Point Symmetries of Generalized Toda Field Theories II Symmetry Reduction

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Abstract

The Lie symmetries of a large class of generalized Toda field theories are studied and used to perform symmetry reduction. Reductions lead to generalized Toda lattices on one hand, to periodic systems on the other. Boundary conditions are introduced to reduce theories on an infinite lattice to those on semi-infinite, or finite ones.

Résumé

Les symétries de Lie d'une grande classe de théories de champs de Toda sont étudiées et utilisées pour faire des réductions par symétries. D'une part, ces réductions nous donnent des treillis de Toda généralisés et, d'autre part, des systèmes périodiques. Nous utilisons des conditions frontières pour réduire les théories définies sur un réseau infini à des cas finis ou semi-infinis.

1 Introduction

In a recent article [1] we determined the Lie point symmetries of a class of equations that we called "generalized Toda field theories". They all involved various types of exponentials and had the form

$$u_{n,xy} = F_n, \qquad F_n = \sum_{m=n-n_1}^{n+n_2} K_{nm} \exp\left(\sum_{l=m-n_3}^{m+n_4} H_{ml} u_l\right), \tag{1.1}$$

where K and H are some real constant matrices and $n_1, \ldots, n_4$ are some finite non-negative integers. The symmetries were obtained independently for three different ranges of the variable n. Namely, we considered the infinite case $-\infty < n < \infty$, the semi-infinite one, $1 \leq n < \infty$, and also the finite case, $1 \leq n \leq N < \infty$.

In the infinite and semi-infinite cases eq. (1.1) was treated as a differential-difference equation. If the range of n is finite, eq. (1.1) is simply a system of N differential equations for N unknowns u_n.

A sizable literature exists on symmetries of difference equations [1]-[19]. Different approaches differ in their treatment of independent variables and also in the degree of generalization of the concept of "point symmetries" that is involved in passing from differential equations to difference ones.

In Ref. [1] we adopted the "differential equation" approach proposed earlier [5]. Essentially, eq. (1.1) was treated as a system of infinitely many differential equations for infinitely many fields $u_n(x, y)$ with $-\infty < n < \infty$, or $1 \leq n < \infty$, respectively.

The purpose of this article is to investigate applications of the obtained Lie point symmetries. In particular we shall show how one can perform various types of symmetry reduction, using these symmetries. The reductions will be from infinite systems to semi-infinite, finite or periodic ones. We will also consider reductions of the number of independent variables, both continuous and discrete. Each time a reduction is performed the resulting equations will inherit some of the symmetries of the original system. We will show how to obtain the "inherited" symmetry group and will compare it with the entire Lie point symmetry group of the reduced system.

The symmetries are point ones, in the sense that we assume that the symmetry algebra, i.e. the Lie algebra of the symmetry group, is realized by vector fields of the form

$$\hat{v} = \xi(x, y, \{u_k\})\partial_x + \eta(x, y, \{u_k\})\partial_y + \sum_j \phi_j(x, y, \{u_k\})\partial_{u_j}, \tag{1.2}$$

where $\{u_k\}$ denotes the set of all fields (with k in an infinite, semi-infinite, or finite range, repectively). The summation in eq. (1.2) is also over the appropriate range of values of j.

Thus, if eq. (1.1) is considered as an equation on a lattice, the coefficients of the vector field $\hat{v}$ can depend on the values of u_k at all points of the lattice.

Whatever the range of the discrete *variable* n, the range of the *interaction* F_n in eq. (1.1) is assumed to be finite, i.e. the integers $n_1, \ldots, n_4$ are finite. Obviously, this allows much more general interactions than nearest neighbour ones. Indeed eq. (1.1) is general enough to include all Toda field theories that, to our knowledge, occur in the literature, be they integrable or not.

Thus, if we put $H_{n,n-1} = -H_{nn} = 1$ and $K_{nn} = -K_{nn+1} = 1$, and all the other components of H and K equal to zero, we obtain the "two-dimensional Toda lattice"

$$u_{n,xy} = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}}, \quad -\infty < n < \infty \tag{1.3}$$

originally introduced by Mikhailov [20] and Fordy and Gibbons [21] and studied further in Ref. [22].

Other well known Toda systems (with nearest neighbour interactions) are obtained if we set $H_{ml} = \delta_{ml}$

$$u_{n,xy} = \sum_{m=n-n_1}^{n+n_2} K_{nm} \exp u_m, \tag{1.4}$$

or vice versa $K_{nm} = \delta_{nm}$

$$u_{n,xy} = \exp \sum_{l=n-n_3}^{n+n_4} H_{nl} u_l. \tag{1.5}$$

All of these systems have been studied from the point of view of their integrability and solutions [20]-[33], usually for a finite number of fields and usually for K, or respectively H, the Cartan matrix of a simple, or affine, Lie algebra.

We shall call the theories (1.3), (1.4) and (1.5) type I, II and III, respectively. We shall use the $n \to \infty$ generalization of an $sl(n+1,\mathbb{C})$ Cartan matrix, i.e. put $K_{n-1n} = K_{n+1n} = -1$, $K_{nn} = 2$, $K_{nm} = 0$ for $m \neq n,\ n \pm 1$ in eq. (1.4) for $-\infty < n < \infty$. The same will be chosen for H in eq. (1.5).

We note that the equations (1.4) and (1.5) are equivalent if the range of n is finite and the matrices H and K are invertible. Indeed, if $u_n(x,y)$ satisfies eq. (1.4), then $w = K^{-1}u$ satisfies eq. (1.5) with $H = K$, and vice-versa. For $-\infty < n < \infty$, or $1 \leq n < \infty$, this is no longer the case and their symmetry groups, in general, are different.

The original Toda lattice [34, 35] is obtained from eq. (1.3) by symmetry reduction, using translational invariance, i.e. looking for solutions invariant under translations generated by

$$P = \partial_x - \partial_y, \quad u_n(x,y) = u_n(t), \quad t = x + y. \tag{1.6}$$

Here and below we use the letter P generically to denote a translation in some direction, specifying the direction in each case. Similarly, D will denote a generic dilation. Its actual form will be specified in each case.

Since the matrices H and K are constant, the same reduction takes eq. (1.1) into the generalized Toda lattice

$$\ddot{u}_n = F_n, \qquad F_n = \sum_{m=n-n_1}^{n+n_2} K_{nm} \exp\left(\sum_{l=m-n_3}^{m+n_4} H_{ml} u_l\right). \tag{1.7}$$

In Section 2, we study the case where the range of n in eq. (1.1) is infinite. We give the symmetry algebra L_f of the general Toda field theory (1.1) as calculated in [1]. The same results concerning the particular theories of types I, II and III are also given. Eq. (1.7) inherits a subgroup of the symmetry group of eq. (1.1). The Lie algebra L_0 of the "inherited symmetry group" will be the normalizer of the vector field $P = \partial_x - \partial_y$ in L_f

$$\mathrm{nor}_{L_f} P = \{x \in L_f | [x,P] = \lambda P\},\ \lambda \in \mathbb{R}. \tag{1.8}$$

Commuting P with a general element of the algebra L_f, imposing the above normalizer condition and letting the resulting vector field act on functions of t and u_n we obtain the inherited symmetry algebra of the Toda lattice. We establish that the entire symmetry algebra of the generalized Toda lattice (1.7) coincides with the one inherited from L_f. From this general result, the symmetry algebras of the Toda lattices corresponding to the types I, II and III are obtained explicitely.

In Section 3, we restrict the range of the discrete variable to be $1 \leq n < \infty$. This means that starting from some value n_0 of n the eq. (1.1) and (1.7) will be the same as in the infinite case but for $1 \leq n < n_0$ they will be modified. Their actual form will depend on the choice of the matrices K and H. Our procedure will be the same in all cases. We start from the already established symmetry algebra in the case of infinitely many fields. Its prolongation will annihilate all the equations in the system that are not modified by the boundary conditions. We apply the prolonged vector field to the equations for $1 \leq n < n_0$ and require that these equations also be annihilated on the solution set. This will provide us with a subgroup of the original symmetry group which is the symmetry group of the semi-infinite Toda field theory. It must then be checked whether this "inherited" symmetry group is indeed the entire symmetry group of the corresponding semi-infinite system, or only a subgroup of it.

In Section 4, we further restrict the range of the variable n and consider Toda field theories and Toda lattices with a finite number of fields, $1 \leq n \leq N$. We shall proceed as in the semi-infinite case, that is start from the symmetries of Toda field theories, or Toda lattices, with $-\infty < n < \infty$. We then impose that the general symmetry generator should also annihilate the modified equations at the beginning and end of the chain. This is equivalent to starting from a semi-infinite Toda theory and requiring that the symmetry generator should also annihilate the equation for $n = N$.

In Section 5, we view the relation between infinite and periodic (generalized) Toda systems as symmetry reduction. Indeed, let us consider the Toda field theory (1.1) for $n \in \mathbb{Z}$ and its symmetry algebra. In order to be able to impose periodicity

$$u_{n+N}(x, y) = u_n(x, y), \tag{1.9}$$

the symmetry algebra of the generalized Toda field theories or of the generalized Toda lattices, should be enlarged by adding the operator

$$\hat{N} = \partial_n \tag{1.10}$$

to the corresponding symmetry algebra, whenever possible. This operator generates shifts ("translations" in n), though the corresponding translation group parameter must be by an integer, $\tilde{n} = n + \lambda, \quad \lambda \in \mathbb{Z}$. Imposing periodicity, as in eq. (1.9), means that $\hat{N}$ must be removed from the symmetry algebra. The symmetry algebra of the periodic system, inherited from the infinite one, will be the normalizer of $\hat{N}$ in the original algebra. The operator $\hat{N}$ of eq. (1.10) can also be used to reduce differential equations on lattices to differential-delay equations [5].

In section 6, we study a further application of the symmetry group. We look at symmetry reductions involving both discrete and continuous variables.

2 Symmetries of infinite generalized Toda field theories and lattices

2.1 General results

Let us now consider the differential-difference eq. (1.1) and for $-\infty < n < \infty$ impose that the matrices H and K are band matrices with finite bands of constant width:

$$H_{nm} = H_{n,n+\sigma} = \begin{cases} h_\sigma(n) & \sigma \in [p_1, p_2] \\ 0 & \sigma \notin [p_1, p_2] \end{cases}, \ h_{p_1}(n) \neq 0, \ \ h_{p_2}(n) \neq 0, \quad p_1 \leq p_2. \tag{2.1}$$

Similarly,

$$K_{nm} = K_{m+\sigma,m} = \begin{cases} k_\sigma(m) & \sigma \in [q_1, q_2] \\ 0 & \sigma \notin [q_1, q_2] \end{cases}, \ k_{q_1}(m) \neq 0, \ \ k_{q_2}(m) \neq 0, \quad q_1 \leq q_2. \tag{2.2}$$

We introduce the quantity ρ_n which, as a function of n, is any particular solution of the inhomogeneous linear finite-difference equation

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\rho_{\sigma+n} = 1 \tag{2.3}$$

and the quantities ψ_n^j, $j = 1, \ldots, p_2 - p_1$ and ϕ_m^l, $l = 1, \ldots, q_2 - q_1$, which are, respectively, linearly independent solutions of the homogeneous linear equations

$$\sum_{\sigma=p_1}^{p_2} h_\sigma(n)\ \psi_{\sigma+n} = 0, \quad \sum_{\sigma=q_1}^{q_2} k_\sigma\ (m)\ \ \phi_{\sigma+m} = 0. \tag{2.4}$$

Without proof we present the following result [1].

Theorem 1 *The symmetry algebra of the infinite Toda field theory (1.1) with $-\infty < n < \infty$ and H and K satisfying (2.1) and (2.2) has a basis consisting of the following vector fields*

$$X = \xi(x)\partial_x + \eta(y)\partial_y - (\xi_x + \eta_y) \sum_{n=-\infty}^{\infty} \rho_n \partial_{u_n}, \tag{2.5a}$$

$$V_j = (r_j(x) + s_j(y)) \sum_{n=-\infty}^{\infty} \psi_n^j \partial_{u_n}, \tag{2.5b}$$

$$Z_{jl} = \left(\sum_{m=-\infty}^{\infty} \phi_m^l u_m \right) \left(\sum_{n=-\infty}^{\infty} \psi_n^j \partial_{u_n} \right), \tag{2.5c}$$

$$j = 1, \ldots, p_2 - p_1, \quad l = 1, \ldots, q_2 - q_1.$$

The functions $\xi(x)$, $\eta(y)$, $r_j\,(x)$ and $s_j\,(y)$ are arbitrary and ρ_n, ψ_n^j and ϕ_n^l are defined in eq. (2.3) and (2.4), respectively.

The operator X reflects the fact that the generalized Toda field theories with $-\infty < n < \infty$ are always conformally invariant, be they integrable or not. If we have $p_2 - p_1 \geq 1$ the theory is invariant under gauge transformations. If we have $q_2 - q_1 \geq 1$ a further type of gauge invariance exists, represented by the operator Z_{jl}. If Z_{jl} is absent the gauge group is abelian, otherwise it is non abelian. The commutation relations are given elsewhere [1].

The generalized Toda Lattice (1.7) is obtained from the generalized Toda field theory by symmetry reduction. Indeed, let us reduce by the translation generator $P = \partial_x - \partial_y$ as in eq. (1.6). Eq. (1.1) reduces to eq. (1.7). We calculate the symmetry subgroup of the symmetry group of eq. (1.1) inherited by eq. (1.7) using the procedure explained in the Introduction. We obtain the following result.

Theorem 2 *The basis for the inherited symmetry algebra for the infinite Toda lattice (1.7) with H and K satisfying (2.1) and (2.2) is given by*

$$\begin{gathered} P = \partial_t, \qquad D = t\partial_t - 2 \sum_{n=-\infty}^{\infty} \rho_n \partial_{u_n} \\ U_j = \sum_{n=-\infty}^{\infty} {\psi_n}^j \partial_{u_n}, \qquad W_j = t \sum_{n=-\infty}^{\infty} {\psi_n}^j \partial_{u_n} \end{gathered} \tag{2.6}$$

and Z_{jl} as in eq. (2.5c). The inherited algebra coincides with the actual symmetry algebra of eq. (1.7).

Just as in the case of differential equations, there is no guarantee that the symmetry algebra inherited from the original equation is the entire symmetry algebra of the reduced equation. However, in this case, a direct calculation of the symmetry algebra of the generalized Toda lattice shows that its symmetry algebra is indeed given by eq. (2.6).

2.2 Special cases

A. The type I system

We have $p_2 - p_1 = q_2 - q_1 = 1$, and obtain

$$\psi_m = \phi_m = 1, \quad \rho_n = -n \tag{2.7}$$

in eqs. (2.5) and (2.6). Thus, (2.5a) is present, as are (2.5b) and (2.5c). The labels $j = l = 1$ can be dropped.

B. The type II system with $K_{n-1n} = K_{n+1n} = -1$, $K_{nn} = 2$

We have $p_2 = p_1 = 0$, $q_2 - q_1 = 2$ and hence

$$\psi_m = 0, \quad \phi_m^1 = 1, \quad \phi_m^2 = m, \quad \rho_n = 1, \tag{2.8}$$

Conformal invariance (2.5a) is present but there are no gauge transformations (2.5b), nor (2.5c).

C. The type III system with $H_{n-1n} = H_{n+1n} = -1$, $H_{nn} = 2$

We have $p_2 - p_1 = 2$, $q_2 - q_1 = 0$ and

$$\psi_n^1 = 1, \quad \psi_n^2 = n, \quad \phi_m = 0, \quad \rho_n = -\frac{1}{2}n^2. \tag{2.9}$$

We have conformal invariance (2.5a), two operators V_1 and V_2, no Z.

The symmetries of the corresponding generalized Toda lattices are obtained by putting the above values of ρ_n, ψ_m and ϕ_m into eq. (2.6).

3 Reduction to semi-infinite theories

We will now calculate the symmetry groups of semi-infinite theories inherited from infinite Toda systems using the procedure explained in the Introduction.

Rather than impose general and arbitrary boundary conditions on the matrices K and H of eq. (1.1) we shall consider several special cases suggested by Lie group theory, that already occured in the literature (usually for n varying in a finite range).

3.1 The semi-infinite field theories of type I related to simple Lie algebras

Let us consider a semi-infinite generalization of the system (1.3). The field equation is given by

$$\mathrm{U}_{xy} = -\mu^2 \sum_{i=1}^{N} \frac{\boldsymbol{\alpha}_i}{\boldsymbol{\alpha}_i^2} \exp(\boldsymbol{\alpha}_i \cdot \mathrm{U}), \tag{3.1}$$

where $\mathrm{U} = (u_1, \ldots, u_N)$ is an N-tuple of real fields and $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N)$ denote the simple roots of some finite simple Lie algebra of rank N. If this Lie algebra is $sl(N+1, \mathbb{R})$ one obtains the usual finite Toda field theory. Instead of this we shall consider the Toda field theory (1.3) and let the Lie algebra root system run through semi-infinite extensions of the classical Cartan Lie algebras A_N, B_N, C_N and D_N with $N \to \infty$ (in one direction). We take the simple roots for the classical simple Lie algebras as in Ref. [27].

For the system (1.3) the symmetry algebra (2.5) reduces to

$$X = \xi(x)\partial_x + \eta(y)\partial_y + (\xi_x + \eta_y) \sum_{n=-\infty}^{\infty} n\partial_{u_n}, \tag{3.2a}$$

$$V = (\beta(x) + \gamma(y)) \sum_{n=-\infty}^{\infty} \partial_{u_n}, \tag{3.2b}$$

$$Z = \left(\sum_{m=-\infty}^{\infty} u_m \right) \left(\sum_{n=-\infty}^{\infty} \partial_{u_n} \right). \tag{3.2c}$$

For the ordinary Toda lattice theory obtained by replacing $u_{n,xy}$ by $u_{n,tt}$ in eq. (1.3), the symmetry algebra (2.6) reduces to

$$P = \partial_t, \quad D = t\partial_t + 2 \sum_{n=-\infty}^{\infty} n\partial_{u_n}, \tag{3.3a}$$

$$U = \sum_{n=-\infty}^{\infty} \partial_{u_n}, \quad W = t \sum_{n=-\infty}^{\infty} \partial_{u_n}, \tag{3.3b}$$

and Z as in (3.2c).

We now turn to the semi-infinite case. For each algebra we show the modified equations (for $n < n_0$). Those not shown coincide with eq. (1.3).

1. The Cartan A series

We have

$$u_{1,xy} = -e^{u_1 - u_2}. \tag{3.4}$$

Taking a general element of the algebra (3.2), applying its prolongation to eq. (3.4) and taking into account that

$$\left(\sum_{i=1}^{\infty} u_i \right)_{xy} = 0 \tag{3.5}$$

we find that the entire symmetry algebra (3.2) leaves eq. (3.4) invariant. Hence the symmetry algebras in the semi-infinite and infinite cases coincide, though in the semi-infinite case all summations are for $1 \leq n < \infty$. Eq. (3.5) was used to show that the Z symmetry also survives.

The same is true for the Toda lattice equations in this case. Namely, the symmetry groups are the same in the infinite and semi-infinite cases.

2. The Cartan B series

In this case, the first equation is

$$u_{1,xy} = 2e^{-u_1} - e^{u_1 - u_2}. \tag{3.6}$$

Applying the same procedure, we find that conformal invariance remains and is realized as in eq. (3.2a). The gauge symmetries (3.2b) and (3.2c) do not survive.

For the Toda lattice, the only surviving symmetries are the translation P and dilation D as in eq. (3.3a).

3. The Cartan C series

We have

$$u_{1,xy} = e^{-2u_1} - e^{u_1 - u_2}. \tag{3.7}$$

In this case the gauge symmetries combine together with the conformal ones. The presence of eq. (3.7) excludes invariance under the Z transformation of eq. (3.2c).

The entire set of equations is invariant under the transformations generated by

$$X = \xi(x)\partial_x + \eta(y)\partial_y + (\xi_x + \eta_y) \sum_{n=1}^{infty} \left(n - \frac{1}{2}\right)\partial_{u_n}, \tag{3.8}$$

The surviving symmetries of the corresponding Toda lattice are P and $D - U$ (see eq. (3.3)).

4. The Cartan D series

In this case, the first two equations must be distinguished. They are

$$u_{1,xy} = e^{-u_1 - u_2} - e^{u_1 - u_2}, \tag{3.9a}$$

$$u_{2,xy} = e^{-u_1 - u_2} + e^{u_1 - u_2} - e^{u_2 - u_3}. \tag{3.9b}$$

The surviving algebra is given by

$$X = \xi(x)\partial_x + \eta(y)\partial_y + (\xi_x + \eta_y) \sum_{n=1}^{\infty} (n-1)\partial_{u_n}. \tag{3.10}$$

Reducing further to the corresponding Toda lattice, we find that it is invariant under translations and dilations generated by P and $D - 2U$ with P, D and U as in eq. (3.3).

Without presenting the proof [1] we state that the above symmetries, inherited from the infinite case, represent the entire symmetry algebra in the semi-infinite case.

3.2 The semi-infinite Toda field theories of type II related to simple Lie algebras

The infinite system is given by eq. (1.1) with $H_{nm} = \delta_{nm}$ and K a Cartan matrix. In other words the infinite system is

$$u_{n,xy} = -e^{u_{n-1}} + 2e^{u_n} - e^{u_{n+1}} \tag{3.11}$$

and the entire symmetry algebra is generated by

$$X = \xi(x)\partial_x + \eta(y)\partial_y - (\xi_x + \eta_y) \sum_{n=-\infty}^{\infty} \partial_{u_n}. \tag{3.12}$$

The gauge transformations discussed in Section 2 are all absent since we have $p_2 = p_1$. It was also shown in Section 2 that the existence of conformal invariance depends only on the matrix H which in the field theory (1.4) is an (infinite) identity matrix. The A, B, C and D Cartan series have different matrices K but this has no influence on conformal invariance. Hence the infinite, semi-infinite and finite theories all have the same symmetry algebra (3.12). Moreover the same symmetries will exist even if K is not a Cartan matrix.

The reduction to the corresponding Toda lattice yields the symmetries

$$P = \partial_t, \quad D = t\partial_t - 2\sum_{n=1}^{\infty} \partial_{u_n} \tag{3.13}$$

in agreement with eq. (2.6).

3.3 The semi-infinite field theories of type III related to simple Lie algebras

We are now restricting the matrix K to satisfy $K_{nm} = \delta_{nm}$ and H to be a Cartan matrix. Thus, in the infinite case, the equations we are studying are

$$u_{n,xy} = e^{-u_{n-1}+2u_n-u_{n+1}}. \tag{3.14}$$

The symmetry algebra in this case is generated by

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y)\sum_{n=-\infty}^{\infty} n^2\partial_{u_n}, \tag{3.15a}$$

$$V = \sum_{n=-\infty}^{\infty} \left(r_1(x) + s_1(y) + n(r_2(x) + s_2(y))\right)\partial_{u_n}, \tag{3.15b}$$

where ξ, η, r_1, r_2, s_1, and s_2 are arbitrary functions.

In the case of the corresponding infinite lattice the symmetry algebra is generated by

$$P = \partial_t, \qquad D = t\partial_t + \sum_{n=-\infty}^{\infty} n^2\partial_{u_n}, \tag{3.16a}$$

$$U_1 = \sum_{n=-\infty}^{\infty} \partial_{u_n}, \quad W_1 = t\sum_{n=-\infty}^{\infty} \partial_{u_n} \tag{3.16b}$$

$$U_2 = \sum_{n=-\infty}^{\infty} n\partial_{u_n}, \quad W_2 = t\sum_{n=-\infty}^{\infty} n\partial_{u_n}. \tag{3.16c}$$

Now let us look at individual semi-infinite cases.

1. The Cartan A series

The first equation is

$$u_{1,xy} = e^{2u_1-u_2}. \tag{3.17}$$

The others are as in (3.14). Requiring that eq. (3.17) also be annihilited by the symmetry algebra, we obtain the constraint $r_1 + s_1 = 0$. Thus the surviving symmetry algebra is the same as in eq. (3.15) but with $r_1 = s_1 = 0$.

For the A Toda lattice, only P, D, U_2 and W_2 in (3.16) survive.

2. The Cartan B series

In this case the two first equations are modified. They are

$$\begin{aligned} u_{1,xy} &= e^{2u_1-u_2}, \\ u_{2,xy} &= e^{-2u_1+2u_2-u_3}. \end{aligned} \tag{3.18}$$

In this case only conformal invariance survives the reduction and takes the form

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y)\sum_{n=1}^{\infty} n(n-1)\partial_{u_n}. \tag{3.19}$$

However a direct calculation of the symmetry algebra shows that there is a further gauge symmetry given by

$$V = (r(x) + s(y))\sum_{n=1}^{\infty} a_n\partial_{u_n}, \quad a_1 = 1,\ a_k = 2 \quad \textit{for} \quad k \geq 2. \tag{3.20}$$

Similarly for the Toda lattice in this case the only inherited symmetries from (3.16) are P and $D-U_2$. From a direct calculation we obtain an additional gauge symmetry $W=(ct+d)\sum_{n=1}^{\infty} a_n\partial_{u_n}$ with a_n as in (3.20) and c and d constants.

3. The Cartan C series

Only the first equation is modified and is

$$u_{1,xy} = e^{2u_1-2u_2}. \tag{3.21}$$

The only surviving symmetries from eq. (3.15) are given by

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x+\eta_y)\sum_{n=1}^{\infty} n(n-2)\partial_{u_n} \tag{3.22}$$

and gauge transformations generated by vector fields of the form (3.15b) with $r_2 = s_2 = 0$.

Similarly for the Toda lattice in this case the symmetries inherited from (3.16) are

$$\begin{aligned} P &= \partial_t, \qquad D = t\partial_t + \sum_{n=-\infty}^{\infty} n(n-2)\partial_{u_n} \\ U_1 &= \sum_{n=-\infty}^{\infty} \partial_{u_n}, \qquad W_1 = t\sum_{n=-\infty}^{\infty} \partial_{u_n} \end{aligned} \tag{3.23}$$

2. The Cartan D series

In this case the three first equations are modified. They are

$$\begin{aligned} u_{1,xy} &= e^{2u_1-u_3}, \\ u_{2,xy} &= e^{2u_2-u_3}, \\ u_{3,xy} &= e^{-u_1-u_2+2u_3-u_4}. \end{aligned} \tag{3.24}$$

In this case only conformal invariance survives the reduction and takes the form

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x+\eta_y)\sum_{n=1}^{\infty}(n^2-3n+2)\partial_{u_n}. \tag{3.25}$$

However a direct calculation of the symmetry algebra shows that there is a further gauge symmetry given by

$$V = (r(x)+s(y))\sum_{n=1}^{\infty} a_n\partial_{u_n}, \quad a_1 = a_2 = 1, \; a_k = 2 \quad \textit{for} \quad k \geq 3. \tag{3.26}$$

Similarly for the Toda lattice in this case the only inherited symmetries from (3.16) are P and $D-3U_2+2U_1$. From a direct calculation we obtain an additional gauge symmetry $W = (ct+d)\sum_{n=1}^{\infty} a_n\partial_{u_n}$ with a_n as in (3.26) and c and d constants.

4 Reduction to finite Toda systems

We will now calculate subgroups of symmetry groups of infinite Toda systems inherited by finite ones. The reduction procedure is explained in the Introduction. It must then be verified that the inherited groups indeed correspond to the complete symmetry groups of the finite systems.

4.1 Finite Toda theories of type I

The symmetries of the semi-infinite Toda field theories and Toda lattices corresponding to A, B, C and D Cartan series were established in Section 3.1 above.

The corresponding finite systems are obtained by setting $u_k = 0$, $k \geq N+1$ and imposing the equation

$$u_{N,xy} = e^{u_{N-1}-u_N}, \quad \text{or} \quad u_{N,tt} = e^{u_{N-1}-u_N} \tag{4.1}$$

respectively. It is easy to check that eq. (4.1) is invariant under the entire algebra (3.2), or (3.3), respectively (with all summations in the range $1 \leq n \leq N$).

We obtain the following result.

Theorem 3 *The symmetries of the finite A_N, B_N, C_N and D_N Toda field theories of type* (1.3) *are all inherited from the symmetries* (3.2) *of the infinite theories and coincide with those of the corresponding semi-infinite ones. The same is true for the A_N, B_N, C_N and D_N Toda lattices.*

4.2 Finite Toda theories of type II

The equations under consideration have the form (3.11). However we modify the matrix K in eq. (1.4), the symmetries remain the same, namely conformal invariance (3.12) for Toda field theories and translations $P = \partial_t$ and dilations $D = t\partial_t - 2\sum_{n=1}^{N} \partial_{u_n}$ for Toda lattices. Thus, the symmetries are the same in the infinite case and in all semi-infinite and finite cases.

4.3 Finite Toda theories of type III

The finite case is obtained from the semi-infinite one of Section 3.3 by setting $u_n = 0$, $n \geq N+1$ and imposing

$$u_{N,xy} = e^{-u_{N-1}+2u_N}, \quad \text{or } u_{N,tt} = e^{-u_{N-1}+2u_N}, \tag{4.2}$$

respectively. The requirement that eq. (4.2) be invariant will restrict the symmetry algebras of Section 3.3.

Let us consider individual cases.

1. The Cartan series A_N

In the $A_{1/2\infty}$ case we have the prolonged symmetry vector

$$\mathrm{pr}\hat{v} = \xi(x)\partial_x + \eta(y)\partial_y + \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{2}(\xi_x + \eta_y)n^2 + (r(x) + s(y))n \right] \partial_{u_n} - (\xi_x + \eta_y)u_{n,xy}\partial_{u_{n,xy}} \right\}. \tag{4.3}$$

This will annihilate the eq. (4.2) only if

$$r(x) + s(y) = -\frac{N+1}{2}(\xi_x + \eta_y). \tag{4.4}$$

Thus, the inherited symmetry algebra of the finite A_N Toda field theory consists of the conformal transformation generated by

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y) \sum_{n=1}^{N} n(n - N - 1)\partial_{u_n}. \tag{4.5}$$

Reducing to the A_N Toda lattice we have

$$P = \partial_t, \quad D = t\partial_t + \sum_{n=1}^{N} n(n-N-1)\partial_{u_n} \tag{4.6}$$

2. The Cartan series B_N

We require that the algebra (3.19), (3.20) should annihilate eq. (4.2) (on its solution set). This requires $r + s = -\frac{1}{2}N(N+1)(\xi_x + \eta_y)$ and restricts the symmetry algebra to

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{4}(\xi_x + \eta_y)\left[-N(N+1)\partial_{u_1} + 2\sum_{n=2}^{N}[n(n-1) - N(N+1)]\partial_{u_n}\right]. \tag{4.7}$$

For the Toda lattice this reduces further to

$$P = \partial_t, \quad D = t\partial_t + \frac{1}{2}\left[-N(N+1)\partial_{u_1} + 2\sum_{n=2}^{N}[n(n-1) - N(N+1)]\partial_{u_n}\right]. \tag{4.8}$$

3. The Cartan series C_N

Starting from the symmetries (3.22) and the remaining gauge transformations, we find that the symmetries of the C_N Toda field theories reduce to conformal transformations, realized as

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{2}(\xi_x + \eta_y)\sum_{n=1}^{N}[n(n-2) - N^2 + 1)]\partial_{u_n}. \tag{4.9}$$

For the C_N Toda lattice this reduces to

$$P = \partial_t, \quad D = t\partial_t + \sum_{n=1}^{N}[n(n-2) - N^2 + 1]\partial_{u_n}. \tag{4.10}$$

4. The Cartan series D_N

Starting from the symmetries (3.25) and (3.26) we find that eq. (4.2) reduces the symmetries to

$$\begin{aligned} X = \ & \xi(x)\partial_x + \eta(y)\partial_y + \frac{1}{4}(\xi_x + \eta_y)\Big[-N(N-1)(\partial_{u_1} + \partial_{u_2}) \\ & +2\sum_{n=3}^{N}[(n-2)(n-1) - N(N-1)]\partial_{u_n}\Big] \end{aligned} \tag{4.11}$$

and respectively

$$\begin{aligned} P = \partial_t, \quad D = t\partial_t + \frac{1}{2}\Big[& -N(N-1)(\partial_{u_1} + \partial_{u_2}) \\ & +2\sum_{n=3}^{N}[(n-2)(n-1) - N(N-1)]\partial_{u_n}\Big]. \end{aligned} \tag{4.12}$$

The result is finally quite simple, namely:

Theorem 4 *Toda field theories of the type (1.5) for a finite number N of fields, based on the Cartan algebras A_N, B_N, C_N and D_N are invariant under conformal transformations only. They are realized by the vector fields (4.5), (4.7), (4.9) and (4.11) for A_N, B_N, C_N and D_N, respectively. The symmetries are inherited from the semi-infinite case (however, for the B and D series the symmetries in the semi-infinite are not all inherited from the infinite case). The finite Toda lattices of type (1.5) are invariant only under a translation, and the appropriate dilations.*

5 Reduction to periodic Toda systems

We will now study reductions from infinite (generalized) Toda systems to periodic ones using the procedure explained in the Introduction. In order to be able to impose periodicity (1.9) and enlarge the symmetry algebra of the infinite system by adding the vector field (1.10), a further condition is necessary, namely that the recursion relations (2.3) and (2.4) should have constant coefficients:

$$h_\sigma(n) = h_\sigma(n+1), \quad k_\sigma(m) = k_\sigma(m+1). \tag{5.1}$$

Let us look at individual cases.

5.1 Periodic Toda systems of type I

The inherited symmetry algebra of the usual periodic Toda field theory (1.3) is given by

$$\begin{gathered} L = x\partial_x - y\partial_y, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \\ V = [\beta(x) + \gamma(y)] \sum_{n=1}^{N-1} \partial_{u_n}, \quad Z = \sum_{m=1}^{N-1} u_m \sum_{n=1}^{N-1} \partial_{u_n}. \end{gathered} \tag{5.2}$$

We see that the infinite dimensional conformal algebra is reduced to the Poincaré one. For the Toda lattice the symmetry algebra (3.3) in the periodic case is reduced to $\{P, U, W, Z\}$.

The periodic Toda field theory (1.3) can be written as eq. (1.1) with

$$K = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & -1 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} -1 & 0 & 0 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}. \tag{5.3}$$

The vector $\bar{1}_N = (1, 1, \dots, 1)$ is not in the image of H, the kernel of H and K^T are one-dimensional. From Theorem 2 of Ref. [1] we conclude that the periodic Toda field theories and lattices of type (1.3) have no further symmetries: all symmetries are inherited.

5.2 Periodic Toda systems of type II

All elements of the symmetry algebra (3.12) commute with $\hat{N}$ of eq. (1.10). Hence, in this case the symmetry algebra is the same in the periodic case as in the infinite one (and also the semi-infinite and all finite ones). Thus, the corresponding Toda field theory is conformally invariant, the Toda lattice is invariant under translations P and dilations D as in eq. (3.13). No new symmetries, due to the reduction, arise.

5.3 Periodic Toda systems of type III

The inherited symmetries from eq. (3.15) in this case are Poincaré and gauge invariance:

$$\begin{gathered} L = x\partial_x - y\partial_y, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \\ V = [\beta(x) + \gamma(y)] \sum_{n=1}^{N-1} \partial_{u_n}. \end{gathered} \tag{5.4}$$

For the corresponding periodic Toda lattice the only inherited symmetries are

$$P = \partial_t, \quad U = \partial_{u_n}, \quad W = t\partial_{u_n}. \tag{5.5}$$

These are the only symmetries of the system.

6 Symmetry reductions involving continuous and discrete variables

In this Section, we will extend the symmetry algebras of the infinite Toda lattices by the generator $\hat{N}$ given in eq. (1.10). It generates transformations of the independent variable n given by

$$n' = n + N. \tag{6.1}$$

The quantity N is to be viewed as a discrete group parameter. We can then act with $\hat{N}$ as if it were an element of the symmetry algebra and calculate group transformations and invariants in exactly the same manner as for differential equations.

6.1 The infinite Toda lattice of type I

The symmetry algebra in this case is given by eq. (3.3). All the possible reductions have been studied in [5]. Let us look at the interesting case of the reduction by the generator

$$\partial_n + a\partial_t, \tag{6.2}$$

where a is a constant [5]. The invariants are $\xi = an - t$ and u_n. We thus consider solutions of the infinite Toda lattice of the form

$$u_n(t) = F(\xi), \;\; \xi = an - t. \tag{6.3}$$

Substituing into (1.3) we find the following equation for F

$$F_{\xi\xi} = \mathrm{e}^{F(\xi-a)-F(\xi)} - \mathrm{e}^{F(\xi)-F(\xi+a)}. \tag{6.4}$$

A soliton solution of (6.4) is given by

$$F(\xi) = \ln\left(\frac{1+\exp\left[2\xi\sinh\frac{\alpha}{2} \pm \alpha\right]}{1+\exp\left[2\xi\sinh\frac{\alpha}{2}\right]}\right), \quad a = \mp\frac{\alpha}{2\sinh\frac{\alpha}{2}}. \tag{6.5}$$

6.2 The infinite Toda lattice of type II

The symmetry algebra in this case is given by eq. (3.13). We first consider reductions by the generator (6.2). Invariant solutions then have the form given by eq. (6.3) and the function F satisfies

$$F_{\xi\xi} = -\mathrm{e}^{F(\xi-a)} + 2\mathrm{e}^{F(\xi)} - \mathrm{e}^{F(\xi+a)}. \tag{6.6}$$

We can also consider reductions by the operator $\hat{N} + aD$ with D given in eq. (3.13). Invariant solutions are then of the form

$$u_n(t) = F(\eta) - 2an, \;\; \eta = t\mathrm{e}^{-an}. \tag{6.7}$$

The function F satisfies the equation

$$F_{\eta\eta} = -\mathrm{e}^{2a}\mathrm{e}^{F(\mathrm{e}^{a}\eta)} + 2\mathrm{e}^{F(\eta)} - \mathrm{e}^{-2a}\mathrm{e}^{F(\mathrm{e}^{-a}\eta)}. \tag{6.8}$$

6.3 The infinite Toda lattice of type III

The symmetry algebra in this case is given by eq. (3.16). We first consider reductions by generators of the form

$$\hat{N} + aP + cU_1 + dU_2 + eW_1 + fW_2, \tag{6.9}$$

where P, U_1, U_2, W_1 and W_2 are given in (3.16). Invariant solutions will have the form

$$u_n(t) = F(\xi) + (c - \xi e)n + \frac{n^2}{2}(d + ae - \xi f) + af\frac{n^3}{3}, \tag{6.10}$$

where $\xi = an - t$. The function F then satisfies the equation

$$F_{\xi\xi} = \exp\left[-F(\xi + a) + 2F(\xi) - F(\xi - a) + ae - d + f\xi\right]. \tag{6.11}$$

In the case when $d = ae$ and $f = 0$, we have a solution quadratic in ξ

$$F = \alpha\xi^2 + \beta\xi + \gamma, \tag{6.12}$$

where α is determined in terms of a by the equation

$$2\alpha = \mathrm{e}^{-2\alpha a^2}. \tag{6.13}$$

The constants β and γ are free.

We also consider reductions by generators of the form

$$\hat{N} + bD + cU_1 + dU_2 + eW_1 + fW_2 \tag{6.14}$$

with b nonzero. In this case invariant solutions have the form

$$u_n(t) = b\frac{n^3}{3} + d\frac{n^2}{2} + cn + \frac{\mathrm{e}^{bn}}{b^2}\eta(be - f + bfn) + F(\eta), \tag{6.15}$$

where $\eta = t\mathrm{e}^{-bn}$. The function F must then satisfy the equation

$$F_{\eta\eta} = \exp\left(-d - F(\mathrm{e}^{b}\eta) + 2F(\eta) - F(\mathrm{e}^{-b}\eta)\right). \tag{6.16}$$

7 Conclusions

The starting point of this article are the symmetries (2.5) of a large class of classical field theories with exponential interactions described by eq. (1.1). The symmetries (2.5), established in our earlier article [1] are present when the discrete variable n, labeling the fields, varies in an infinite range $-\infty < n < \infty$. The "interaction matrices" K and H in eq. (1.1) are very general band matrices with bands of constant width (see eq. (2.1) and (2.2)). From eq. (2.5a) we see that the theory is always conformally invariant. In view of eq. (2.5b), the theory is gauge invariant, unless the matrix H is diagonal. If K is diagonal, the gauge group is abelian. If both H and K are non diagonal, the gauge group is nonabelian.

If the generalized Toda field theories (1.1) correspond to difference equations with constant coefficients, then the symmetry group includes a further transformation, namely translations of n

$$\tilde{n} = n + N, \quad N \in \mathbb{Z}, \tag{7.1}$$

formally generated by the operator (1.10). This occurs if the band matrices H and K, specified in eq. (2.1) and (2.2), satisfy condition (5.1), that is, if the corresponding difference equations have constant (n-independent) coefficients.

The symmetry group of the infinite ($-\infty < n < \infty$) generalized Toda field theory is applied to study different types of reductions of the system.

It should be emphasized that when we perform symmetry reduction, for differential equations, difference equations, or differential-difference ones, we obtain a symmetry algebra inherited from the original symmetry algebra. This may not be the entire symmetry algebra of the reduced system. A prime example for this is provided by the Laplace equation in three-dimensional euclidian space E_3. It is invariant under the conformal group O(4,1). If we reduce it by translational invariance to the Laplace equation in E_2, we obtain the inherited symmetry group O(3,1). However, the reduced equation is invariant under a much larger group, the infinite-dimensional conformal group E_2. The presence of this noninherited symmetry group is an indication of the fact that two-dimensional theories differ qualitatively from three-dimensional ones. We have seen above that no such radical difference between "inherited symmetries" and "all symmetries" occurs in reductions of Toda systems.

Among open problems, not addressed in this article, we mention two related ones. They are: under what conditions on the matrices H and K are the generalized Toda systems (1.1) and (1.7) integrable? When do these equations allow higher symmetries that depend on the derivatives of u_n, or that are nonlocal?

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