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# Control-Oriented Modeling of Distributed Parameter Systems[1]

*In this paper the use of linear, time-invariant, distributed parameter systems (LTI-DPS) as models of physical processes is considered from a control viewpoint. Specifically, recent theoretical results obtained by the authors for the control-oriented modeling of LTI-DPS are concisely reviewed and then a series of applications is given in order to illustrate the practical ramifications of these results.*

## 1 Introduction

The suitability of linear, time-invariant, distributed parameter systems (LTI-DPS) for use as models in the design of feedback control systems has recently been investigated by Helmicki et al. (1991). There it is argued that a plant model is *ill-posed* for use in feedback compensator design if (i) there exists no stabilizing feedback compensation for the plant model which renders the open-loop map of the feedback system strictly proper or (ii) there exists no strictly proper stabilizing feedback compensation for the plant model. This argument is based on (*a*) the need for stability robustness in the face of unmodeled high frequency dynamics and (*b*) the desire to be able to test the implemented controller in a hardware-in-the-loop simulation against the plant model, respectively. The principal contribution of the paper by Helmicki et al. is the characterization of these classes of ill-posed models. This characterization is carried out within a mathematical framework which encompasses a large number of models of engineering interest, with stability taken to mean exponential stability. The key qualitative conclusion which follows from this analytical characterization is that *the class of LTI-DPS models which are not ill-posed for use in feedback control system design is precisely the class of LTI-DPS models whose instabilities are essentially lumped in nature.*

To make more concrete the qualitative conclusion cited above, consider for a moment a simple undamped wave equation:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

with boundary conditions:

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} = -\alpha_2 u(t), \quad \frac{\partial v(1,t)}{\partial x} = 0.$$

This equation is considered in detail in the sequel. Regarding $u(t)$ as the control input and

$$y(t) = \frac{\partial v(x^*,t)}{\partial t}$$

as the controlled output, where $x^* \in [0, 1]$ is arbitrarily chosen, it is easy to show that the transfer function from $u$ to $y$ has an infinite number of poles lying on the $j\omega$-axis. As such, the instabilities of this system are *not* essentially lumped in nature, and it follows from the results of Helmicki et al. that this system is ill-posed for use in feedback control system design. Indeed, although this system can be stabilized by static output feedback, this constitutes infinite bandwidth compensation, and in fact there exists no finite-bandwidth (strictly proper) stabilizing dynamic output feedback compensation for this system. Note, however, that if damping is added to the above model, so that the PDE given above is replaced by:

$$\frac{\partial^2 v(x,t)}{\partial t^2} + \epsilon_1 \frac{\partial v(x,t)}{\partial t} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

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then in this case the instabilities of the system are essentially lumped in nature, and hence the system is not ill-posed in the sense defined by Helmicki et al.

In the spirit of the above simple example, the aim of the current paper is to delineate, by explicit examples, the widespread applicability and practical ramifications of the theoretical results given by Helmicki et al. Specifically, examples are given here demonstrating that: (i) the mathematical framework of models formulated by Helmicki et al. does indeed encompass a large number of models of engineering interest, and (ii) models violating the conditions established by Helmicki et al. have been proposed for use in control design in the literature. An additional aim of the current paper is to clarify the theoretical results derived by Helmicki et al. by demonstrating that: (i) the criteria used by Helmicki et al. to delineate ill-posed models are compatible with criteria used by other authors in investigating control-oriented modeling issues for LTI-DPS, in the sense that models deemed ill-posed in other works are also "tagged" as ill-posed under the criteria adopted here, and (ii) if stability is taken to mean BIBO stability instead of exponential stability, the qualitative conclusion stated above no longer holds, and must be replaced by a slightly weaker qualitative conclusion.

Accordingly, this paper is organized as follows: In Section 2 the basic notation and definitions used in the sequel are given and the main results from the paper by Helmicki et al. are concisely summarized. In Section 3 engineering interpretations of these results are given. In Section 4 a series of examples illustrating the points discussed in Section 3 is presented. In Section 5 results analogous to those given in the previous sections under the requirement of closed-loop exponential stability are presented for the weaker requirement of closed-loop BIBO stability. Finally, conclusions are given in Section 6.

## 2 Review

The purpose of this section is to review the theoretical results on control-oriented modeling of distributed parameter systems obtained by Helmicki et al. (1991). The reader is directed to the series of papers by these authors (1991, 1990, 1989) and the references therein for details, proofs, and alternative expositions.

We begin by establishing some basic notation. Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{C}$ denote the set of complex numbers. Let $\mathbf{R}_+ := \{x \in \mathbf{R} | x \geq 0\}$. For $\sigma \in \mathbf{R}$, let ORHP $(\sigma) = \{s \in \mathbf{C} | \mathrm{Re}\, s > \sigma\}$, with ORHP := ORHP(0), and let CRHP($\sigma$) $= \{s \in \mathbf{C} | \mathrm{Re}\, s \geq \sigma\}$, with CRHP := CRHP(0). For $p \in [1,\infty]$, let $L_p$ denote the usual spaces of Lebesgue measurable functions with support on $\mathbf{R}_+$. For any Laplace transformable distribution $f$ with support on $\mathbf{R}_+$, let $\hat{f}$ denote the Laplace transform of $f$ (Callier and Desoer, 1979). Finally, for any set $X$, let $M(X)$ denote the set of all matrices with elements in $X$.

The following convolution algebras and their associated properties will also be needed in the sequel.

Definition 1 $L_{1,\sigma} := \{f: \mathbf{R}_+ \rightarrow \mathbf{R} | e^{-\sigma(\cdot)} f(\cdot) \in L_1\}$.

Definition 2 For $\sigma \in \mathbf{R}$, $f \in \mathcal{A}(\sigma)$ *if*

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ f_a(t) + f_{ap}(t), & \text{for } t \geq 0' \end{cases}$$

where (i) $f_a \in L_{1,\sigma}$, and (ii) $f_{ap}(t) = \Sigma_{i=0}^{\infty} f_i \delta(t - t_i)$, with $\delta(\cdot)$ denoting the Dirac delta, $f_i \in \mathbf{R}$ for all $i$, $0 = t_0 < t_1 < \ldots$, and $\Sigma_{i=0}^{\infty} |f_i| e^{-\sigma t_i} < \infty$.

Fact 1 For $f \in \mathcal{A}(\sigma)$, $\hat{f}$ is analytic in ORHP($\sigma$) and the function $\omega \rightarrow \hat{f}(\sigma + j\omega)$ is uniformly continuous and bounded on $\mathbf{R}$.

Fact 2 For $f \in \mathcal{A}(\sigma)$, $f \in L_{1,\sigma}$ iff $|\hat{f}(s)| \rightarrow 0$ as $|s| \rightarrow \infty$ in ORHP($\sigma$).

We next delineate the mathematical framework of LTI-DPS models assembled by Helmicki et al. (1991). This framework is given in terms of the algebra of impulse responses:

$$\mathcal{W} := \bigcup_{\sigma < \infty} \mathcal{A}(\sigma).$$

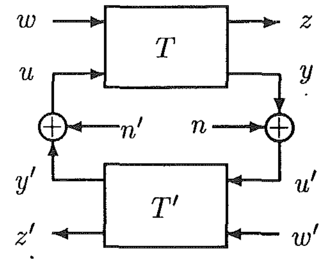

Fig. 1 The control systems configuration $\Sigma(T, T')$

Specifically, the set of all models considered is given by the set of all systems, $\Sigma(T, T')$, of the form shown in Fig. 1, where $T \in M(\mathcal{W})$ models the plant and $T' \in M(\mathcal{W})$ models the controller. For ease of reference, the models $T$ and $T'$ are partitioned conformably with respect to the signals shown in Fig. 1 as follows:

$$\begin{bmatrix} z \\ y \end{bmatrix} = T \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = T' \begin{bmatrix} w' \\ u' \end{bmatrix} = \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} \begin{bmatrix} w' \\ u' \end{bmatrix}.$$

In addition, the following input/output maps are defined for $\Sigma(T, T')$:

$$\begin{bmatrix} z \\ y \\ z' \\ y' \end{bmatrix} = \Phi \begin{bmatrix} w \\ n' \\ w' \\ n \end{bmatrix}, \quad \begin{bmatrix} u \\ u' \end{bmatrix} = \Psi \begin{bmatrix} w \\ n' \\ w' \\ n \end{bmatrix}.$$

Within this framework of models the subclasses of models corresponding to stable systems and strictly proper systems are defined similarly in terms of the subalgebras:

$$\mathcal{S} := \bigcup_{\sigma < 0} \mathcal{A}(\sigma) \quad \text{and} \quad \mathcal{SP} := \bigcup_{\sigma < \infty} L_{1,\sigma},$$

respectively.

Finally, we identify two subclasses of models which are potentially ill-posed for use in feedback design:

Definition 3 For a model $T \in M(\mathcal{W})$, $T \in [IP]_{\mathrm{OLM}}$ if there does not exist a $T' \in M(\mathcal{W})$ such that (i) $\Phi, \Psi \in M(\mathcal{S})$ and (ii) $T_{22}T'_{22}$, $T'_{22}T_{22} \in M(\mathcal{SP})$.

Definition 4 For a model $T \in M(\mathcal{W})$, $T \in [IP]_C$ if there does not exist a $T' \in M(\mathcal{W})$ such that (i) $\Phi, \Psi \in M(\mathcal{S})$ and (ii) $T'_{22} \in M(\mathcal{SP})$.

The main result of the paper by Helmicki et al. (1991) is the characterization of the elements of $[IP]_{\mathrm{OLM}}$ and $[IP]_C$. In order to state this characterization we must first define yet another subalgebra:

$\mathcal{WP} := \{f \in \mathcal{W} |$ there exists $g \in \mathcal{S}$ and rational, strictly proper $\hat{r}$ such that $f = g + r.\}$,

and then define:

Definition 5 A model $T \in M(\mathcal{WP})$ is said to be admissible if $T_{22}$ and $T$ share the same CRHP poles in terms of both location and McMillan degree.[2]

The following theorem is the main result of Helmicki et al. (1991). It shows that modulo an admissibility condition, $M(\mathcal{WP})$ is the complement of $[IP]_C \cup [IP]_{\mathrm{OLM}}$ in $M(\mathcal{W})$, i.e., impulse responses in $\mathcal{WP}$ effectively comprise the class of models which are not ill-posed.

Theorem 1 For a model $T \in M(\mathcal{W})$ the following are equiv-

[2]Suppose $X \in M(\mathcal{WP})$ and $p \in$ CRHP is a pole of $X$. The McMillan degree of $p$ as a pole of $X$ is the highest order it has as a pole of any minor of $X$.

alent: (i) $T \in [IP]_C$, (ii) $T \in [IP]_{\text{OLM}}$, (iii) either $T \notin M(\mathcal{WP})$ or $T$ is not admissible.

The final result of this section, also proven by Helmicki et al. (1991), is a direct consequence of Theorem 1:

Corollary 1 If $T \in M(\mathcal{W})$ is such that $T_{22} \in M(\mathcal{SP})$ and $T \notin M(\mathcal{WP})$, then there does not exist a $T' \in M(\mathcal{W})$ such that $\Phi$, $\Psi \in M(\mathcal{S})$.

## 3 Discussion

The purpose of this section is to provide an engineering interpretation corresponding to the summary of the results of Helmicki et al. (1991) given in the previous section.

As alluded to above, the sets $\mathcal{W}$, $\mathcal{S}$, and $\mathcal{SP}$ correspond to, respectively, the world of all impulse responses, the subclass of stable impulse responses, and the subclass of strictly proper impulse responses to be considered in this paper. Examples given later in this paper serve to demonstrate the generality of the set $\mathcal{W}$ as a global modeling class. The notion of stability corresponding to the set $\mathcal{S}$ is known to encompass BIBO as well as exponential stability. This claim is supported by the following well-known facts:

Fact 3 (Desoer and Vidyasagar, 1975, p. 247) For $f \in \mathcal{S}$, $p \in [1, \infty]$, if $u \in L_p$, then $y = f * u \in L_p$, i.e., systems in $\mathcal{S}$ are $L_p$ stable for all $p$.

Fact 4 (Callier and Winkin, 1986, Fact 2.2) For $f \in \mathcal{S}$, $u(t) = 1(t)$, and $y = f * u$, there exists $\rho < \infty$ and $\sigma < 0$ such that $|y(t) - \hat{f}(0)| \le \rho e^{\sigma t}$ for all $t \ge 0$, i.e., systems in $\mathcal{S}$ track step inputs exponentially.[3]

Fact 5 (Cheng and Desoer, 1982, p. 369) For $f \in \mathcal{S}$, $u \in L_\infty$ with compact support, and $y = f * u$, there exists $\delta < \infty$, $K < \infty$, and $\sigma < 0$ such that $|y(t)| \le K \|u\|_\infty e^{\sigma t}$ for all $t \ge \delta$, i.e., responses of systems in $\mathcal{S}$ to $L_\infty$ inputs with compact support decay exponentially.

Finally, the notion of strict properness given by $\mathcal{SP}$ agrees with the intuitive notion that the class of strictly proper impulse responses is exactly the sub-class of impulse responses for which the outputs of the corresponding convolution systems do not depend instantaneously on any portion of the applied inputs, past or present.[4] The characterization given in Fact 2 also provides further justification for this definition by showing that the elements of $\mathcal{SP}$ are precisely those elements in $\mathcal{W}$ which have transforms that "roll-off" in some half plane. Note, however, that transforms of elements of $\mathcal{SP}$ need not "roll-off" along the imaginary axis; equivalently, elements of $\mathcal{SP}$ need not be bandlimited. Indeed, as will be shown later, there exist $f \in \mathcal{SP}$ such that $|\hat{f}(j\omega)| \nrightarrow 0$ as $|\omega| \to \infty$. Thus, though strict properness of $f$ is necessary for the bandlimitedness of $f$, it is not sufficient, in general. However, from Fact 2 it follows that bandlimitedness and stict properness are equivalent for elements of the class of stable systems $\mathcal{S}$.

Given these interpretations it follows that the set $[IP]_{\text{OLM}}$ contains those models within our framework which cannot be stabilized in such a way that the resulting feedback system open-loop map is strictly proper (hence the subscript "OLM"). The set $[IP]_C$ contains those models within our framework which cannot be stabilized with strictly proper compensation (hence the subscript "$C$"). The rationale for considering the elements of $[IP]_{\text{OLM}}$ as ill-posed stems from the fact these models cannot be stabilized robustly in the face of unmodeled high frequency domains (see Chen and Desoer, 1982; Doyle and Stein, 1981; Helmicki et al., 1991). The rationale for considering the elements of $[IP]_C$ as ill-posed stems from the fact that all physical systems exhibit strictly proper behavior. Thus, models in $[IP]_C$ have the rather paradoxical property that although stabilizing feedback compensation may exist for the model, no physical implementation of this compensation could possibly stabilize the model. Clearly, this attribute of the model would render it rather suspect for use in a hardware-in-the-loop simulation of the model against a hardware implementation of a controller design based on the model (see Helmicki et al., 1991).

In order to fully understand the practical implications of Theorem 1, it is first necessary to at least heuristically understand the concept of model admissibility put forth in Definition 5. Such an interpretation can be given as follows: A plant $T \in M(\mathcal{WP})$ is admissible iff the unstable dynamics of $T$ are "built into" $T_{22}$ and hence, built into the loop of $\Sigma(T, T')$ (see Fig. 1). Roughly speaking then, plant model admissibility implies that the sensors and actuators selected for control are "fully connected to" the unstable plant dynamics.[5] Hence, it follows that the condition on admissibility in Theorem 1 merely reflects the fact that in order for $\Sigma(T, T')$ to be stabilizable the plant sensors and actuators must be chosen so that $T_{22}$ "fully contains" the unstable plant dynamics. Thus, it is the condition that $T \notin M(\mathcal{WP})$ which places the interesting constraint on the form of the impulse responses comprising the classes of ill-posed models. In particular, this result indicates that $\mathcal{WP}$ is precisely the class of LTI-DPS impulse responses which should be considered in order to avoid the difficulties associated with ill-posedness as given by $[IP]_{\text{OLM}}$ and/or $[IP]_C$. It therefore follows that the models within our framework which are not ill-posed are precisely those which possess at most a finite number of singularities in some open half plane containing the CRHP, with these singularities all poles of finite multiplicity. In other words *the class of LTI-DPS models which are not ill-posed for use in feedback control system design is precisely the class of LTI-DPS models whose instabilities are essentially lumped in nature.*

Finally, we note that Corollary 1 delineates a set of (possibly admissible) plant models for which there exists no stabilizing feedback compensation, independent of ill-posedness considerations.

## 4 Examples

In this section the discussion given above is amplified by applying the results presented in Section 2 to a series of LTI-DPS models. Some of these models are drawn from the area of flexible structures, and many are taken directly from recent literature (see Datko, 1988; Hara et al., 1988; Morris and Vidyasagar, 1988; Piche, 1985; Piche, 1987; Vidyasagar and Morris, 1987). Interestingly, several of the models are shown to be ill-posed for use in feedback design.

**4.1 State-Space Model Formulations.** Consider the class of models given by

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where the following are assumed:

(i) $x(t) \in H$, a Hilbert space, $u(t) \in \mathbf{R}^n$, $y(t) \in \mathbf{R}^m$,

(ii) $A$ is the generator of a $C_0$-semigroup of bounded linear operators on $H$ denoted by $\{e^{At}\}_{t \ge 0}$,

(iii) $B$, $C$, $D$ are bounded linear operators, i.e., $B \in \mathcal{L}(\mathbf{R}^n, H)$, $C \in \mathcal{L}(H, \mathbf{R}^m)$, and $D \in \mathcal{L}(R^n, R^m)$.

Given any $x(0) \in \text{dom}(A)$, where $\text{dom}(A)$ denotes the domain of the operator $A$, and any input $u$, the response of this system can be described by the convolution relation:

[3] The symbol $1(t)$ denotes the unit step function.

[4] This follows as a consequence of the fact the elements of $\mathcal{SP}$ do not contain any impulsive terms.

[5] The concept of admissibility also has interpretations in terms the state-space notions of joint stabilizability/detectability and the fractional representation notions of right/left coprimeness (see Nett, 1986).

$$y(t) = h_{zi} * \delta(t) + h_{zs} * u(t),$$

where $h_{zi}(t) := Ce^{At}x(0)$, and $h_{zs}(t) := Ce^{At}B + D\delta(t)$.

In addition, suppose the operators $C' \in \mathcal{L}(H, \mathbf{R}^k)$ and $D' \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^k)$ are chosen so that the kernel $h'_{zs}(t) := C'e^{At}B + D'\delta(t)$ denotes the transfer map from $u$ to any $k$ critical (but not necessarily measured) responses of the plant. In this case, $h'_{zi}(t) := C'e^{At}x(0)$ denotes the corresponding initial condition response kernel.

We begin by showing that under the assumptions above $h_{zi}$, $h_{zs}$, $h'_{zi}$, $h'_{zs} \in M(\mathcal{W})$. For this purpose we recall the following key fact concerning $C_0$-semigroups (Balakrishnan, 1982, p. 166): By assumption (ii) there exist real constants $K$ and $\omega$ so that $\|e^{At}\| \le Ke^{\omega t}$, $t \ge 0$. Thus, it follows from (iii) that $\|h_{zs}\| \le \|C\| \cdot \|B\| \cdot Ke^{\omega t}$, $\|h_{zi}\| \le \|C\| \cdot \|x(0)\| \cdot Ke^{\omega t}$, and likewise for $h'_{zs}$ and $h'_{zi}$. These inequalities guarantee that $h_{zi}$, $h_{zs}$, $h'_{zi}$, $h'_{zs} \in M(\mathcal{Q}(\sigma))$ for any $\sigma > \omega$.

Second, we demonstrate that a model of the form discussed in Section 2 can be given which accounts for all of the effects discussed above. Specifically, consider:

$$T := \begin{bmatrix} h'_{zi} & h'_{zs} \\ h_{zi} & h_{zs} \end{bmatrix} = \begin{bmatrix} C' \\ C \end{bmatrix} e^{At}[x(0) \quad B] + \begin{bmatrix} 0 & D'\delta(t) \\ 0 & D\delta(t) \end{bmatrix} \in M(\mathcal{W}).$$

Here $u$ and $y$ have their usual roles, $z$ represents unmeasurable outputs, and $w(t) = \delta(t)$ represents unmanipulated inputs in the form of initial condition effects.[6] Hence, it follows immediately that all differential equations of the form considered here (i.e., with bounded sensing and control operators) can be characterized fairly completely in terms of models of the form discussed in Section 2.

Third, we show that the notion of stability typically associated with the class of state-space systems described above is compatible with the notion of input/output stability adopted here. Suppose (i)–(iii) above hold and further suppose that

(iv) the $C_0$-semigroup generated by $A$ is exponentially stable, i.e., there exist $K \in [1,\infty)$ and $\omega \in (-\infty,0)$ such that $\|e^{At}\| \le Ke^{\omega t}$.

Under this additional assumption, it follows from the inequalities $\|h_{zs}\| \le \|C\| \cdot \|B\| \cdot Ke^{\omega t}$, etc., that $h_{zi}$, $h_{zs}$, $h'_{zi}$, $h'_{zs} \in M(\mathcal{S})$.[7]

Finally, we show that under some mild additional assumptions these models are actually non-ill-posed. Specifically, assume in addition to (i)–(iii) above that:

(v) the pair $(A, B)$ is stabilizable and the pair $(C, A)$ is detectable.[8]

Under these conditions, it can be shown (Jacobson and Nett, 1988, Lemma 24) that the spectrum of the semigroup generator $A$ decomposes into a stable component and a finite dimensional unstable component, and hence that $h_{zi}$, $h_{zs}$, $h'_{zi}$, $h'_{zs} \in M(\mathcal{W}\mathcal{P})$. In fact, it follows from this decomposition property, Definition 5, and assumption (v) that the $T$ given above is also admissible.

**4.2 The Undamped Wave Equation.** Next we consider a typical LTI-DPS model of a flexible structure to show that models of the form discussed in Section 2 can also be used to describe systems with point and/or boundary sensing and/or actuation. Such systems normally give rise to unbounded sensing and/or control operators in a state-space formulation and so are not included in the class of systems considered in Section 4.1.

Consider a uniform rod of unit length free to move at one end ($x = 1$) and subject to an axial force, $u(t)$, applied at the other end ($x = 0$). Suppose that damping effects are assumed negligible and therefore that the rod behaves in a perfectly elastic fashion. Let $v(x,t)$ denote the longitudinal displacement of a particular cross section $x$ at time $t$. Under these assumptions, the longitudinal motion of the rod can be described by the PDE (Fung, 1965; Graff, 1975):

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

with boundary conditions:

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} = -\alpha_2 u(t), \quad \frac{\partial v(1,t)}{\partial x} = 0.$$

If we take as outputs the displacements at various points $x_i$, $i = 1 \ldots m (0 \le x_i \le 1)$, along the rod, then the output equations become

$$y_i(t) = v(x_i,t), \quad i = 1\ldots m.$$

A convolution description of this system can be found from the associated Green's function (Butkovskiy, 1982, p. 86). The corresponding impulse responses are given by:

$$h_{x_i}(t) = \left(\alpha_2 t + \frac{2\alpha_2}{\pi\alpha_1} \sum_{k=1}^{\infty} \frac{\cos k\pi x_i}{k} \sin\alpha_1 k\pi t\right) 1(t).$$

We begin by showing that these impulse responses belong to $\mathcal{W}$. Rewriting the series portion of $h_{x_i}(t)$ as:

$$\sum_{k=1}^{\infty} \frac{\cos k\pi x_i}{k} \sin\alpha_1 k\pi t = \sum_{k=1}^{\infty} \frac{1}{k} \left(\sin(k\pi(x_i + \alpha_1 t)) - \sin(k\pi(x_i - \alpha_1 t))\right),$$

we note that each of the terms on the right side of this equality yields a uniformly bounded function a.e. (Apostol, 1957, p. 366), and so we have, by an application of Dirichlet's test (Apostol, 1957, p. 366), that the series portion of $h_{x_i}$ converges uniformly a.e. on $[0,\infty)$. It therefore follows that there exists $N$ such that for almost all $t \in [0,\infty)$, $|\Sigma_{k=N}^{\infty} \cos k\pi x_i / k \sin \alpha_1 k\pi t| < 1$. Hence, for $\sigma > 0$:

$$\int_0^{\infty} e^{-\sigma t} |h_{x_i}(t)| dt \le \alpha_2 \int_0^{\infty} te^{-\sigma t} dt + \frac{2\alpha_2}{\pi\alpha_1} \int_0^{\infty} \times e^{-\sigma t} \left| \sum_{k=1}^{N-1} \frac{\cos k\pi x_i}{k} \sin\alpha_1 k\pi t \right| dt + \frac{2\alpha_2}{\pi\alpha_1} \int_0^{\infty} e^{-\sigma t} \left| \sum_{k=N}^{\infty} \frac{\cos k\pi x_i}{k} \sin\alpha_1 k\pi t \right| dt$$

$$\le \alpha_2 \int_0^{\infty} te^{-\sigma t} dt + \frac{2N\alpha_2}{\pi\alpha_1} \int_0^{\infty} e^{-\sigma t} dt < \infty.$$

Thus, $h_{x_i} \in \mathcal{Q}(\sigma)$ for all $\sigma > 0$.

While the discussion above pertains specifically to the zero-state case, we note that impulse responses describing the zero-input response corresponding to any specific initial condition can also be derived from the corresponding Green's functions (Butkovskiy, 1982). By analysis similar to that performed above these impulse responses can also be shown to be in $\mathcal{W}$. Thus, by using the various sub-blocks of $T$ as in the last section, a "complete" model $T \in M(\mathcal{W})$ can be assembled describing the longitudinal motion considered here. However, it can clearly be seen that the impulse responses $h_{x_i}$ comprising this model contain an infinite number of unstable modes. Thus, it follows that $T \notin M(\mathcal{W}\mathcal{P})$.

In fact, using Corollary 1, we can show that the situation for this model is actually much worse. Consider the corresponding transfer functions:

[6]Note that responses for any finite number of initial conditions $\{x_i(0)\}_{i=0}^{l}$ can be incorporated into the entries of $T_{12}$ and $T_{21}$. Then any initial condition in the span of $\{x_i(0)\}_{i=0}$ can be triggered by applying the appropriate unmanipulated input $w(t) = x\delta(t)$, $x \in \mathbf{R}^l$.

[7]For details on the converse of this result see Jacobson and Nett (1988).

[8]Here stabilizability of the pair $(A, B)$ is defined in terms of the existence of an operator $F \in \mathcal{L}(H, \mathbf{R}^n)$ such that the $C_0$-semigroup generated by $A - BF$ is exponentially stable, and detectability of the pair $(C, A)$ corresponds to stabilizability of the adjoint pair $(A^*, C^*)$.

$$\hat{h}_{x_i}(s) = \frac{\alpha_2 \cosh\left(\dfrac{(x_i - 1)}{\alpha_1} s\right)}{\alpha_1 s \sinh\left(\dfrac{s}{\alpha_1}\right)}.$$

As a result of the $s$ term in the denominator above we have that $|\hat{h}_{x_i}(s)| \to 0$ as $|s| \to \infty$ in the ORHP. Hence, by Fact 2, we have that $h_{x_i} \in \mathcal{SP}$. Thus, it follows from Corollary 1 that any model $T$ based on the above choices for $u$ and $y$ (i.e., force actuation and displacement sensing) cannot be stabilized.

Before leaving this example two more points should be noted. First, even though we have shown that $h_{x_i} \in \mathcal{SP}$, it is clear from the expression for $\hat{h}_{x_i}$ that $|\hat{h}_{x_i}(j\omega)| \not\to 0$ as $|\omega| \to \infty$. Hence, the $h_{x_i}$'s given here are examples of impulse responses that are not bandlimited even though they are strictly proper. Second, we note that in Piche (1985) it is shown that the model considered above can be stabilized if velocity outputs are measured and proportional feedback is used. However since the corresponding transfer functions differ from those considered above by only a factor of $s$, it follows that the undamped wave equation results in an ill-posed model even if velocity outputs are measured.

**4.3 Damped Wave Equation Models.** In the last section we ruled out as unsuitable for use in feedback design an undamped wave equation model for the longitudinal motion of a uniform rod. Below a series of examples is given which explore the effect that the addition of damping would have on the suitability of this model.

*4.3.1 External Damping Mechanisms.* Suppose that to the model of the rod considered in Section 4.2 we add the assumption of an external damping mechanism which operates according to a viscous relation, i.e., acting on each rod cros-section there is an external restoring force proportional to the crossectional velocity. An LTI-DPS model which incorporates such effects is given by (Fung, 1965; Graff, 1975):

$$\frac{\partial^2 v(x,t)}{\partial t^2} + \epsilon_1 \frac{\partial v(x,t)}{\partial t} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

with boundary conditions:

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} = -\alpha_2 u(t), \quad \frac{\partial v(1,t)}{\partial x} = 0.$$

A convolution description similar to the one obtained in the undamped case can be derived from the corresponding Green's functions (Butkovskiy, 1982, p. 92). The impulse responses corresponding to the transfer maps from the force input to the displacement outputs are given by:

$$h_{x_i}(t) = \alpha_2 (h_{x_i,1}(t) + 4e^{-\epsilon_1 t/2}(h_{x_i,2}(t) + h_{x_i,3}(t)))1(t),$$

where

$$h_{x_i,1}(t) = \frac{1}{\epsilon_1}(1 - e^{-\epsilon_1 t}),$$

$$h_{x_i,2}(t) = \sum_{1 \le k < N} \frac{\cos(k\pi x_i)}{\gamma_{k,2}} \sinh(\gamma_{k,2} t),$$

$$h_{x_i,3}(t) = \sum_{k > N} \frac{\cos(k\pi x_i)}{\gamma_{k,3}} \sin(\gamma_{k,3} t),$$

with $N = \epsilon_1/(2\pi\alpha_1)$ and

$$\gamma_{k,2} = \sqrt{\left(\frac{\epsilon_1}{2}\right)^2 - (k\pi\alpha_1)^2}, \quad \gamma_{k,3} = \sqrt{(k\pi\alpha_1)^2 - \left(\frac{\epsilon_1}{2}\right)^2}.$$

A cursory inspection of $h_{x_i}$ reveals that the addition of external damping appears to have taken care of the troublesome vibratory modes which plagued the undamped case. Certainly each of the terms in $h_{x_i,1}$ is in $\mathcal{WP}$, and likewise for each of the finite number of terms in $e^{-\epsilon_1(\cdot)/2}h_{x_i,2}(\cdot)$. The term resulting from $h_{x_i,3}$, however, requires closer inspection. Rewriting

$$\sin(\gamma_{k,3}t) = \sin(\gamma_{k,3}t) - \sin(\alpha_1 k\pi t) + \sin(\alpha_1 k\pi t)$$
$$= \frac{1}{2}\sin(2(\gamma_{k,3} - \alpha_1 k\pi)t)\cos(2\alpha_1 k\pi t)$$
$$- \sin^2((\gamma_{k,3} - \alpha_1 k\pi)t)\sin(2\alpha_1 k\pi t) + \sin(\alpha_1 k\pi t),$$

and noting that $\gamma_{k,3} \to k\pi\alpha_1$ as $k \to \infty$, it follows by reasoning analogous with that in Section 4.2 (i.e., an application of Dirichlet's test, etc.) that $e^{-\epsilon_1(\cdot)/2}/h_{x_i,3}(\cdot) \in \mathcal{A}(\sigma)$ for all $\sigma > -\epsilon_1/2$. Thus $h_{x_i}$, being the sum of three terms each in $\mathcal{WP}$, is in $\mathcal{WP}$. Hence, by proper choice of sensors and actuators (i.e., by proper choice of $T_{22}$ so that $T$ is admissible) we see from Theorem 1 that externally damped rod models $T$ can be found which avoid the difficulties associated with ill-posedness.

*4.3.2 Internal Damping Mechanisms.* Next we explore the effect of the addition of internal damping. Specifically, it is assumed that the rod has the viscoelastic characteristics of a Voigt solid (Fung, 1965; Graff, 1975). The resulting PDE is given by:

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \epsilon_2 \frac{\partial^3 v(x,t)}{\partial x^2 \partial t} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

with boundary conditions:

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} + \epsilon_2 \frac{\partial^2 v(0,t)}{\partial x \partial t} = -\alpha_2 u(t),$$

$$\alpha_1^2 \frac{\partial v(1,t)}{\partial x} + \epsilon_2 \frac{\partial^2 v(1,t)}{\partial x \partial t} = 0.$$

Unlike the previous cases, the Green's function for this PDE is not readily available. However, by solving the two point boundary value problem associated with the transformed PDE, and inverting the resulting transfer function we obtain:

$$h_{x_i}(t) = \alpha_2 (t + h_{x_i,1}(t) + h_{x_i,2}(t))1(t),$$

where

$$h_{x_i,1}(t) = \sum_{1 \le k < N} \frac{\cos(k\pi x_i)}{\gamma_{k,1}} e^{-\frac{\eta}{2}t} \sin(\gamma_{k,1} t),$$

$$h_{x_i,2}(t) = \sum_{k > N} \frac{\cos(k\pi x_i)}{\gamma_{k,2}} e^{-\frac{\eta}{2}t} \sinh(\gamma_{k,2} t),$$

with $N = 2\alpha_1/(\pi\epsilon_2)$ and

$$\gamma_{k,1} = k\pi\sqrt{\alpha_1^2 - \left(\frac{k\pi\epsilon_2}{2}\right)^2}, \quad \gamma_{k,2} = k\pi\sqrt{\left(\frac{k\pi\epsilon_2}{2}\right)^2 - \alpha_1^2},$$

$$\eta = (k\pi)^2 \epsilon_2.$$

As with the externally damped case, it appears as though these impulse responses have the desired properties, i.e., $h_{x_i} \in \mathcal{WP}$. Clearly, $h_{x_i,1} \in \mathcal{WP}$. Straightforward analysis of $h_{x_i,2}$ (i.e., an application of the M-test (Apostol, 1957, p. 396) to show uniform convergence, ...) reveals that $h_{x_i,2} \in \mathcal{WP}$. Thus, as expected, the addition of internal damping effects also results in a model of longitudinal motion which (modulo the suitable selection of sensors and actuators) is not ill-posed.

**4.4 Longitudinal Motion of a Semi-Infinite Rod.** The results of the last section seem to indicate that the addition of damping to flexible structure models has a beneficial effect on the suitability of the corresponding models for use in feedback design. In this section we demonstrate that this is not a universal rule by considering the longitudinal motion of a semi-infinite rod.

*4.4.1 The Undamped Case.* Consider a perfectly elastic rod of semi-infinite length. An LTI-DPS model for the longitudinal motion corresponding to this situation is given by (Fung, 1965; Graff, 1975):

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

with boundary conditions:

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} = -\alpha_2 u(t), \quad \lim_{x \to \infty} \frac{\partial v(x,t)}{\partial x} = 0.$$

Again assuming displacement outputs, straightforward application of Laplace transform techniques yields the corresponding impulse responses:

$$h_{x_i}(t) = \frac{\alpha_2}{\alpha_1} 1\left(t - \frac{x_i}{\alpha_1}\right).$$

Clearly, $h_{x_i} \in \mathcal{W}\mathcal{P}$. Thus, we have the peculiar result that whereas the undamped model for the finite rod is ill-posed the undamped model for the semi-infinite rod is not.[9]

*4.4.2 The Damped Case.* Next we consider the effect of adding damping to the model considered above.[10] The PDE describing the externally damped semi-infinite rod is given as follows (Fung, 1965; Graff, 1975):

$$\frac{\partial^2 v(x,t)}{\partial t^2} + \epsilon_1 \frac{\partial v(x,t)}{\partial t} - \alpha_1^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0,$$

$$\alpha_1^2 \frac{\partial v(0,t)}{\partial x} = -\alpha_2 u(t), \quad \lim_{x \to \infty} \frac{\partial v(x,t)}{\partial x} = 0.$$

Applying Laplace transform techniques, the transfer functions:

$$\hat{h}_{x_i}(s) = \frac{\alpha_2}{\alpha_1 \sqrt{s^2 + \epsilon_1 s}} e^{-\frac{x_i}{\alpha_1}\sqrt{s^2 + \epsilon_1 s}},$$

are obtained. The reader should note that unlike any of the previous examples these transfer functions possess branch points at $s = 0, -\epsilon_1$. Using transform tables (Beyer, 1978), the corresponding impulse responses are found to be:

$$h_{x_i}(t) = \frac{\alpha_2}{\alpha_1} e^{-\frac{\epsilon_1}{2}t} I_0\left(\frac{\epsilon_1}{2}\sqrt{t^2 - \left(\frac{x_i}{\alpha_1}\right)^2}\right) 1\left(t - \frac{x_i}{\alpha_1}\right),$$

where $I_0(x) = 1 + x^2/2^2(1!)^2 + x^4/2^4(2!)^2 + \ldots$ That $h_{x_i} \in \mathcal{W}$ follows immediately from the fact that $|I_0(x)| = I_0(x)$ for all $x$. Note however that because $s = 0$ is a branch point of $\hat{h}_{x_i}$, it follows that $h_{x_i} \notin \mathcal{W}\mathcal{P}$. Thus we have the even more peculiar result that unlike the finite length case, the *addition* of external damping to the semi-infinite rod results in a model which is ill-posed. In fact, Corollary 1 can be used to show that this model, like the undamped finite length case, is not even stabilizable.[11]

**4.5 Repetitive Control.** The discussions above have illustrated some of the ramifications of Theorem 1 and Corollary 1 in the selection of models for flexible structures. However, the results of Helmicki et al. (1991) are not limited to this case. In this section we discuss a recently published paradigm for the design of a servo system which has the capability of tracking any periodic signal of fixed period $L$. This paradigm, termed repetitive control, is illustrated in Fig. 2 (Hara et al., 1988). Here $\hat{p}$, which is assumed to be rational, represents the plant model, and the goal is to design $\hat{k}$ so as to stabilize the augmented plant $\hat{p}\hat{a}$, where $\hat{a}(s) := 1/(e^{Ls} - 1)$. The desired tracking property is then assured as a direct consequence of the well-known internal model principle (Francis and Wonham, 1975). However, straightforward analysis reveals that $\hat{a}$ has poles at $s = \pm j2\pi m/L$, $m = 0, 1, \ldots$ and thus that $p * a \notin \mathcal{W}\mathcal{P}$. Furthermore, it follows from an application of Corollary 1 that for strictly proper $\hat{p}$ there exists no stabilizing compensation $\hat{k}$ for the augmented plant $\hat{p}\hat{a}$. While we note that the aforementioned aspects of repetitive control have not gone unnoticed by the authors mentioned above, the discussion here does serve to indicate that Theorem 1 and Corollary 1 have applications beyond those of merely assessing the suitability of flexible structure models.

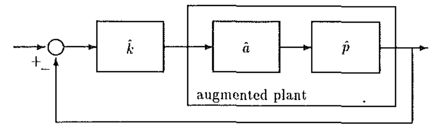


**Fig. 2 Repetitive control system**

**4.6 Other Notions of Ill-Posedness.** Up to this point we have studied the suitability of various models for use in feedback control system design based on the criterion set forth by the notions ill-posedness given in Definitions 3 and 4 and the associated characterization given in Theorem 1. However, we note that our authors have addressed the issue of model suitability from a control viewpoint for various specific models based on different criteria (Datko, 1988; Morris and Vidyasagar, 1988; Piche, 1987; Vidyasagar and Morris, 1987). The purpose of this section is to show that the notions of ill-posedness considered here are compatible with those considered in the works cited in the sense that the models deemed unsuitable in these works are also "tagged" as ill-posed according to the criteria adopted here.

*4.6.1 Delays in the Loop.* Since many modern control designs are implemented digitally, robustness with respect to small delays in the feedback loop is essential. Correspondingly, it is reasonable to regard those models for which any stabilizing controller can be made destabilizing by an arbitrarily small delay in the feedback loop as unsuitable for use in feedback compensator design. Studies of model suitability from this standpoint are documented in Piche (1987) and Datko (1988) for certain beam models and certain hyperbolic partial differential equations subject to boundary control, respectively.

Specifically, in Piche (1987) the slewing of a uniform undamped beam is considered using the transfer function:

$$\hat{p}(s) := \frac{\cos\sqrt{2\alpha s} + \cosh\sqrt{2\alpha s} + 2}{\sinh\sqrt{2\alpha s} - \sin\sqrt{2\alpha s}}.$$

It is shown that the closed-loop stability of any feedback design based on this model will be extremely sensitive to delays in the feedback path. However, since straightforward analysis contained in Piche (1987) reveals that this transfer function contains poles on the imaginary axis that tend to $\pm i(n + 1/4)$ as $n \to \infty$, it follows that $p \notin \mathcal{W}\mathcal{P}$.

Similar results are obtained in Datko (1988) for various examples of hyperbolic partial differential equations subject to boundary feedback. One such example is given by the PDE:

[9]Physically, this is a direct consequence of the fact that for the semi-infinite case there are no reflections from the far boundary and so the vibratory modes appearing in the finite length case are not present here.

[10]As analysis of both the externally damped and the internally damped semi-infinite rods lead to similar conclusions we shall present here only the externally damped case, and it should be understood that all conclusions drawn apply to the internally damped case as well.

[11]The authors note that physical intuition to support this behavior is, at this point, still not fully understood and is the subject of current research.

$$\frac{\partial^2 v(x,t)}{\partial t^2}+\frac{\partial^4 v(x,t)}{\partial x^4}=0,\quad 0<x<1,\quad t>0,$$
$$v(0,t)=\frac{\partial v(0,t)}{\partial x}=\frac{\partial^3 v(0,t)}{\partial x^3}=0,$$
$$\frac{\partial^2 v(1,t)}{\partial x^2}=f(t),$$

with feedback $f(t) = -\partial^2 v(1,t)/\partial x\partial t$. Again, straightforward analysis of the open-loop system reveals infinitely many poles on the imaginary axis. Hence the open-loop system cannot be described by an impulse response in $\mathcal{WP}$. Thus, in both of these cases the models considered are also ill-posed in the sense discussed in Section 2.

*4.6.2 Finite Dimensional Compensation.* Since the design of controllers employing only a finite number of scalars, adders, and integrators (i.e., controllers with rational transfer functions) is a typical constraint imposed upon control system engineers, the existence of stabilizing finite dimensional compensation yields yet another criterion by which the suitability of a given model for control system design can be judged. In Morris and Vidyasagar (1988) and Vidyasagar and Morris (1987) it is demonstrated that the undamped Euler-Bernoulli beam model is deficient in this regard and further, that the incorporation of Rayleigh damping rectifies the aberrant behavior. The specific transfer functions considered in these papers are given by:

$$\hat{p}(s)=\frac{L(\sinh(\beta\sqrt{s})+\sin(\beta\sqrt{s}))}{I_b s^2\beta\sqrt{s}(1+\cos(\beta\sqrt{s})\cosh(\beta\sqrt{s}))},$$

and

$$\hat{p}_R(s)=\frac{L((\sin\lambda(s))/\lambda(s)+(\sinh\lambda(s))/\lambda(s)-1-\cos\lambda(s)\cosh\lambda(s))}{I_b s^2(1+\cos\lambda(s)\cosh\lambda(s))},$$

respectively, where

$$\lambda(s):=\beta\left[\frac{s^2+c_v s}{1+c_d s}\right]^{1/4}$$

Straightforward analysis given in these papers verifies that $p \notin \mathcal{WP}$ and $p_R \notin \mathcal{WP}$. Hence, these results too are consistent with those given in Section 2.

## 5 Exponential Versus BIBO Stability

As noted in Section 1, the characterizations given in Theorem 1 and Corollary 1 are contingent upon the use of $\mathcal{S}$ as the set of stable impulse responses, i.e., the requirement of closed-loop exponential stability.[12] While we note that the requirement of closed-loop exponential stability is well rooted in engineering applications, it is true that other, less demanding, forms of closed-loop stability have been considered in the literature. In this section we will study the effect on the results discussed previously of relaxing the closed-loop stability requirement from exponential stability to BIBO stability (MacCluer, 1988; MacCluer, 1990). Specifically, we consider the alternative framework of models generated by using the classes of impulse responses $\mathcal{W}$, $\mathcal{S}'$, and $\mathcal{SP}$ where

$$\mathcal{S}':=\mathcal{A}(0).$$

It is well known in the literature (Callier and Desoer, 1978; Desoer and Vidyasagar, 1975) that the notion of stability corresponding to $\mathcal{S}'$ is precisely BIBO stability.

Using this alternative notion of stability the definitions of $[IP]_{\mathrm{OLM}}$ and $[IP]_C$ given previously can easily be adapted by simply making the appropriate substitutions. Although technical difficulties preclude a complete characterization of these classes of ill-posed models, straightforward application of the proof techniques employed in (Helmicki et al., 1991) yield a weakened version of Theorem 1. This result is stated in terms of the quotient algebra:

$$\mathcal{WP}':=\frac{\mathcal{A}(0)}{\mathcal{A}^{\infty}(0)}\subset\mathcal{W},$$

where

$$\mathcal{A}^{\infty}(0):=\{f\in\mathcal{A}(0)\mid \text{there exists } \gamma,\rho>0 \text{ such that } |\hat{f}(s)| >\gamma \text{ for all } s\in\Gamma(\rho)\},$$

with $\Gamma(\rho):=\{s\in \mathrm{ORHP}: |s|>\rho\}$, i.e., $\mathcal{A}^{\infty}(0)$ denotes those elements of $\mathcal{A}(0)$ whose Laplace transforms are bounded away from zero at infinity in the ORHP.

Theorem 2 If $T\in M(\mathcal{W})$ and $T\notin M(\mathcal{WP}')$, then $T\in [IP]_{\mathrm{OLM}}\cup[IP]_C$.

The following partial characterizations of $\mathcal{WP}'$ can be obtained by using Fact 1 and the elementary properties of analytic functions (Hille, 1962).

Fact 6 If $f\in\mathcal{WP}'$, then $\hat{f}$ cannot have a limit point of CRHP poles at infinity, i.e., $\hat{f}$ cannot have poles $p_k$, $k=1,2,\ldots$ with $\mathrm{Re}\, p_k\geq 0$, such that $p_k\to\infty$ as $k\to\infty$.

Fact 7 If $f\in\mathcal{WP}'$, then for any $\sigma>0$, $\hat{f}$ cannot have more than a finite number of singularities in ORHP $(\sigma)$, and these singularities must all be poles.

As a result of Fact 7, we see that even for the case of BIBO stability a significant restriction is placed on the nature of permissible transfer function singularities in order to avoid the difficulties associated with ill-posedness. In addition, we see from Fact 6 that models like the undamped wave equation and schemes like direct repetitive control still give rise to potential difficulties under the weaker stability requirement. However, we are careful to point out that Facts 6 and 7 cannot be used together with Theorem 2 to infer that under the BIBO stability requirement models with nonlumped instabilities are ill-posed. In fact, it is easy to construct impulse responses with nonlumped instabilities which are not ill-posed. Consider for example the kernel

$$h(t):=\sum_{k=1}^{\infty}(.9)^k e^{-t/k},\quad t\geq 0.$$

Since $h\in\mathcal{S}'$, it is stabilized by the null compensator which is clearly strictly proper, and thus $h$ is not ill-posed with respect to BIBO closed-loop stability despite the fact that $\hat{h}$ has a sequence of singularities on the negative real axis with a limit point at $s=0$.[13] Similar situations occur in the modeling of physical systems as can be seen by inspecting the $\mathcal{S}'$ stable transfer function

$$\hat{h}(s):=e^{-a\sqrt{s}}$$

which arises from the heat equation (MacCluer, 1990). Thus, it therefore follows that the qualitative conclusions obtained previously for the case of exponential stability must be weakened for the case of BIBO stability: Only those LTI-DPS models whose instabilities in the ORHP are lumped can be nonill-posed.

In any case, we note that under the relaxation to BIBO stability only a partial characterization of ill-posedness is pos-

[12]Obviously, the characterizations obtained in Theorem 1 and Corollary 1 remain valid when the stronger notion of exponential stability corresponding to the convolution algebras proposed by Callier and Winkin (1986) is utilized.

[13]We note that this condition implies that $h$ is ill-posed with respect to exponential closed-loop stability, i.e., $h\notin\mathcal{WP}$.

sible. We contrast this with the complete characterization of ill-posedness afforded by Theorem 1 for the case of exponential stability. This fact provides additional motivation for adopting the notion of exponential stability utilized in (Helmicki, 1991).

## 6 Conclusions

In this paper the control-oriented modeling issues considered by Helmicki et al. (1991) have been amplified and clarified by way of a series of concrete examples. These examples serve to demonstrate that the framework within which these issues were initially delineated encompasses a wide range of models of engineering interest, including state-space models with bounded sensing and control operators (see Section 4.1), models with unbounded sensing and control operators (see Sections 4.2 through 4.4), and even models whose instabilities are infinite-dimensional in nature (see Section 4.2 and Section 4.4.2). Within this framework of models, it has been argued that when stability is taken to mean exponential stability, those models whose instabilities are essentially finite-dimensional in nature are precisely the models which are viable for use in feedback control system design. In addition, the examples provided demonstrate that this condition cannot be ensured merely by accounting for such physical effects as damping. Furthermore, the examples given serve to demonstrate that the notions of ill-posedness described here are compatible with other notions of ill-posedness considered in the literature. Finally, the dependence of the characterizations of ill-posedness on the particular form of closed-loop stability required has also been addressed, and it has been shown that when the closed-loop stability requirement is relaxed from exponential stability to BIBO stability, many though not all of the systems ill-posed in the former case remain ill-posed in the latter case.

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