Bichromatic Compatible Matchings

Greg Aloupis*† Luis Barba*‡§ Stefan Langerman*¶ Diane L. Souvaine‖

Abstract

For a set R of n red points and a set B of n blue points, a *BR-matching* is a non-crossing geometric perfect matching where each segment has one endpoint in B and one in R. Two BR-matchings are compatible if their union is also non-crossing. We prove that, for any two distinct BR-matchings M and M', there exists a sequence of BR-matchings $M = M_1, \ldots, M_k = M'$ such that M_{i-1} is compatible with M_i. This implies the connectivity of the *compatible bichromatic matching graph* containing one node for each BR-matching and an edge joining each pair of compatible BR-matchings, thereby answering the open problem posed by Aichholzer et al. in [6].

1 Introduction

A planar straight line graph (PSLG) is a geometric graph in which the vertices are points embedded in the plane and the edges are non-crossing line segments. There are many special types of PSLGs of which we name a few. A triangulation is a PSLG to which no more edges may be added between existing vertices. A *geometric matching* of a given point set P is a 1-regular PSLG consisting of pairwise disjoint line segments in the plane joining points of P. A geometric matching is *perfect* if every point in P belongs to exactly one segment.

Two branches of study on PSLGs include those of geometric augmentation and geometric reconfiguration. A typical augmentation problem on a PSLG $G = (V, E)$ asks for a set of new edges E' such that the graph $(V, E \cup E')$ retains or gains some desired properties (see survey by Hurtado and Tóth [12]).

A typical reconfiguration problem on a pair of PSLGs G and G' sharing some property asks for a sequence of PSLGs $G = G_0, \ldots, G_k = G'$ where each successive pair of PSLGs G_{i-1}, G_i jointly satisfy some geometric constraints. In some situations, a bound on the value of k is desired as well [2, 3, 5, 7, 9, 11, 15].

One such solved problem is that of reconfiguring triangulations: given two triangulations T and T', one can compute a sequence of triangulations $T = T_0, \ldots, T_k = T'$ on the same point set such that T_{i-1} can be reconfigured to T_i by flipping one edge. Furthermore, bounds on the value of k are known: $O(n^2)$ edge flips are always sufficient [11] and $\Omega(n^2)$ edge flips are sometimes necessary [9].

Two PSLGs (not necessarily disjoint) on the same vertex set are *compatible* if their union is planar. Compatible geometric matchings have been the object of study in both augmentation and reconfiguration problems. For example, the *Disjoint Compatible Matching Conjecture* [5] was recently solved in the affirmative [13]: every perfect planar matching M of $2n$ segments on $4n$ points can be augmented by $2n$ additional segments to form a PLSG that is the union of simple polygons.

Let M and M' be two perfect planar matchings of a given point set. The reconfiguration problem asks for a *compatible sequence* of perfect matchings $M = M_0, \ldots, M_k = M'$ such that M_{i-1} is compatible with M_i for all $i \in \{1, \ldots, k\}$. Aichholzer et al. [5] proved that there is always a compatible sequence of $O(\log n)$ matchings that reconfigures any given matching into a canonical matching.

*Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium aloupis.greg@gmail.com, {lbarbafl,slanger}@ulb.ac.be

†Chargé de recherches du F.R.S.-FNRS.

‡School of Computer Science, Carleton University, Ottawa, Canada

§Boursier FRIA du FNRS

¶Directeur de recherches du F.R.S.-FNRS.

‖NSF Grant #CCF-0830734., Department of Computer Science, Tufts University, Medford, MA. dls@cs.tufts.edu

Thus, the *compatible matching graph*, that has one node for each perfect planar matching and an edge between any two compatible matchings, is connected with diameter $O(\log n)$. Razen [15] proved that the distance between two nodes in this graph is sometimes $\Omega(\log n/\log\log n)$.

A natural question to extend this research is to ask what happens with bichromatic point sets in which the segments must join points from different colors. Let $P = B \cup R$ be a set of points in the plane in general position where $|R| = |B| = n$. A straight-line segment with one endpoint in B and one in R is called a *bichromatic segment*. A perfect planar matching of P where every segment is bichromatic is called a *BR-matching*. Sharir and Welzl [16] proved that the number of BR-matchings of P is at most $O(7.61^n)$. Hurtado et al. [10] showed that any BR-matching can be augmented to a crossing-free bichromatic spanning tree in $O(n \log n)$ time. Aichholzer et al. [6] proved that for any BR-matching M of P, there are at least $\lceil \frac{n-1}{2} \rceil$ bichromatic segments spanned by P that are compatible with M. Furthermore, there are BR-matchings with at most $3n/4$ compatible bichromatic segments.

At least one BR-matching can always be produced by recursively applying *ham-sandwich cuts*; see Fig. 1 for an illustration. A BR-matching produced in this way is called a *ham-sandwich matching*. Notice that the general position assumption is sometimes necessary to guarantee the existence of a BR-matching. However, not all BR-matchings can be produced using ham-sandwich cuts. Furthermore, some point sets admit only one BR-matching, which must be a ham-sandwich matching.

Two BR-matchings M and M' are *connected* if there is a sequence of BR-matchings $M = M_0, \ldots, M_k = M'$, such that M_{i-1} is compatible with M_i, for $1 \leq i \leq k$. An open problem posed by Aichholzer et al. [6] was to prove that all BR-matchings of a given point set are connected[1]. We answer this in the affirmative by using a ham-sandwich matching H as a canonical form. Consider the first ham-sandwich cut line ℓ used to construct H. We show how to reconfigure any given BR-matching via a compatible sequence so that the last matching in the sequence contains no segment crossing ℓ. We use this result recursively, on every ham-sandwich cut used to generate H, to show that any given BR-matching is connected with H.

2 Ham-sandwich matchings

In this paper, a *ham-sandwich cut* of P is a line passing through no point of P and containing exactly $\lfloor \frac{n}{2} \rfloor$ blue and $\lfloor \frac{n}{2} \rfloor$ red points to one side. Notice that if n is even, then this matches the *classical* definition of ham-sandwich cuts (see Chapter 3 of [14]). However, when n is odd, a ham-sandwich cut ℓ according to the classical definition will go through a red and a blue point of P. In this case, we obtain a ham-sandwich cut according to our definition by slightly moving ℓ away from these two points without changing its slope and without reaching another point of P. By the general position assumption, this is always possible.

Recall that P admits at least one ham-sandwich matching resulting from recursively applying ham-sandwich cuts. Moreover, note that P may admit several ham-sandwich matchings.

Let M be a BR-matching of P. In this section we prove that M is connected with a ham-sandwich matching H of P. Consider a ham-sandwich cut ℓ used to construct H. The idea of the proof is to show the existence of a BR-matching M', compatible with M, such that M' has "fewer" crossings with ℓ according to some measure defined on BR-matchings. By repeatedly applying this result, we end up with a matching N, connected with M, such that no segment of N crosses ℓ. Once we know how to avoid a ham-sandwich cut, we can apply the same result recursively on every ham-sandwich cut used to generate H. In this way, we obtain a sequence of compatible BR-matchings that connect M with H.

The main ingredient to obtain these results is Lemma 2.2 below. Before stating this lemma, we need a few more definitions.

Given a line ℓ that contains no point of P, let M_ℓ be the set of segments of M that cross ℓ. We say that ℓ is a *chromatic cut* of M if $|M_\ell| \geq 2$ and not all endpoints of M_ℓ on one side of ℓ have the same color. Without loss of generality, we can assume that if a chromatic cut ℓ exists, then it is vertical and no segment of M is parallel to ℓ. The following observation shows the relation between chromatic and ham-sandwich cuts.

[1] This problem was also posed during the EuroGIGA meeting that took place after EuroCG 2012.

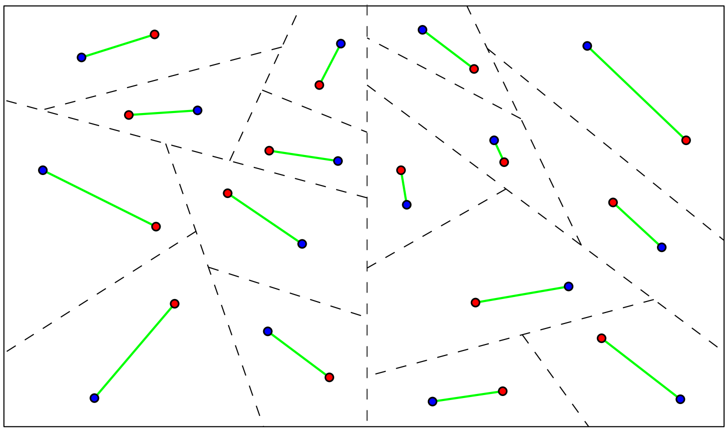

Figure 1: A ham-sandwich matching obtained by recursively applying ham-sandwich cuts.

Lemma 2.1. *Given any BR-matching M of P, every ham-sandwich cut of P either crosses no segment of M, or is a chromatic cut of M.*

Proof. Let ℓ be a ham-sandwich cut of M. Recall that the number of blue and red points to the right of ℓ must be the same. Moreover, every segment that is not crossed by ℓ has both of its endpoints on one of its sides. Therefore, if ℓ crosses a segment of M having a red endpoint to the right of ℓ, then, to maintain the balance of red and blue points, ℓ must cross another segment having a blue endpoint to the right of ℓ. That is, ℓ is a chromatic cut of M. □

We say that a point lies above (*resp.* below) a segment if it lies above (*resp.* below) the line extending that segment. We proceed to state Lemma 2.2. However, its proof is deferred to Section 4 for ease of readability.

Lemma 2.2. *Let M be a BR-matching of P and let ℓ be a chromatic cut of M. There exists a BR-matching M' of P compatible with M with the following properties: There is a segment s in $M \setminus M'$ that crosses ℓ such that all segments of M that cross ℓ below s belong also to M'. Moreover, these are the only segments of M' crossing ℓ below s.*

In other words, Lemma 2.2 states that we can find a BR-matching M', compatible with M, such that a segment s from M that crosses ℓ does not appear in M'. Moreover, every segment of M that crosses ℓ below s is preserved in M', and no new segment crossing ℓ below s is introduced. However, we have no control over what happens above s.

Lemma 2.3. *Given a BR-matching M of P and a ham-sandwich cut ℓ, there is a BR-matching N connected with M such that no segment of N crosses ℓ.*

Proof. Assume that ℓ is a chromatic cut of M, i.e., that it crosses at least one segment of M. Otherwise, the result follows trivially by Lemma 2.1. Given a BR-matching W of P such that ℓ is a chromatic cut of W, let $\textsc{Next}(W)$ be a BR-matching compatible with W that exists as a consequence of Lemma 2.2 when applied to W.

We construct a sequence φ of BR-matchings as follows. Let $M_0 = M$ be the first element of φ. If ℓ is a chromatic cut of M_i, then let $M_{i+1} = \textsc{Next}(M_i)$. Otherwise, M_i has no segment that crosses ℓ by Lemma 2.1 and is the final element of φ. We claim that $\varphi = (M_0, M_1, \dots, M_h)$ is finite and hence, that $N := M_h$ has no segment that crosses ℓ.

Assume without loss of generality that ℓ is a vertical line. Let $\mathcal{C}_P = \{z_0, z_1, \dots, z_m\}$ be the set of all $O(n^2)$ bichromatic segments that cross ℓ with endpoints in P. Assume that the segments of $\mathcal{C}_P$ are sorted, from top to bottom, according to their intersection with ℓ. Given a BR-matching W of P, let $\chi_W = b_m b_{m-1} \dots b_1 b_0$ be a binary number where b_i is defined as follows:

$$b_i = \begin{cases} 1 & \text{If } z_i \text{ belongs to } W \\ 0 & \text{Otherwise} \end{cases}$$

Let M_i and M_{i+1} be two consecutive matchings in φ. By Lemma 2.2, there is a segment $z_k \in M_i \setminus M_{i+1}$. Moreover, if z_j is a segment that crosses ℓ below z_k, then $z_j \in M_i$ if and only if $z_j \in M_{i+1}$. Therefore, the k-th digit of χ_{M_i} is 1 while the k-th digit of $\chi_{M_{i+1}}$ is 0. Moreover, for every $j > k$, the j-th digit of χ_{M_i} is identical to the j-th digit of $\chi_{M_{i+1}}$. This implies that $\chi_{M_i} > \chi_{M_{i+1}}$. Therefore, $\Phi = \chi_{M_0}, \chi_{M_1}, \ldots, \chi_{M_h}$ is a strictly decreasing sequence. This means that no BR-matching is repeated and that Φ converges to zero yielding our claim, i.e., we reach a BR-matching containing no segment that crosses ℓ. □

Theorem 2.4. *Let $P = B \cup R$ be a set of points in the plane in general position where $|R| = |B|$. If M is a BR-matching of P and H is a ham-sandwich matching of P, then M and H are connected, i.e., there is a sequence of BR-matchings $M = M_0, \ldots, M_r = H$ such that M_{i-1} is compatible with M_i for $1 \leq i \leq r$.*

Proof. The proof uses induction on the size of P. Notice that the result follows trivially if $|P| = 2$ as it contains a unique BR-matching.

Assume that the result holds for any bichromatic point set with fewer than n points. Let ℓ be the first ham-sandwich cut line used to construct H. By Lemma 2.3, there is a matching N such that M and N are connected, and no segment of N crosses ℓ. Let Π_1 and Π_2 be the two halfplanes supported by ℓ. For $i \in \{1, 2\}$, let P_i be the set of points of P that lie in Π_i and let N_i and H_i be, respectively, the set of segments of N and H that are contained in Π_i. Because $|P_i| < |P|$, N_i and H_i are connected by the induction hypothesis.

Since every BR-matching of P_1 is compatible with every BR-matching of P_2, we can merge the two compatible sequences obtained by the recursive construction that certify that N_i and H_i are connected. Thus, N is connected with H and because M is connected to N, M and H are also connected. □

Let V be the set of all BR-matchings of P and let G_P be the *compatible bichromatic matching graph* of P with vertex set V, where there is an edge between two vertices if their corresponding BR-matchings are compatible.

Corollary 2.5. *Given a set of points $P = B \cup R$ in general position such that $|R| = |B| = n$, the graph G_P is connected.*

While Theorem 2.4 implies the connectedness of G_P, it does not provide a non-trivial upper bound on its diameter. Since Sharir and Welzl [16] proved that the number of vertices in G_P is at most $O(7.61^n)$, we obtain an exponential upper bound on its diameter. Lemma 2.6 below, depicted in Fig. 2, shows a linear lower bound on the diameter of G_P.

Lemma 2.6. *There exists a bichromatic set $P = B \cup R$ of $4n$ points that admits two BR-matchings at distance $\Omega(n)$ in G_P.*

Proof. Let P be the set of vertices of a regular $4n$-gon Q. Partition P into four disjoint sets $P_0, \ldots, P_3$, each of n consecutive points along the boundary of Q. Assume without loss of generality that the unique edge on the boundary of Q joining a point from P_3 with a point in P_0 is parallel to the x-axis. Moreover, assume that this edge is the topmost edge of Q. Let $B = P_0 \cup P_2$ and let $R = P_1 \cup P_3$. Note that for any $0 \leq i \leq 3$, both bichromatic point sets $P_i \cup P_{i+1}$ and $P_i \cup P_{i-1}$ have unique BR-matchings where sum is taken modulo 3; see Fig. 2(a).

Let M be the union of the unique BR-matchings of $P_0 \cup P_1$ and $P_2 \cup P_3$. Analogously, let M' be the union of the BR-matchings of $P_1 \cup P_2$ and $P_0 \cup P_3$; see Fig. 2(a) for an illustration. Let p be a point of P and let $M(p)$ and $M'(p)$ be the points matched with p in M and M', respectively. We claim that in any BR-matching W of P, the point p is matched with either $M(p)$ or $M'(p)$. If this is not the case, then the segment s joining p with its neighbor in W separates P into two sets, each of them having an unbalanced number of blue and red points—a contradiction as the points on each side of s support a BR-matching that does not cross s.

Let $s_1, \ldots s_n$ be the last n segments of M when sorted from left to right. We claim that to connect M with a BR-matching that does not contain s_i, we need a compatible sequence of BR-matchings of length at least i. The proof uses induction on i.

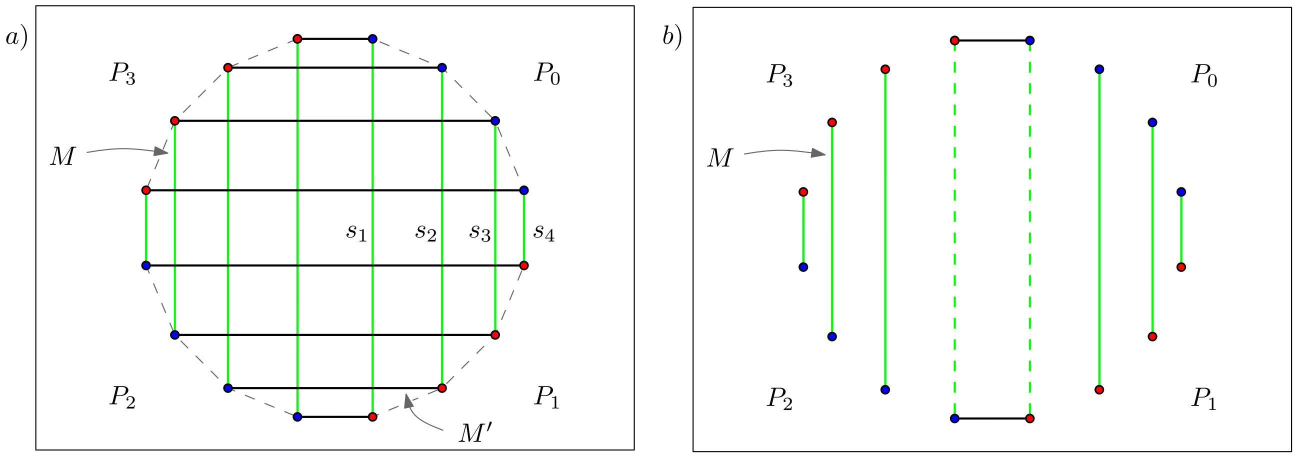

Figure 2: *a*) Two BR-matchings M and M' at distance $\Omega(n)$ in the graph G_P that are both ham-sandwich matchings. *b*) The only two segments compatible with M are the topmost and bottommost segments of M'.

For the base case, note that only the topmost and bottommost segments of M' are compatible with M, any other bichromatic segment that could be part of a BR-matching either crosses an edge of M or splits P into two unbalanced sets. By adding these two edges and removing the segments of M that form a cycle with them, we obtain the unique BR-matching compatible with M where s_1 is not present; see Fig. 2(*b*)

Assume that the result holds for any $j < i$. Let p be an endpoint of s_i and note that the segment connecting p with $M'(p)$ crosses s_{i-1}. That is, p is matched through s_i in any BR-matching that contains s_{i-1}. Consequently, to reach a BR-matching where s_i is not present, we need to first remove s_{i-1}, which requires at least $i-1$ steps by the induction hypothesis, and then at least one step more to remove s_i. Therefore, a sequence of length at least i is needed to connect M with a BR-matching that does not contain s_i.

Thus, to connect M with M' where s_n is not present, we need a compatible sequence of $\Omega(n)$ BR-matchings. □

3 Well-colored graphs and basic tools

In this section, we introduce some tools that will help us prove Lemma 2.2 in Section 4.

Given a bounded face F of a PSLG, we denote its interior by $int(F)$ and its boundary by ∂F. In the remainder, we only consider bounded faces when we refer to a face of a PSLG. A vertex v is *reflex in* F if there is a non-convex connected component in the intersection of $int(F)$ with any disk centered at v. Notice that a vertex can be reflex in at most one face of a PSLG. A vertex of a PSLG is *reflex* if it is reflex in one of its bounded faces.

Let F be a face of a given PSLG whose reflex vertices are colored either blue or red. We say that F is *well-colored* if the sequence of reflex vertices along its boundary alternates in color. In the same way, a PSLG is well-colored if all its faces are well-colored. Since a vertex is reflex in at most one face, a well-colored PSLG has an even number of reflex vertices.

Let G be a well-colored PSLG. The *boundary of* G, denoted by ∂G, is the union of all the edges in G. The interior of G is the union of the interiors of its faces. Let M be a BR-matching such that all segments of M are contained in the interior of G, i.e., no segment of M crosses an edge of G or lies on the boundary of G. We show how to *glue* the segments of M to G in such a way that every endpoint of M becomes a reflex vertex. We then provide a technique to construct a new BR-matching, compatible with M, by matching the reflex vertices of G after the gluing.

3.1 Coloring a PSLG

Throughout, we consider each segment either bichromatic or on ∂G to have two sides. Moreover, we assign a color, either red or blue, to each of these sides; see Fig. 3 for an illustration.

Assume that F is a well-colored face of G and let s be a segment along ∂F. Let π be an open halfplane supported by s. The side of s facing π has color c if for any $y \in \pi$, the first reflex vertex met when walking in a straight line from y to the midpoint of s, and then counterclockwise along ∂F has color c. If F contains no reflex vertex, then we chose a color c, either red or blue, and for each segment along ∂F, we permanently fix the color of each of its sides facing the interior of F to c. In this case, we use the freedom on the choice of c to guarantee a color scheme that matches our needs.

This coloring scheme can be used for bichromatic segments as well. For $r \in R$ and $b \in B$, let $s = [r, b]$ and consider an open halfplane π supported by s. The side of s facing π is *blue* if for any $y \in \pi$ the triple y, r, b makes a left turn. Otherwise, this side is *red*. Note that s has one side colored red and its opposite side colored blue.

A point on a segment s has color c when viewed from another point z if the side of s facing the halfplane that contains z supported by s has color c. Therefore, the color of a point on s depends on the position from which it is viewed.

Let x and y be two points such that each one lies either on ∂G or on a segment of M. We say that x and y are *visible* if the open segment (x, y) is contained in the interior of G and crosses no segment of M. We say that x and y are *color-visible* if they are visible and the color of x when viewed from y is equal to the color of y when viewed from x; see Fig. 3 for an example.

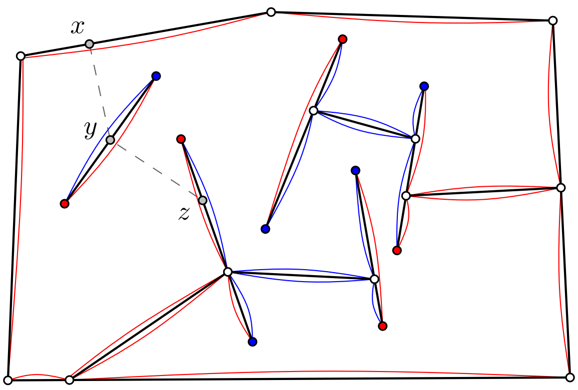

Figure 3: The coloring of the segments of a well-colored PSLG, as well as of a bichromatic segment. The point y is blue when viewed from x but red when viewed from z. Moreover, y and z are color-visible.

3.2 Basic operators for well-colored PSLGs

Let z be a point in the interior of a segment $s = [a, b]$ of M and let z' be a non-reflex point on ∂G such that z and z' are color-visible. The GLUE operator produces a new PSLG by attaching s to ∂G using z and z' as points of attachment. Formally, if z' is not a vertex of G, then insert it as a vertex by splitting the edge of G that contains z'. Add the vertices z, a and b and the edges $[z, z']$, $[z, a]$ and $[z, b]$ to G. In the resulting PSLG, denoted by $\text{GLUE}(G, z, z')$, a and b are both reflex vertices of degree one; see Fig. 4.

Let y and y' be two color-visible points on ∂G such that neither y nor y' is a reflex vertex. The CUT operator joins y with y' in the following way. Let F be the face of G that contains the segment $[y, y']$. If either y or y' is not a vertex of G, insert it by splitting the edge where it lies on. Thus, $[y, y']$ is a chord of F, so adding the edge $[y, y']$ to G splits F into two new faces. In this way, we obtain a new PSLG $\text{CUT}(G, y, y')$ with one face more than G; see Fig. 4 for an illustration of this operation.

Since both operators join two points by adding the edge between them, we can define an operator GLUECUT on G, z and z', that behaves like GLUE when z belongs to a segment in M, or behaves like CUT if both z and z' belong to ∂G. The PSLG output by this operator is denoted by $\text{GLUECUT}(G, z, z')$.

A *Glue-Cut Graph (GCG)* is a well-colored PSLG where every reflex vertex has degree one. Although this definition is more general, we can think of a GCG as a PSLG obtained by repeatedly applying GLUECUT operations between a polygon (initially convex) and the segments of a BR-matching contained in its interior; see Fig. 4 for an example.

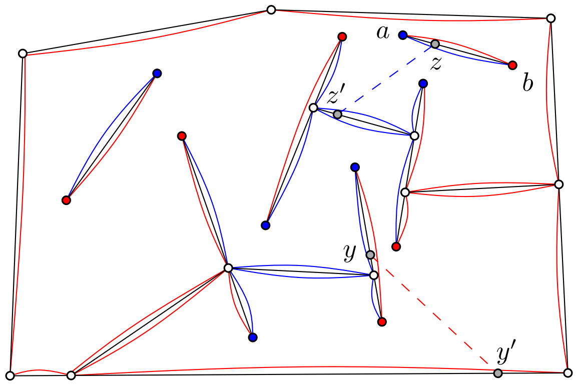

Figure 4: Two pairs of color-visible points z, z' and y, y', where z and z' can be joined by the GLUE operator and y and y' by the CUT operator.

Lemma 3.1. *The family of Glue-Cut Graphs is closed under the* GLUECUT *operator.*

Proof. Let G be a GCG and z be a point in a bichromatic segment s contained in the interior of G. Let z', y and y' be points on ∂G such that z and z' (*resp.* y and y') are color-visible. When constructing GLUE(G, z, z'), the endpoints of s become reflex vertices of degree one. That is, we add one red and one blue reflex vertex to G. Therefore, to prove that GLUE(G, z, z') is a GCG, it suffices to show that it is well-colored. This, however, is guaranteed by the color-visibility of z and z'; see Fig. 4.

On the other hand, CUT(G, y, y') neither adds nor removes reflex vertices of G. This operation divides a well-colored face of G into two, by inserting a new edge. Consider either of the new faces. Let a, b be the first reflex vertices found when following the boundary from this edge in each direction. Since y and y' are color-visible when CUT is invoked, we know that a and b are of different colors. Thus, each new face, and therefore CUT(G, y, y'), is well-colored; see Fig. 4. □

3.3 Simplification of a GCG

Let $F = (v_1, v_2, \ldots, v_k, v_1)$ be a face of a GCG given as a sequence of its vertices in clockwise order along its boundary. For each vertex v_i, if the triple v_{i-1}, v_i, v_{i+1} makes a right turn, let x_i be a point at distance $\varepsilon > 0$ from v_i, lying on the bisector of the convex angle formed by $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$. If v_i is reflex in F, let $x_i = v_i$. Otherwise, if v_{i-1}, v_i, v_{i+1} are collinear, do nothing. Let $\mathcal{P}_F = (x_1, \ldots, x_k, x_1)$ (consider only the indices where x_i is defined). By choosing ε sufficiently small, we obtain the following result; see Fig. 5.

Observation 3.2. *For every bounded face F of a GCG, $\mathcal{P}_F$ is a simple polygon contained in F such that F and $\mathcal{P}_F$ share the same set of reflex vertices.*

We call $\mathcal{P}_F$ a *simplification* of F. Though the simplification of a face F is not unique as it depends on the choice of ε, the results presented in this paper hold for any simplification. Therefore, when referring to $\mathcal{P}_F$, we refer to any simplification of F.

Let $F_1, \ldots, F_k$ be the bounded faces of a GCG G. We call $\mathcal{P}_G = \bigcup \mathcal{P}_{F_i}$ the *simplification of* G. Note that $\mathcal{P}_G$ is the union of a set of disjoint simple polygons.

3.4 Merging a matching with a GCG

The following result is a special case of Lemma 5 of [1].

Lemma 3.3. *Let $\mathcal{P}$ be a simple polygon with an even number of reflex vertices. There exists a perfect planar matching M of the reflex vertices of $\mathcal{P}$ such that each segment of M is contained in $\mathcal{P}$ (or on its boundary).*

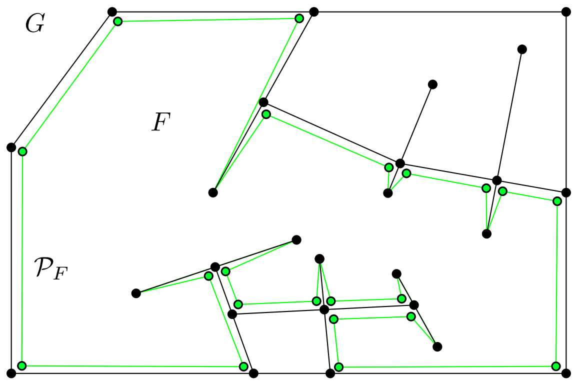

Figure 5: A face F of a GCG G and its simplification $\mathcal{P}_F$, contained in F, with the same set of reflex vertices.

Let $C = \{r_0, \ldots, r_k\}$ be the set of reflex vertices of a simple polygon $\mathcal{P}$ sorted along its boundary. Let M be a perfect planar matching of C that exists by Lemma 3.3. Let $[r_i, r_j]$ be a segment of M, and note that this segment splits $\mathcal{P}$ into two sub-polygons. Notice that if $[r_i, r_j]$ is contained in the boundary of $\mathcal{P}$, then one sub-polygon is a segment and the other one is $\mathcal{P}$ itself. In order for M to be perfect and planar, each sub-polygon must contain an even number of reflex vertices. Therefore, if a segment $[r_i, r_j]$ belongs to M, then $i \bmod 2 \neq j \bmod 2$. This implies that if $\mathcal{P}$ is well-colored, then M is a BR-matching of the reflex vertices of $\mathcal{P}$.

The main tool to construct BR-matchings of the reflex vertices of a GCG is provided by the following lemma; see Fig. 6 for an illustration.

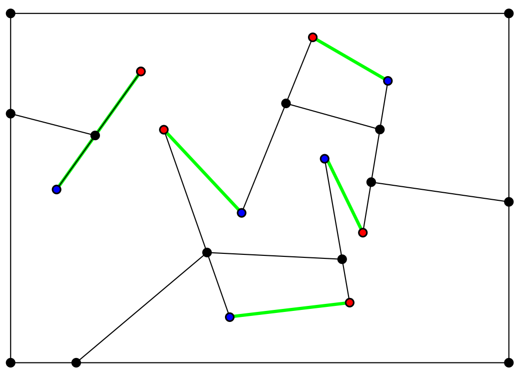

Figure 6: A GCG and a BR-matching of its reflex vertices.

Lemma 3.4. *If G is a GCG, then there is a BR-matching M of the reflex vertices of G, such that each segment of M is contained in $\mathcal{P}_G$ (or on its boundary).*

Proof. Let $F_1, \ldots, F_k$ be the well-colored faces of G. By Observation 3.2, each F_i and its simplification $\mathcal{P}_{F_i}$ share the same set of reflex vertices. By Lemma 3.3, there is a matching M_i of the reflex vertices of $\mathcal{P}_{F_i}$, such that each segment lies either in the interior or on the boundary of $\mathcal{P}_{F_i}$. Since F_i is well-colored, M_i is a BR-matching. Note that a vertex can be reflex in at most one face of G. Therefore, $M = \bigcup M_i$ is a BR-matching of the reflex vertices of G and each segment of M lies either in the interior or on the boundary of $\mathcal{P}_G$. □

3.5 Gluing BR-matchings

Let X be a GCG and let M be a BR-matching contained in the interior of X. In this section, we show how to glue the segments of M to the boundary of X. In this way, we obtain a GCG G such that the endpoints of the segments of M are all reflex vertices of G. Thus, by Lemma 3.4, we can obtain a BR-matching M' of the reflex vertices of G where every segment is contained in $\mathcal{P}_G$, i.e. we can obtain a BR-matching M' whose union with M contains no crossings.

Assume that the vertices of X and the endpoints of M are in general position and that no two points have the same x-coordinate.

Let s be the segment with the rightmost endpoint among all segments of M. We may assume that the left (*resp.* right) endpoint of s is blue (*resp.* red) and hence, that s is blue (*resp.* red) when viewed from below (*resp.* above).

Extend s to the right until it reaches the interior of a segment s' on ∂X at a point y and choose a point y' in the interior of s' above (*resp.* below) y if s' is red (*resp.* blue) when viewed from s. Choose y' sufficiently close to y so that the whole segment s is visible from y'. This is always possible because y is visible from the right endpoint of s. Let m be the midpoint of s and note that m and y' are color-visible by construction.

Let $X' = \text{GLUE}(X, m, y')$. By Lemma 3.1, X' is a GCG. Moreover, the endpoints of s become reflex vertices of X'; see Fig. 7. Remove s from M, let $X = X'$ and repeat this construction recursively until M is empty. We obtain the following result.

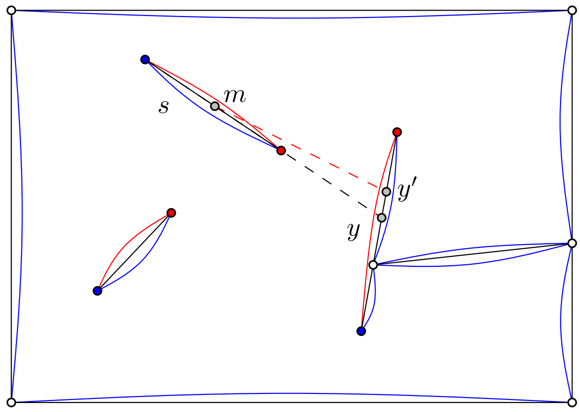

Figure 7: Gluing a bichromatic segment with the boundary of a GCG.

Lemma 3.5. *Let X be a GCG and M be a BR-matching contained in the interior of X. There is a GCG G augmenting X such that all reflex vertices of X and all endpoints in M are reflex vertices in G. Moreover, every segment of M is contained in ∂G.*

4 Removing crossings with chromatic cuts

In this section, we provide the proof of Lemma 2.2 presented in Section 2.

Let M be a BR-matching of P and let ℓ be a chromatic cut of M. Recall that M_ℓ denotes the set of segments of M that cross ℓ. We show that it is possible to obtain a new BR-matching M' with at least one segment s of M_ℓ absent. Furthermore, when examining segments of M that cross ℓ below s, all segments of M_ℓ are preserved in M' and no new segments are introduced.

The following observation is depicted in Fig. 8.

Observation 4.1. *Let v be a vertex of a GCG X. If no line through v, intersecting the interior of X, supports a closed halfplane containing all the neighbors of v, then v lies outside of $\mathcal{P}_X$. Moreover, the open segments joining v with its neighbors also lie outside of $\mathcal{P}_X$.*

A vertex v of a GCG is *isolable* if it satisfies the conditions of Observation 4.1.

Let ℓ be a chromatic cut of M and assume that $M_\ell = \{s_1, \ldots, s_k\}$ is sorted from bottom to top according to the intersection, x_i, of s_i with ℓ.

The idea of the proof is to construct a GCG X augmenting M, using Lemma 3.5, in such a way that $x_1, \ldots, x_j$ become isolable vertices of X for some $1 \leq j \leq k$. Furthermore, we require X to contain the edge between x_i and x_{i+1} for every $1 \leq i < j$. By Observation 4.1, these edges will lie outside of $\mathcal{P}_X$ and so will the portion of ℓ lying below s_j. Thus, this portion of ℓ will not be crossed by a BR-matching, compatible with M, obtained by applying Lemma 3.4 to X.

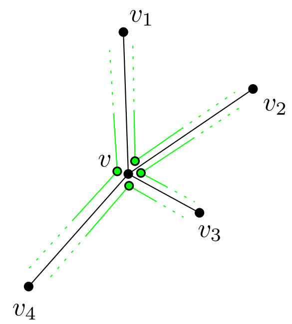

Figure 8: An isolable vertex v that lies outside of the simplifications of each of its adjacent faces.

Notice that we can only glue x_i with x_{i+1} if they are color-visible. Although the following lemma shows that there is at least one pair of consecutive color-visible points among $x_1, \ldots, x_k$, we may not be able to glue all of them. Thus, we will resort to a different strategy that allows us to alter the color of a segment.

Lemma 4.2. *There exist two consecutive segments s_i and s_{i+1} in M_ℓ such that x_i and x_{i+1} are color-visible.*

Proof. Because ℓ is a vertical chromatic cut, there exist two segments s_j and s_h in M_ℓ such that the left endpoint of s_j is of different color than the left endpoint of s_h. Therefore, there must exist segments s_i and s_{i+1} whose left endpoints have different colors. This implies that the color of s_i when viewed from above is the same as the color of s_{i+1} when viewed from below. Finally, since s_i and s_{i+1} are consecutive segments in M_ℓ, x_i and x_{i+1} are visible. □

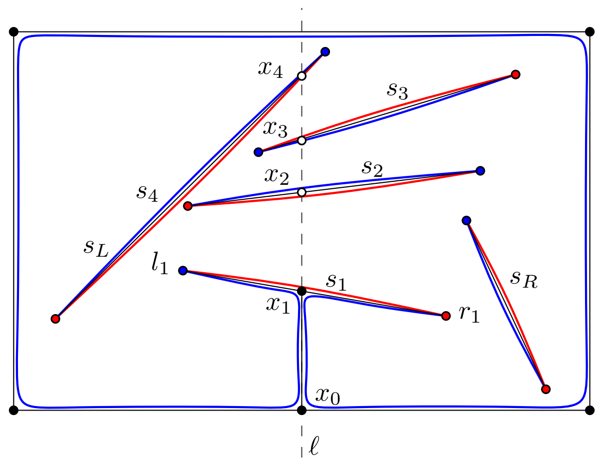

Figure 9: Case where there is at least one point, lying above s_1, on segment s_2, s_L or s_R that is color-visible with x_1.

We proceed to describe the construction of the GCG that augments M. Let $\mathcal{R}$ be a convex polygon strictly containing all segments of M. Assume without loss of generality that the left endpoint of s_1 is blue, implying that s_1 is red from above and blue from below. Let x_0 be the bottom intersection between ℓ and $\mathcal{R}$. Since the bounded face of $\mathcal{R}$ contains no reflex vertex, we can assume that x_0 is blue when viewed from x_1. That is, x_0 and x_1 are color-visible. Finally, let $X_1 = \textsc{Glue}(\mathcal{R}, x_1, x_0)$ be the GCG obtained by joining x_0 with x_1; see Fig. 9.

Consider the edge of $\mathcal{R}$ containing x_0 to be a segment s_0. The following invariants on the GCG X_i hold initially for $i = 1$ and are maintained throughout.

- The points x_i and x_{i+1} are visible while x_i and x_{i-1} are neighbors.
- Besides x_{i-1}, vertex x_i neighbors two vertices on s_i, one to the left and one to the right of ℓ.
- The endpoints of s_i are reflex vertices of X_i.

- The endpoints of s_{i-1} are not reflex in X_i and x_{i-1} is an isolable vertex.
- The color of s_i, when viewed from a point lying above s_i, is given by the color of the right endpoint of s_i.

Our objective is to find a point, color-visible with x_i, that lies above the line extending s_i. If such a point exists, by gluing it to x_i and then merging the remaining segments of M using Lemma 3.5, we obtain a GCG X that augments M with the desired properties (a full explanation is presented in Section 4.2, an example is depicted in Fig. 9).

As long as no such point exists, we iteratively augment X_i, maintaining the above properties as an invariant. This is done using the AUGMENT(i) procedure (defined in more detail below), which takes a GCG X_i and adds edges (including the edge between x_i and x_{i+1}) to produce a new GCG X_{i+1} where the above properties hold again. After several augmentations, we will produce a GCG where the desired color-visible point will be found.

Procedure AUGMENT(i)

Refer to Fig. 10 for an illustration of this procedure. Let l_i and r_i be the left and right endpoints of s_i, respectively. Assume without loss of generality that l_i is colored blue (and r_i is red). Thus, s_i is red when viewed from above. Extend s_i on both sides and let s_L (*resp.* s_R) be the first segment reached to the left (*resp.* right). This procedure is only used when the points in s_{i+1}, s_L and s_R appear blue when viewed from x_i. Otherwise, AUGMENT is not required as there is a point in either s_{i+1}, s_L or s_R, lying above s_i, that is color-visible with x_i.

Notice that s_{i+1}, s_L, and s_R could belong either to M, or to ∂X_i. Let y_L and y_R be the points where the line extending s_i intersects s_L and s_R, respectively. Let X'_i be the PSLG obtained by adding the edges $[l_i, y_L]$ and $[r_i, y_R]$ to X_i (y_L and y_R are added as vertices).

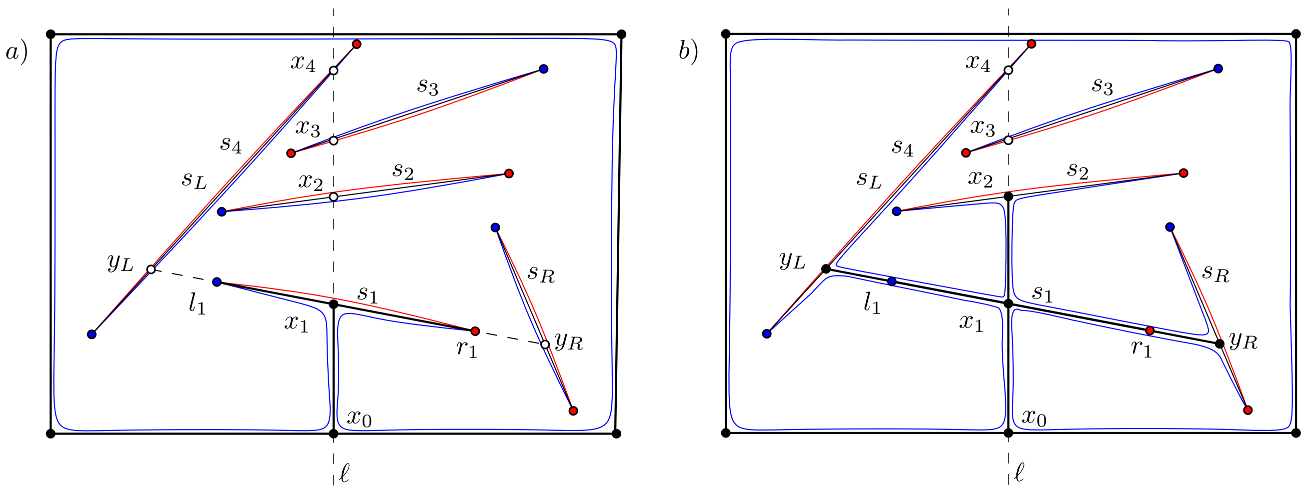

Figure 10: a) Example where procedure AUGMENT(1) is required. Points above s_1 that lie on segments s_2, s_L and s_R are not color-visible with x_1. b) The construction obtained by extending s_1, where two reflex vertices l_1, r_1 disappear to let x_1 and x_2 become color-visible.

This may create new faces depending on whether s_L or s_R belong to M. Vertices y_L, l_i, x_i, r_i, y_R are collinear, meaning l_i and r_i are no longer reflex vertices in X'_i. Thus, the color of s_i will now be blue when viewed from above or from below. Furthermore, if s_L or s_R belong to M, then their endpoints are now reflex vertices of X'_i. One can verify that X'_i is well-colored since y_L and y_R are both blue when viewed from x_i, hence X'_i is a GCG. See Fig. 10(b) for an illustration. Notice that, when viewed from above, the color of x_i is now blue, in contrast with the red color that x_i had in X_i. Therefore, since x_{i+1} is blue when viewed from below, x_{i+1} and x_i are now color-visible in X'_i and can be glued.

Let $X_{i+1} = $ GLUECUT(X'_i, x_{i+1}, x_i). This way, the endpoints of s_{i+1} become (if they were not already) reflex vertices of the GCG X_{i+1} and x_i becomes an isolable vertex. Notice that no vertex on

s_{i+1} neighbors a point lying above s_{i+1}. Therefore, the color of every point on segment s_{i+1}, when viewed from above, is given by the color of the right endpoint of s_{i+1}. In fact, the invariant properties are maintained, should there be a subsequent use of AUGMENT.

4.1 Analysis of AUGMENT

Observation 4.3. *On each iteration of* AUGMENT, *all reflex vertices of X_i are preserved in X_{i+1}, except for the two endpoints of s_i that become non-reflex.*

Lemma 4.4. *The procedure* AUGMENT *will only be used $O(n)$ times before producing a GCG X_j where there exists a point, lying above the segment s_j, that is color-visible with x_j (for some $1 \leq j \leq k-1$).*

Proof. By Lemma 4.2, there exist segments $s_h, s_{h+1} \in M_\ell$ such that x_h and x_{h+1} are color-visible before executing AUGMENT on X_1. We claim that AUGMENT can only go as far as to construct X_h. If X_h is not constructed, it is because a GCG X_j was constructed (for some $0 \leq j < h$) where there exists a point, lying above the segment s_j, that is color-visible with x_j. Otherwise, if X_h is constructed, then, by the preserved invariants, the color of x_h, when viewed from above, remains unchanged and hence x_h and x_{h+1} are color-visible. □

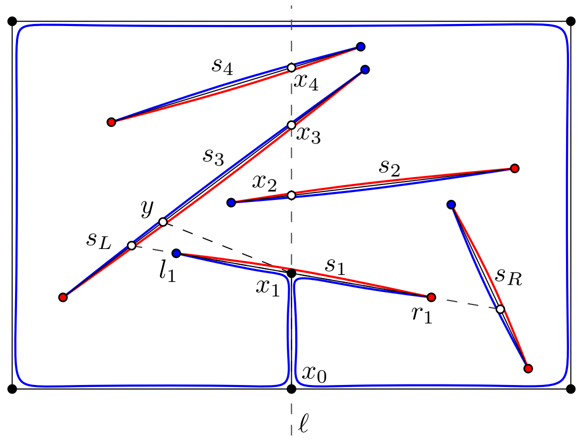

Figure 11: Case where x_1 and x_2 are not color-visible, but a point y can be found in s_L so that x_1 and y are color-visible.

4.2 Processing after AUGMENT

By Lemma 4.4, we know that after the last call to AUGMENT we obtain a GCG X_j such that there is a point in either s_{j+1}, s_L or s_R, lying above s_j, that is color-visible with x_j. Assume without loss of generality that x_j is red when viewed from above. If s_{j+1} is red when viewed from below, then x_j and x_{j+1} are color-visible. In this case, we define $G_{M,\ell} = \text{GLUECUT}(X_j, x_{j+1}, x_j)$.

If x_{j+1} is blue when viewed from x_j, we follow a different approach. Recall that the endpoints of s_j are reflex vertices. If s_L is red when viewed from the left endpoint of s_j, choose a point y, slightly above y_L on s_L, such that the whole segment s_j is visible from y. Since x_j is red when viewed from above, x_j and y are color-visible. Let $G_{M,\ell} = \text{GLUECUT}(X_j, y, x_j)$; see Fig. 11. An analogous construction of $G_{M,\ell}$ follows if s_R is red when viewed from the right endpoint of s_j. We call $G_{M,\ell}$ the *extension of X_j*.

Lemma 4.5. *If $G_{M,\ell}$ is an extension of X_j, then the following properties hold:*

- *The endpoints of s_j are reflex vertices of $G_{M,\ell}$, but s_j is not contained in $\mathcal{P}_{G_{M,\ell}}$.*
- *The downwards ray with apex at x_j does not intersect $\mathcal{P}_{G_{M,\ell}}$.*
- *For every $1 \leq i < j$, the endpoints of s_i are not reflex vertices of $G_{M,\ell}$. Moreover, s_i is not contained in $\mathcal{P}_{G_{M,\ell}}$.*

Proof. By the invariants of AUGMENT, x_j neighbors x_{j-1} as well as two vertices on s_j, one to the left and one to the right of ℓ. Since x_j also neighbors a vertex in $G_{M,\ell}$ lying above the segment s_j, x_j is an isolable vertex in $G_{M,\ell}$. Thus, by the preserved invariants and by Observation 4.1, for every $1 \leq i \leq j$, x_i lies outside of $\mathcal{P}_{G_{M,\ell}}$ and hence the segment s_i is not contained in $\mathcal{P}_{G_{M,\ell}}$. Furthermore, the segment joining x_i with x_{i-1} also lies outside of $\mathcal{P}_{G_{M,\ell}}$ and so does the downwards ray with apex at x_j. Finally, Observation 4.3 tells us that, for every $1 \leq i < j$, no endpoint of s_i is a reflex vertex of X_j (nor of $G_{M,\ell}$). ☐

We are now ready to provide the proof of Lemma 2.2 which is restated for ease of readability.

Lemma 2.2. *Let M be a BR-matching of P and let ℓ be a chromatic cut of M. There exists a BR-matching M' of P compatible with M with the following properties: There is a segment s of $M \setminus M'$ that crosses ℓ such that all segments of M that cross ℓ below s also belong to M'. Moreover, these are the only segments of M' crossing ℓ below s.*

Proof. Let $G_{M,\ell}$ be the GCG obtained using the construction presented in this section on M and ℓ. Recall that M_ℓ is the set of segments of M that cross ℓ. Lemma 4.5 states that there is a segment $s_j \in M_\ell$, such that its endpoints are reflex vertices of $G_{M,\ell}$ but s_j is not contained in $\mathcal{P}_{G_{M,\ell}}$. Let W be the set of segments in M that are contained in the interior of $G_{M,\ell}$ and let $Z_\ell = \{s_1, \ldots, s_{j-1}\}$ be the set of segments of M_ℓ that cross ℓ below x_j. By Lemma 4.5, we know that $Z_\ell \cap W = \emptyset$.

By Lemma 3.5 and from the fact that W is contained in the interior of $G_{M,\ell}$, we can extend $G_{M,\ell}$ by gluing the segments of W to its boundary such that the endpoints of every segment in W become reflex vertices in $G_{M,\ell}$. Moreover, the reflex vertices of $G_{M,\ell}$ are preserved.

By Lemma 3.4, there exists a BR-matching W' of the reflex vertices of $G_{M,\ell}$ such that each segment in W' is contained in $\mathcal{P}_{G_{M,\ell}}$. Notice that the endpoints of s_j are re-matched in W'. However, since s_j is not contained in $\mathcal{P}_{G_{M,\ell}}$, s_j does not belong to W'. Moreover, Lemma 4.5 implies that the ray, shooting downwards from x_j, lies outside $\mathcal{P}_{G_{M,\ell}}$. Thus, no segment in W' crosses ℓ below x_j.

Let $M' = W' \cup Z_\ell$ be a set of bichromatic segments. Every point in P is matched in M' since every point in P is either a reflex vertex of $G_{M,\ell}$, or an endpoint of a segment in Z_ℓ. Lemma 4.5 implies that the endpoints of the segments in Z_ℓ are not reflex vertices in $G_{M,\ell}$. Therefore, M' is a BR-matching of P. Since W and W' are compatible, M and M' are compatible BR-matchings. ☐

5 Remarks

Although the techniques developed in this paper appear tailored for this specific problem, they have a more general underlying scope. At a deeper level, our tools generate a *balanced* convex partition of the plane. Roughly speaking, in Lemma 3.3 a simple polygon is partitioned into a set of convex polygons obtained by shooting rays from the reflex vertices towards the interior of the polygon until hitting its boundary. Once the polygon is partitioned, a matching can be found on each convex piece. By using Lemma 3.3 in the bichromatic setting, we generate a convex partition of the GCGs, where each convex face is in charge of matching a balanced number of red and blue points. Convex partitions, usually constructed by extending segments until they reach another segment or a previously extended section, have been extensively used to solve several augmentation and reconfiguration problems [5, 6, 13, 10, 8]. Therefore, the techniques provided in this paper are of independent interest. In conjunction with Lemma 3.4, operators like GLUE and CUT can be used to find special convex partitions that provide new ways to construct compatible PSLGs.

Recently, Aichholzer et al. [4] adapted some of the tools presented in this paper and showed that there is always a sequence of length $O(n)$ that connects any two BR-matchings. This result, together with the lower bound of $\Omega(n)$ presented in this paper yield a tight bound of $\Theta(n)$ for the length of any compatible sequence that connects two given BR-matchings.

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