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Modules with Unique Coclosure Relative to a Torsion Theory

Y. Talebi and T. Amoozegar

Department of Mathematics, Faculty of Science,
University of Mazandaran, Babolsar, Iran
talebi@umz.ac.ir
t.amoozegar@umz.ac.ir

Abstract

We investigate modules in which every submodule has a unique coclosure with respect to a cohereditary torsion theory τ.

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1 Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and M will be unitary right R-module. Submodules of M will be right R-submodules, while one sided ideals for these rings will be right ideals for R and right ideal for S, respectively. We reserve the term "ideal" for the two-sided ideals in both rings.

The notation $N \leq^{\oplus} M$ denotes that N is a direct summand in M; $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). A module N is said to be *small* if $N \ll L$, for some module L. For $N, L \leq M, N$ is *supplement* of L in M if $N + L = M$ with $N \cap L \ll N$. Following [9], a module M is called supplemented if every submodule of M has a supplement in M. On the other hand, the module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A in M such that $P \leq B$. Module M is called a *weakly supplemented* module if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. A submodule A of M is called *coclosed* if A has no proper cosmall submodule.

Furthermore, $\tau = (\mathcal{T}, \mathcal{F})$ denotes a cohereditary torsion theory on R-Mod, where $\mathcal{T}$ and $\mathcal{F}$ denote the classes of all τ-torsion and τ-torsionfree modules respectively. If $\tau(M)$ denotes the sum of the τ-torsion submodules of M, then $\tau(M)$ is necessarily the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$ for an R-module M. $\tau(M)$ is referred to as the *τ-torsion submodule* of M and it follows that $\mathcal{T} := \{M \in Mod{-}\mathrm{R}; \tau(M) = M\}$ and $\mathcal{F} := \{M \in Mod{-}\mathrm{R}; \tau(M) = 0\}$. For every torsion theory τ, both the torsion class $\mathcal{T}$ and the torsion-free class $\mathcal{F}$ of R-modules contain the zero module and both are closed under isomorphism; that is, if $N \in \mathcal{T}$ and $N' \cong N$, then $N' \in \mathcal{T}$, and similarly for $\mathcal{F}$. A *$\mathcal{T}$-submodule*(or *$\mathcal{F}$-submodule*) of M is a submodule N of M such that N belong to $\mathcal{T}$(or $\mathcal{F}$).

For a torsion theory $\tau := (\mathcal{T}, \mathcal{F}), \mathcal{T} \cap \mathcal{F} = 0$ and the torsion class $\mathcal{T}$ is closed under homomorphic images, direct sums and extension; and $\mathcal{F}$ is closed under submodules, direct products and extensions. Torsion theory τ are assumed to be cohereditary, that is, we assume that homomorphic images of τ-torsionfree modules is τ-torsionfree.

Let N be a submodule of a module M. Then N is called *τ-small* in M if it is $N \in \mathcal{F}$ and small in M. In this case we write $N \ll_\tau M$. Let $B \leq A \leq M$. If $A/B \ll_\tau M/B$, then B is called a *τ-cosmall* submodule of A in M (denoted by $B \overset{\tau\text{-}cs}{\hookrightarrow} A$). A submodule A of M is called *τ-coclosed* (denoted by $A \overset{\tau\text{-}cc}{\hookrightarrow} M$), if A has no proper τ-cosmall submodule. A *τ-coclosure* of a submodule B of M is a τ-cosmall submodule of B in M which is also a τ-coclosed submodule of M. A module M is called a *unique τ-coclosure module* (denoted by τ-UCC module) if every submodule of M has a unique τ-coclosure in M. A submodule K of M is said to be a *τ-supplement of N in M* if K is minimal with the respect to $\leq$ in the class $\{L \leq M | L + N = M, L \cap N \in \mathcal{F}\}$. A submodule K of M is called a *τ-supplement submodule* if it is a τ-supplement of some submodule L of M.

In [4] Doğruöz, Harmanci, and Smith studied modules in which every submodule has a unique closure with respect to a hereditary torsion theory τ. Dually, in this paper we investigate in which every submodule has a unique coclosure with respect to a cohereditary torsion theory τ and this give some generalizations of results in [5]. If ϱ is the cohereditary torsion theory for which every module is torsion free, then ϱ-UCC modules are precisely the UCC modules discussed in [5].

2 Preliminary Notes

2.1 Lemma. *Let K, L and L' be submodule M. Then*

(1) *If $K \leq L$, then $L \ll_\tau M$ if and only if $K \ll_\tau M, L/K \ll_\tau M/K$.*

(2) *If $K \ll_\tau M$, then $L \overset{\tau\text{-}cs}{\hookrightarrow} K + L$ in M.*

(3) *Let $K \leq L \leq L'$. $K \overset{\tau\text{-}cs}{\hookrightarrow} L$ in M and $L \overset{\tau\text{-}cs}{\hookrightarrow} L'$ in M if and only if $K \overset{\tau\text{-}cs}{\hookrightarrow} L'$ in M.*

Proof. (1) See [8, Proposition 3.1(1)].

(2) Let $(K+L)/L + D/L = M/L$, where $L \leq D \leq M$. Then $K + L + D = M$. By assumption $L + D = M$, thus $D = M$. Thus $(K + L)/L \ll M/L$. Moreover, Since $K \in \mathcal{F}$, then $(K + L)/L \in \mathcal{F}$.

(3) By assumption L/K and L'/La are both τ-torsion free modules. Consider the short exact sequence $0 \to L/K \to L'/K \to L'/L \to 0$. Thus $L'/K \in \mathcal{F}$. Moreover, it is easily checked, since K is cosmall in L and L is cosmall in L' then K is cosmall in L'. Then K is τ-cosmall in L'. Conversely is easy.

□

2.2 Corollary. *Let $K \leq K'$ and $L \leq L'$ be submodules of a module M such that K is τ-cosmall in K' and L is τ-cosmall in L'. Then $K + L$ is τ-cosmall in $K' + L'$.*

Proof. Note that $K \overset{cs}{\hookrightarrow} K'$ follows that $K + L \overset{cs}{\hookrightarrow} K' + L$. Moreover, $(K' + L)/(K+L) \simeq K'/(K' \cap (L+K)) = K'/((K' \cap L)+K)$. Since $K \leq (K' \cap L)+K$, we have natural epimorphism $f : K'/K \to K'/((K' \cap L) + K),$. Therefore, since $\mathcal{F}$ is closed under homomorphism image, $K'/((K' \cap L) + K) \in \mathcal{F}$. Then $(K' + L)/(K + L) \in \mathcal{F}$. Thus $K + L$ is τ-cosmall in $K' + L$. Similarly, $K' + L$ is τ-cosmall in $K' + L'$. By Lemma 2.1, $K + L$ is τ-cosmall in $K' + L'$. □

2.3 Theorem. *Let $K \leq L \leq M$. If $K \overset{\tau\text{-}cc}{\hookrightarrow} L$ and $L \overset{\tau\text{-}cc}{\hookrightarrow} M$, then $K \overset{\tau\text{-}cc}{\hookrightarrow} M$.*

Proof. Suppose that N is a τ-cosmall submodule of K in M, i.e. $K/N \ll_\tau M/N$. Since $L \overset{\tau\text{-}cc}{\hookrightarrow} M$, then $L/N \overset{\tau\text{-}cc}{\hookrightarrow} M/N$.

Now by [8, Lemma 3.12] we have $K/N \ll_\tau L/N$. But K is τ-coclosed in L, therefore $N = K$. □

3 τ-UCC-Modules

Given an R-module M we define a relation θ_τ on the lattice of submodules of M as follows: given submodules K and L of M, $K\theta_\tau L$ provided K is τ-cosmall in $K+L$ in M and L is τ-cosmall in $K+L$ in M.

3.1 Lemma. *With the above notation, for any module M, θ_τ is an equivalence relation on the lattice of submodules of M.*

Proof. The relation θ_τ is clearly reflexive and symmetric. Now let K, L and H be submodules of M such that $K\theta_\tau L$ and $L\theta_\tau H$. We will show that $K\theta_\tau H$. Since $K\theta_\tau L$, it follows that both L and K are τ-cosmall submodules of $K+L$ in M. Similarly, both H and L are τ-cosmall submodule of $H+L$ in M. By Corollary 2.2, $K+L$ is τ-cosmall submodule of $K+L+H$. But this implies that K is τ-cosmall in $K+L+H$ by Lemma2.1(3). Similarly H is τ-cosmall in $K+L+H$. Thus $H\theta_\tau K$.

□

3.2 Lemma. *The following are equivalent for an R-module M;*

(1) *M is a $\tau-UCC$ module;*

(2) *given $N \leq M$ there exists a τ-coclosure K of N in M such that if $L \overset{\tau\text{-}cs}{\hookrightarrow} N$ in M, then $K \leq L$.*

Proof. Clear. □

3.3 Lemma. *Let M be a module and A, B, C be submodules of M such that $B \leq C$ and $C/B \in \mathcal{F}$. Then $(A\cap C)/(A\cap B) \in \mathcal{F}$.*

Proof. Since $(A\cap C)/(A\cap B) \simeq (B+(A\cap C))/B$, and $(B+(A\cap C))/B \subseteq C/B$, then $(A\cap C)/(A\cap B) \in \mathcal{F}$. □

3.4 Lemma. *(1) Let M be a module and let $B \leq C$ submodules of M such that C/B is a τ-supplement submodule of M/B and B is a τ-supplement submodule of M. Then C is a τ-supplement submodule of M.*

(2) Let M be a weakly τ-supplemented module and $B \leq C$ submodules of M such that C/B is τ-coclosed in M/B and B is τ-coclosed in M. Then C is τ-coclosed in M.

Proof. (1) Let C/B be a τ-supplement of C'/B in M/B and let B be a τ-supplement of B' in M. Then $M/B = C/B + C'/B, C/B \cap C'/B \ll_\tau C/B$. Also $M = B + B'$ and $B \cap B' \ll_\tau B$. It is easily checked that $M = C + (B' \cap C'), (C \cap C' \cap B') \ll C$. But $(C \cap C')/B = (C/B) \cap (C'/B) \in \mathcal{F}$ and so by Lemma 3.3 $(C \cap C' \cap B')/(B \cap B') \in \mathcal{F}$. Since $B \cap B' \in \mathcal{F}$, thus $C \cap C' \cap B' \in \mathcal{F}$. It is easily checked that $M = C + (B' \cap C'), (C \cap C' \cap B') \ll C$. Thus C is a τ-supplement of $B' \cap C'$ in M.

(2) It is clear by [8, Lemma 3.12] and (1).

□

Unless otherwise defined we denote the unique τ-coclosure of $A \leq M$ by $\overline{A}$.

3.5 Lemma. *The following are equivalent for an R-module M:*

(1) *M is $\tau - UCC$ module*

(2) *every factor module of M is a $\tau - UCC$ module.*

Proof. (1) ⇒ (2) Suppose $K \leq M$ and $A/K \leq M/K$. Now A has a unique τ-coclosure, say, $\overline{A}$ in M. We claim that $(\overline{A}+K)/K$ is the unique τ-coclosure of A/K in M/K. We have $\overline{A} \subseteq \overline{A}+K \subseteq A \subseteq M$ and $\overline{A} \overset{\tau\text{-}cs}{\hookrightarrow} A$ in M. Hence by Corollary 2.2, $\overline{A}+K \overset{\tau\text{-}cs}{\hookrightarrow} A$ in M and so $(\overline{A}+K)/K \overset{\tau\text{-}cs}{\hookrightarrow} A/K$ in M/K. Next we show that $(\overline{A}+K)/K$ is τ-coclosed in M/K. Let $X/K \overset{\tau\text{-}cs}{\hookrightarrow} (\overline{A}+K)/K$ in M/K. Then $X/K \overset{\tau\text{-}cs}{\hookrightarrow} A/K$ in M/K and hence $X \overset{\tau\text{-}cs}{\hookrightarrow} A$ in M. But $\overline{A}$ is the unique τ-coclosure of A in M. By Lemma 3.2 $\overline{A} \subseteq X$, proving $X/K = (\overline{A}+K)/K$. Thus $(\overline{A}+K)/K$ is a τ-coclosure of A/K in M/K. Now we prove that it is unique. Let $Y/K \overset{\tau\text{-}cs}{\hookrightarrow} A/K$ in M/K. Then $Y \overset{\tau\text{-}cs}{\hookrightarrow} A$ in M implying $\overline{A} \subseteq Y$ (3.2). So $(\overline{A}+K)/K \subseteq Y/K$, proving that $(\overline{A}+K)/K$ is the unique τ-coclosure of A/K in M/K.

(2) ⇒ (1) is trivial. □

3.6 Lemma. *Consider the following statements for an R-module M:*

(1) *M is a $\tau - UCC$ module*

(2) *every submodule of M has a τ-coclosure in M and for any $A \subseteq B \subseteq M$ and a τ-coclosure B_1 of B in M, there exists a τ-coclosure A_1 of A in M such that $A_1 \subseteq B_1$.*

Then (2) ⇒ (1). The converse is true if M is weakly τ-supplemented.

Proof. (2) $\Rightarrow$ (1) Let $A \subseteq M$. Suppose that A_1 and A_2 are two τ-coclosures of A in M. Now $A_1 \subseteq A$ and A_2 is a τ-coclosure of A in M. So by (2) there exists a τ-coclosure C of A_1 in M such that $C \subseteq A_2$. But $A_1 \overset{\tau\text{-}cc}{\hookrightarrow} M$ implies $C = A_1 \subseteq A_2$, and so $A_1 \overset{\tau\text{-}cs}{\hookrightarrow} A_2$. Since A_2 is τ-coclosure, then $A_1 = A_2$.

(1) $\Rightarrow$ (2) Let $H = \overline{A}$. By Lemma 3.5 M/H is a UCC module. Suppose that $L/H = \overline{B/H}$ in M/H. As M is weakly τ-supplemented, $H \overset{\tau\text{-}cc}{\hookrightarrow} M$ and $L/H \overset{\tau\text{-}cc}{\hookrightarrow} M/H$ we have $L \overset{\tau\text{-}cc}{\hookrightarrow} M$ (Lemma3.4). Since $L \overset{\tau\text{-}cs}{\hookrightarrow} B$ in M, $\overline{B} = L$. So $\overline{A} = H \subseteq L = \overline{B}$.

□

3.7 Corollary. *Suppose M is a weakly τ-supplemented UCC module. If $A \subseteq B \subseteq M$ and $\overline{A}, \overline{B}$ are their respective unique τ-coclosures in M, then $\overline{A} \subseteq \overline{B}$.*

3.8 Lemma. *Consider the following statements for an R-module M.*

(i) *M is a τ-UCC module.*

(ii) *for all sets I and for all submodules $K_i \overset{\tau\text{-}cs}{\hookrightarrow} L_i$ in M for all $i \in I$, $\bigcap_{i\in I} K_i \overset{\tau\text{-}cs}{\hookrightarrow} \bigcap_{i\in I} L_i$ in M.*

(iii) *every submodule of M has a τ-coclosure in M and if $K \overset{\tau\text{-}cs}{\hookrightarrow} K'$ in M and $L \overset{\tau\text{-}cs}{\hookrightarrow} L'$ in M, then $(K \cap L) \overset{\tau\text{-}cs}{\hookrightarrow} (K' \cap L')$.*

(iv) *every submodule of M has a τ-coclosure in M and if $K\theta_\tau K'$ and $L\theta_\tau L'$, for submodules K, K', L and L' of M, then $(K \cap L)\theta_\tau(K' \cap L')$.*

(v) *every submodule of M has a τ-coclosure in M and if $L \overset{\tau\text{-}cs}{\hookrightarrow} K + L$ then $(K \cap L) \overset{\tau\text{-}cs}{\hookrightarrow} K$.*

Then $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$ if M is amply τ-supplemented, then $(i) \Rightarrow (ii)$.

Proof.

$(ii) \Rightarrow (iii)$ Suppose $N \subseteq M$. Then $N \overset{\tau\text{-}cs}{\hookrightarrow} N$ in M and the intersection of all τ-cosmall submodules of N in M is a τ-cosmall submodule of N in M and hence is a τ-coclosure of N in M. The rest of (iii) follows trivially from (ii).

$(iii) \Rightarrow (iv)$ Note that both K and K' are τ-cosmall in $K + K'$ and also both L and L' are τ-cosmall in $L + L'$. By (iii), both $K \cap L$ and $K' \cap L'$ are τ-cosmall in $(K + K') \cap (L + L')$, and so both $K \cap L$ and $K' \cap L'$ are τ-cosmall in $(K \cap L) + (K' \cap L')$. Thus $(K \cap L)\theta_\tau(K' \cap L')$.

$(iv) \Rightarrow (v)$ If $L \overset{\tau\text{-}cs}{\hookrightarrow} K+L$, then $L\theta_\tau(K+L)$ and since $K\theta_\tau K$, by (iv) $(L\cap K)\theta_\tau K$. So $(K\cap L) \overset{\tau\text{-}cs}{\hookrightarrow} K$.

$(v) \Rightarrow (i)$ Let K and K' be τ-coclosure of N in M. Then $K' \overset{\tau\text{-}cs}{\hookrightarrow} K+K'$. By (v) $K\cap K' \overset{\tau\text{-}cs}{\hookrightarrow} K$ and so $K\cap K' = K$. Thus $K \subseteq K'$. Similarly $K' \subseteq K$ and so $K = K'$.

$(i) \Rightarrow (ii)$ Let $K_i \overset{\tau\text{-}cs}{\hookrightarrow} L_i$ in M for every $i \in I$. Since $\overline{K}_i = \overline{L}_i$. Now $\bigcap L_i \subseteq L_i$ for every $i \in I$. By Corollary 3.7 $\overline{\bigcap L_i} \subseteq \overline{L}_i$ for every $i \in I$. Therefore $\overline{\bigcap L_i} \subseteq \bigcap \overline{L}_i = \bigcap \overline{K}_i \subseteq \bigcap K_i \subseteq \bigcap L_i$ proving (ii). □

3.9 Definition. Let M be an R-module. A submodule K of M is called a strongly τ-coclosed submodule of M (denoted by $K \overset{\tau\text{-}scc}{\hookrightarrow} M$) if for every $X \subseteq M$ with $K \nsubseteq X, X \overset{\tau-cs}{\not\hookrightarrow} (K+X)$ in M.

3.10 Lemma. *Suppose M is an R-module and $K \subseteq M$. Then the following hold:*

(i) *if $K \overset{\tau\text{-}scc}{\hookrightarrow} M$, then $K \overset{\tau\text{-}cc}{\hookrightarrow} M$*

(ii) *if $K_i \overset{\tau\text{-}scc}{\hookrightarrow} M$, for all $i \in I$, then $\sum_{i\in I} K_i \overset{\tau\text{-}scc}{\hookrightarrow} M$.*

Proof. (i) Let X be a proper submodule of K. Then obviously $K \nsubseteq X$. As $K \overset{\tau\text{-}scc}{\hookrightarrow} M$, $X \overset{\tau-cs}{\not\hookrightarrow} K+X = K$,proving $K \overset{\tau\text{-}cc}{\hookrightarrow} M$.

(ii) Let $X \subseteq M$ be such that $\sum_{i\in I} K_i \nsubseteq X$. Then there is an $i \in I$ such that $K_i \nsubseteq X$. As $K_i \overset{\tau\text{-}scc}{\hookrightarrow} M, X \overset{\tau-cs}{\not\hookrightarrow} (X+K_i)$ in M. Now $X \subseteq (X+K_i) \subseteq (X+\sum_{i\in I} K_i)$ implies that $X \overset{\tau-cs}{\not\hookrightarrow} X+\sum_{i\in I} K_i$ by Lemma 2.1(3). □

3.11 Lemma. *The following statements are equivalent for an amply τ-supplemented module M.*

(i) *M is a $\tau-UCC$ module.*

(ii) *every τ-coclosed submodule of M is strongly τ-coclosed.*

(iii) *$N^+ = \bigcap\{T | T \overset{\tau\text{-}cs}{\hookrightarrow} N+T\}$ is a τ-coclosure of N in M, for every submodule N of M.*

Proof. $(i) \Rightarrow (ii)$ Let K be any τ-coclosed submodule of M. Assume that T is τ-cosmall in $K+T$ for some $T \subseteq M$. By Lemma 3.8, $T\cap K$ is τ-cosmall in K. Hence $T\cap K = K$. So $K \subseteq T$. Thus (ii) hold.

$(ii) \Rightarrow (iii)$ Let N be any submodule of M and K be a τ-coclosure of N in M. Let $T \subseteq M, T \overset{\tau\text{-}cs}{\hookrightarrow} N+T$. We have $T \subseteq K+T \subseteq N+T$, then $T \overset{\tau\text{-}cs}{\hookrightarrow} K+T$. By (ii) $K \subseteq T$ and so $K \subseteq \bigcap\{T|T \overset{\tau\text{-}cs}{\hookrightarrow} N+T\}$. Therefore $K \subseteq N^+$. On the other hand, since $K \overset{\tau\text{-}cs}{\hookrightarrow} N$, then $K \overset{\tau\text{-}cs}{\hookrightarrow} N+K$. Thus $N^+ \subseteq K$.

$(iii) \Rightarrow (i)$ Let N be any submodule of M. Note that N^+ is τ-cosmall in N. Suppose that L is a τ-coclosure of N in M. Then $L \overset{\tau\text{-}cs}{\hookrightarrow} N+L$ and so $N^+ \subseteq L \subseteq N$. Thus N^+ is τ-cosmall in L. But L is τ-coclosed , therefore $N^+ = L$. □

3.12 Lemma. *Suppose M is an amply τ-supplemented R-module. Then the following are equivalent:*

(1) *M is a $\tau - UCC$ module;*

(2) *every τ-coclosed submodule of M is strongly τ-coclosed;*

(3) *the sum of any family of τ-coclosed submodule of M is τ-coclosed;*

(4) *the sum of two τ-coclosed submodule of M is τ-coclosed.*

Proof. (1) ⇒ (2) By Lemma 3.11.

(2) ⇒ (3) Suppose $K_i \overset{\tau\text{-}cc}{\hookrightarrow} M$, for all $i \in I$. Then $K_i \overset{\tau\text{-}scc}{\hookrightarrow} M$, for all $i \in I$, by Lemma 3.11. We get $\sum_{i\in I} K_i \overset{\tau\text{-}scc}{\hookrightarrow} M$ and so $\sum_{i\in I} K_i \overset{\tau\text{-}cc}{\hookrightarrow} M$ (3.10).

(3) ⇒ (4) It is obvious.

(4) ⇒ (1) Let $A \leq M$ with two τ-coclosure A_1 and A_2 in M. By (4) $A_1+A_2 \overset{\tau\text{-}cc}{\hookrightarrow} M$. But $A_1 \overset{\tau\text{-}cs}{\hookrightarrow} A$ in M and $A_1 \leq A_1+A_2 \leq A$ implies $A_1+A_2 = A_1 = A_2$.

□

3.13 Theorem. *Let R be a ring and let τ be a cohereditary torsion theory on Mod-R. If M is an amply τ-supplemented R-module, then the following statements are equivalent.*

(1) *M is a $\tau - UCC$ module;*

(2) *for all sets I and for all submodules $K_i \overset{\tau\text{-}cs}{\hookrightarrow} L_i$ in M for all $i \in I$, $\bigcap_{i\in I} K_i \overset{\tau\text{-}cs}{\hookrightarrow} \bigcap_{i\in I} L_i$ in M.*

(3) *if $K \overset{\tau\text{-}cs}{\hookrightarrow} K'$ in M and $L \overset{\tau\text{-}cs}{\hookrightarrow} L'$ in M, then $(K \cap L) \overset{\tau\text{-}cs}{\hookrightarrow} (K' \cap L')$.*

(4) *if* $K\theta_\tau K'$ *and* $L\theta_\tau L'$, *for submodules* K, K', L *and* L' *of* M, *then* $(K \cap L)\theta_\tau(K' \cap L')$.

(5) *if* $L \overset{\tau\text{-}cs}{\hookrightarrow} K + L$ *then* $(K \cap L) \overset{\tau\text{-}cs}{\hookrightarrow} K$.

(6) *for any* $A \subseteq B \subseteq M$ *and a* τ*-coclosure* B_1 *of* B *in* M, *there exists a* τ*-coclosure* A_1 *of* A *in* M *such that* $A_1 \subseteq B_1$.

(7) *the sum of any family of* τ*-coclosed submodule of* M *is* τ*-coclosed;*

(8) *the sum of two* τ*-coclosed submodule of* M *is* τ*-coclosed.*

(9) *every* τ*-coclosed submodule of* M *is strongly* τ*-coclosed.*

(10) $N^+ = \bigcap\{T | T \overset{\tau\text{-}cs}{\hookrightarrow} N + T\}$ *is a* τ*-coclosure of* N *in* M, *for every submodule* N *of* M.

Moreover, in this case N^+ *is the unique* τ*-coclosure of* N *in* M, *for every submodule* N *of* M.

Proof. For the eqivalence of (1)-(10) see Lemmas 3.6, 3.8, 3.11 and 3.12. Now suppose that M is a τ-UCC module. Let N be any submodule of M. Then N^+ is the unique τ-coclosure of N in M by the proof of Lemma 3.11. □

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