Geometry, Univ. Comibra,* Comibra, 1999.
[16] M. Popescu and P. Popescu, Projectable on-linear connections on submanifolds and on distributions on manifolds, Conference on Applied Differential Geometry, General Relativity, Workshop on Global Analysis, Differential Geometry, Lie Algebras August 27 - September 1, 2000, Aristotle University of Thessaloniki, Greece, BSG Proceedings 9, 107–112.
[17] P. Popescu and M. Popescu, Lagrangians adapted to submersions and foliations, *Differential Geom. Appl.* **27** (2009), no. 2, 171–178.

[18] B. L. Reinhart, Foliated manifolds with bundle-like metric, *Ann. of Math.* **69** (1959) 119–132.

[19] W. Sarlet, The Helmholtz conditions revisited, a new approach to the inverse problem of Lagrangian dynamics, *J. Phys.* **15** (1982), no. 5, 1503–1517.

[20] J. Szilasi and Sz. Vattamány, On the Finsler-metrizabilities of spray manifolds, *Period. Math. Hungar.* **44** (2002), no. 1, 81–100.

[21] J. Szilasi and Z. Muzsnay, Nonlinear connections and the problem of metrizability, *Publ. Math. Debrecen* **42** (1993), no. 1-2, 175–192.

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