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An Educational Elasticity Problem With Friction

Part 3: General Load Paths

This concluding paper treats general load paths when the two components of the concentrated force are allowed to change independently with time. It is shown that there are two kinds of dependence on the load path. For certain directions of the forward tangent, the dependence is strict in that the deformations depend on the full details of the path. For other directions, however, the dependence is loose, and the deformations do not depend on the exact nature of the path as long as the forward tangent falls within given bounds. The problem also shows that, given an initial state, the load space can be subdivided into different regions each corresponding to a certain mode of deformations.

Introduction

To conclude the discussion of the example for friction in elasticity, which was treated earlier for a constant direction of the applied force [1, 2], we consider the more general case when the two components of the force can change independently. This will illustrate the dependence of deformations on the load path. A particularly interesting aspect of friction in elasticity, which emerges from the analysis, is that there are two types of dependence. For certain directions of the load path the dependence is *strict* in that the deformations depend on the exact nature of the path. For other directions, however, the dependence is *loose*, and the slip process does not depend on the detailed nature of the path, as long as the forward tangent of the path falls within certain bounds.

The geometry of the problem is shown in Fig. 1. The two components of the force $P(t)$ and $Q(t)$ can vary independently, but the restriction $P(t) \geq 0$ is imposed to avoid singular normal and shear tractions on the cut because they require separate considerations [1]. The extent of the separation zone $0 < x < a$ and the distribution of normal tractions on the cut do not depend on the slip process, and from [1]

$$a = \frac{\kappa - 1}{\pi(\kappa+1)} \frac{P}{p^\infty} \quad (1)$$

$$N(x) = -p^\infty \left(\frac{x-a}{x}\right)^{1/2} H(x-a), \quad 0 < x < \infty \quad (2)$$

Suppose that there are initial shear tractions $S_i(x)$ on the cut due to some previous loading. The solution for the next segment of the load path can be constructed by distributing additional dislocations with the density $B_x^c(x)$ on a finite interval. This dislocation distribution gives the corrective shear tractions

$$S_c(x) = -\frac{2\mu}{\pi(\kappa+1)} \int_0^{\max(b)} \frac{B_x^c(\xi)d\xi}{\xi - x}, \quad 0 < x < \infty \quad (3)$$

and the total shear tractions, which must satisfy the appropriate boundary conditions on the separation and slip zones, are then

$$S(x) = S_i(x) + S_c(x) \quad (4)$$

The upper limit of integration in (3) denoted symbolically by $\max(b)$ must be chosen so that all of the new dislocations are included, even if they become locked into a stick zone in case a slip zone shrinks or there is no slip zone [2].

The friction law can be written as

$$S(x) = -fN(x)\mathrm{sng}\,V(x), \quad 0 < x < b \quad (5)$$

valid in both the separation and slip zones. Substituting from (2),

$$S(x) = fp^\infty \left(\frac{x-a}{x}\right)^{1/2} H(x-a)\mathrm{sgn}\,V(x), \quad 0 < x < b \quad (6)$$

and consequently

$$\int_0^{\max(b)} \frac{B_x^c(\xi)d\xi}{\xi - x} = \frac{\pi(\kappa+1)}{2\mu} \left\{ S_i(x) - fp^\infty \left(\frac{x-a}{x}\right)^{1/2} H(x-a)\mathrm{sgn}\,V(x) \right\}, \quad 0 < x < b \quad (7)$$

where $0 < x < a$ is the separation zone and $a < x < b$ is the active slipe zone.

Expanding Slip Zone

We take for the first initial state that reached by loading with a force in a constant direction to the point (P_1, Q_1) in the P,Q plane. It can be assumed without loss of generality that $Q_1 > 0$. Then the extents of the initial separation and slip zones, and the normal and shearing tractions are given by [1]

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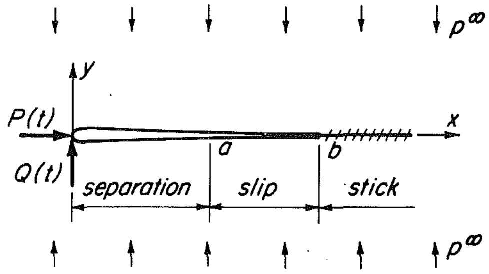

Fig. 1 Geometry of the problem

$$a_1 = \frac{\kappa-1}{\pi(\kappa+1)} \frac{P_1}{p^\infty} \tag{8}$$

$$b_1 = a_1 + \frac{\kappa-1}{\pi(\kappa+1)} \frac{Q_1}{fp^\infty} = \frac{\kappa-1}{\pi(\kappa+1)} \frac{fP_1+Q_1}{fp^\infty} \tag{9}$$

$$N_1(x) = -p^\infty \left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1), \quad 0 < x < \infty \tag{10}$$

$$S_1(x) = fp^\infty \left\{ -\left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0 < x < \infty \tag{11}$$

It will be seen that the same initial state can be reached by other quite different load paths.

For an expanding slip zone

$$\frac{db}{dt} > 0 \tag{12}$$

and the upper limit of the integral in (7) can be taken as b. It is not known, however, where b falls in relation to a_1 and b_1, and the right side of (7) must be modified by introducing the Heaviside step functions $H(b-a_1)$ and $H(b-b_1)$ to obtain the governing integral equation. Thus,

$$\int_0^b \frac{B_x^c(\xi)d\xi}{\xi - x} = \frac{\pi(\kappa+1)}{2\mu} fp^\infty \left\{ -\left(\frac{x-a}{x}\right)^{1/2} H(x-a)\operatorname{sgn} V - \left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1)H(b-a_1) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1)H(b-b_1) \right\}, \quad 0<x<b \tag{13}$$

The solution of the integral equation that is singular at $x = 0$ but bounded at $x = b$ is

$$B_x^c(x) = \frac{\kappa+1}{2\mu} fp^\infty \left\{ -\left(\frac{a-x}{x}\right)^{1/2} H(a-x)\operatorname{sng} V + \left(\frac{b-x}{x}\right)^{1/2} [\operatorname{sgn} V + H(b-a_1) - H(b-b_1)] - \left(\frac{a_1-x}{x}\right)^{1/2} H(a_1-x)H(b-a_1) + \left(\frac{b_1-x}{x}\right)^{1/2} H(b_1-x)H(b-b_1) \right\}, \quad 0<x<b \tag{14}$$

The corrective shear tractions computed from (3) are in turn

$$S_c(x) = fp^\infty \left\{ \left(\frac{x-a}{x}\right)^{1/2} H(x-a)\operatorname{sgn} V + \left(\frac{x-b}{x}\right)^{1/2} H(x-b)[-\operatorname{sgn} V - H(b-a_1) + H(b-b_1)] + \left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1)H(b-a_1) - \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1)H(b-b_1) \right\}, \quad 0<x<\infty \tag{15}$$

yielding the total shear tractions

$$S(x) = fp^\infty \left\{ \left(\frac{x-a}{x}\right)^{1/2} H(x-a)\operatorname{sgn} V + \left(\frac{x-b}{x}\right)^{1/2} H(x-b)[\operatorname{sgn} V + H(b-a_1) - H(b-b_1)] - \left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1)[1-H(b-a_1)] + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1)[1-H(b-b_1)] \right\}, \quad 0<x<\infty \tag{16}$$

The conditions remaining to be enforced are that the slip velocity derived from the dislocation distribution as

$$V(x) = -\frac{\kappa-1}{4\mu}\dot{Q} - \int_0^x \dot{B}_x^c(\xi)d\xi \tag{17}$$

should not be inconsistent because of the $\operatorname{sgn} V$ term in $B_x^c(x)$, and that the tangential shift corresponding to $B_x^c(x)$ vanishes at the end of the active slip zone, or

$$h_c(b) = \frac{\kappa-1}{4\mu}(Q_1 - Q) - \int_0^b B_x^c(\xi)d\xi = 0 \tag{18}$$

The time derivative of the dislocation distribution is

$$\dot{B}_x^c(x) = \frac{\kappa+1}{4\mu} fp^\infty \left\{ -\frac{\dot{a}}{(ax-x^2)^{1/2}} H(a-x)\operatorname{sgn} V + \frac{\dot{b}}{(ax-x^2)^{1/2}} [\operatorname{sgn} V + H(b-a_1) - H(b-b_1)] \right\}, \quad 0<x<b \tag{19}$$

which from (17) yields the slip velocity

$$V(x) = -\frac{\kappa-1}{4\mu}\dot{Q} - \frac{\kappa+1}{4\mu} fp^\infty \left\{ -\pi\dot{a}\operatorname{sgn} V + \dot{b}[\operatorname{sgn} V + H(b-a_1) - H(b-b_1)]\cos^{-1}\left(1-\frac{2x}{b}\right) \right\}, \quad a<x<b \tag{20}$$

Since $V(b) = 0$, we can express $\dot{b}$ in terms of $\dot{a}$ and $\dot{Q}$, and substituting for $\dot{a}$ from (1),

$$V(x) = \frac{\kappa-1}{4\pi\mu}(f\dot{P}\operatorname{sgn} V - \dot{Q})\left\{\pi - \cos^{-1}\left(1-\frac{2x}{b}\right)\right\}, \quad a<x<b \tag{21}$$

It is seen from (21) that $V(x)$ is of the same algebraic sign over the whole slip zone, and that for its sign to be consistent with the previous relations we must have

$$\operatorname{sgn}(f\dot{P}\operatorname{sgn} V - \dot{Q}) = \operatorname{sgn} V \tag{22}$$

Thus

$$V(x) > 0 \text{ requires } f\dot{P} - \dot{Q} > 0 \quad \text{or } dQ < fdP \tag{23}$$

$$V(x) < 0 \text{ requires } f\dot{P} + \dot{Q} > 0 \quad \text{or } dQ > -fdP \tag{24}$$

These are conditions on the forward tangent of the load path, and they are shown graphically in Fig. 2. It is seen that the conditions for $V(x) > 0$ and $V(x) < 0$ overlap in the right sector, and that the left sector remains empty.

The condition (18) that the tangential shift vanishes at $x = b$ gives upon substitution of (14) and (9)

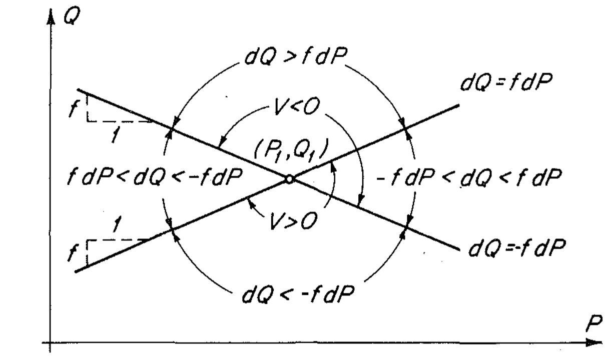

Fig. 2 Conditions on the directions of the forward tangent for loading from the initial state

$$b[\operatorname{sgn}V+H(b-a_1)-H(b-b_1)]=-\frac{\kappa-1}{\pi(\kappa+1)}\frac{Q}{fp^{\infty}}+a\operatorname{sgn}V$$

$$-a_1[1-H(b-a_1)]+b_1[1-H(b-b_1)] \qquad (25)$$

The evaluation of this relation is tedious because a large number of cases must considered individually. Most of them lead to contradictions and are not possible. Consider, for instance, the possibility $V(x)<0$ and $a<b<a_1<b_1$, for which (25) gives

$$b=\frac{\kappa-1}{\pi(\kappa+1)}\frac{Q}{fp^{\infty}}+a+a_1-b_1$$

Then using (1), (8) and (9), it is seen that

$$a<b \text{ requires } Q>Q_1$$

$$b<a_1 \text{ requires } Q<Q_1+f(P-P_1)$$

$$a<a_1 \text{ requires } P<P_1$$

which makes the point (P, Q) fall in the empty sector, so that it cannot be reached under the restriction (24) for $V(x)<0$.

A full investigation of the various possibilities reveals that only two cases can be realized under the assumption of $db/dt>0$. They will be called regimes I and II and are described in the next section.

Regimes *I* and *II*

Regime I corresponds to the choice $V(x)<0$, which requires $dQ>-fdP$, and $b_1<b$ with $\max(a_1,a)<b$. Relation (25) then yields

$$b=\frac{\kappa-1}{\pi(\kappa+1)}\frac{Q}{fp^{\infty}}+a \qquad (26)$$

and

$$b_1<b \text{ requires } Q>Q_1-f(P-P_1) \qquad (27)$$

$$a_1<b \text{ requires } Q>-f(P-P_1) \qquad (28)$$

$$a<b \text{ requires } Q>0 \qquad (29)$$

These inequalities contain no contradictions and regime I is achieved if the load path starting at (P_1, Q_1) satisfies (24) at all points. The extent of regime I in the P, Q plane determined by (27) and (29) is shown in Fig. 3. Using the inequalities that describe regime I in (14)–(16), we obtain

$$B_x^c(x)=\frac{\kappa+1}{2\mu}fp^{\infty}\left\{\left(\frac{a-x}{x}\right)^{1/2}H(a-x)\right.$$

$$-\left(\frac{a_1-x}{x}\right)^{1/2}H(a_1-x)+\left(\frac{b_1-x}{x}\right)^{1/2}H(b_1-x)-$$

$$\left.-\left(\frac{b-x}{x}\right)^{1/2}\right\},\quad 0<x<b \qquad (30)$$

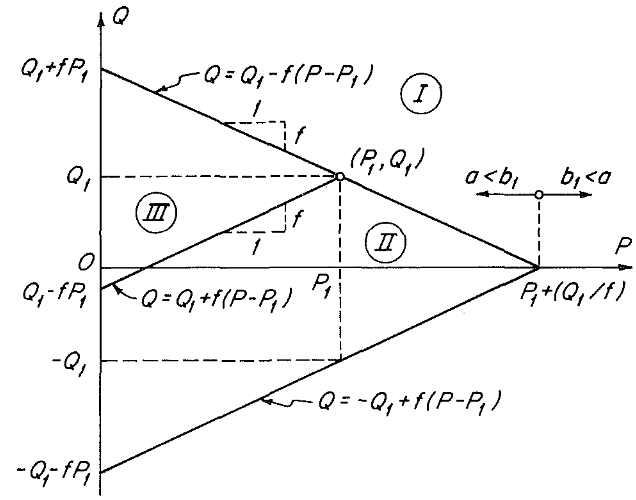

Fig. 3 Regimes achieved by loading from the point (P_1, Q_1)

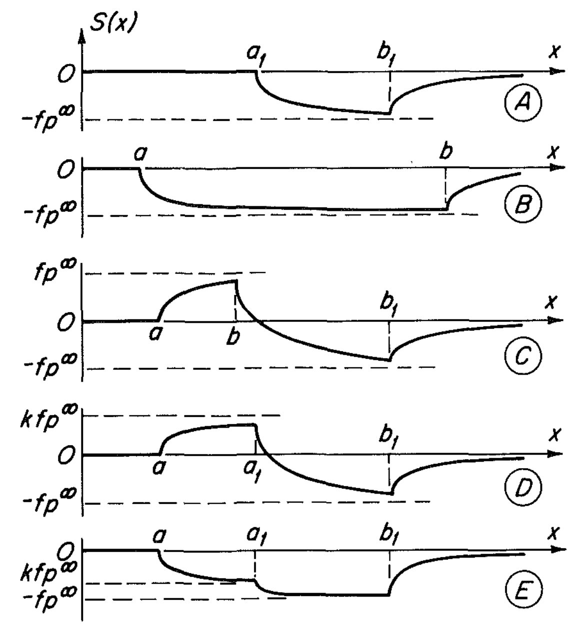

Fig. 4 Distributions of shear tractions for loading from (P_1, Q_1): *(A)*-initial distribution; *(B)*-regime *I*; *(C)*-regime *II*; *(D)*-regime *III* with $k>0$; *(E)* regime *III* with $k<0$

$$S_c(x)=fp^{\infty}\left\{-\left(\frac{x-a}{x}\right)^{1/2}H(x-a)+\left(\frac{x-a_1}{x}\right)^{1/2}H(x-a_1)\right.$$

$$\left.-\left(\frac{x-b_1}{x}\right)^{1/2}H(x-b_1)+\left(\frac{x-b}{x}\right)^{1/2}H(x-b)\right\},$$

$$0<x<\infty \qquad (31)$$

$$S(x)=fp^{\infty}\left\{-\left(\frac{x-a}{x}\right)^{1/2}H(x-a)+\left(\frac{x-b}{x}\right)^{1/2}H(x-b)\right\},$$

$$0<x<\infty \qquad (32)$$

The distribution of the shear tractions for regime I is shown schematically in Fig. 4. Loading from (P_1,Q_1) to the boundary $P=0$, $Q>Q_1+fp_1$ leads to

$$a=0,\quad b=\frac{\kappa-1}{\pi(\kappa+1)}\frac{Q}{fp^{\infty}} \qquad (33)$$

$$S(x)=fp^\infty\left\{-1+\left(\frac{x-b}{x}\right)^{1/2}H(x-b)\right\},\quad 0<x<\infty \tag{34}$$

Similarly, loading to the boundary $Q = 0$, $P > P_1 + Q_1/f$ gives

$$b=a=\frac{\kappa-1}{\pi(\kappa+1)}\frac{P}{p^\infty}>b_1 \tag{35}$$

$$S(x)=0,\quad 0<x<\infty \tag{36}$$

thus erasing all shear tractions.

Comparing (11) with (32) it is clear that the initial state corresponding to (P_1,Q_1) can be reached from the origin along any path that satisfies (24).

Regime *II* corresponds to $V(x) > 0$, requiring $dQ < fdP$, and $\max(a_1,a) < b < b_1$. These conditions reduce (25) to

$$2b=-\frac{\kappa-1}{\pi(\kappa+1)}\frac{Q}{fp^\infty}+a+b_1 \tag{37}$$

and

$$a_1<b \quad \text{requires} \quad Q<Q_1+f(P-P_1) \tag{38}$$

$$a<b \quad \text{requires} \quad Q<Q_1-f(P-P_1) \tag{39}$$

$$b<b_1 \quad \text{requires} \quad Q>-Q_1+f(P-P_1) \tag{40}$$

Again, these inequalities contain no contradictions if the load path starting at (P_1,Q_1) falls within the three boundaries given by (38)–(40). The extent of regime *II* in the P,Q plane is shown in Fig. 3. The results obtained from (14)–(16) are for this regime

$$B_x^c(x)=\frac{\kappa+1}{2\mu}fp^\infty\left\{-\left(\frac{a-x}{x}\right)^{1/2}H(a-x)-\left(\frac{a_1-x}{x}\right)^{1/2}H(a_1-x)+2\left(\frac{b-x}{x}\right)^{1/2}\right\},\quad 0<x<b \tag{41}$$

$$S_c(x)=fp^\infty\left\{\left(\frac{x-a}{x}\right)^{1/2}H(x-a)+\left(\frac{x-a_1}{x}\right)^{1/2}H(x-a_1)-2\left(\frac{x-b}{x}\right)^{1/2}H(x-b)\right\},\quad 0<x<\infty \tag{42}$$

$$S(x)=fp^\infty\left\{\left(\frac{x-a}{x}\right)^{1/2}H(x-a)-2\left(\frac{x-b}{x}\right)^{1/2}H(x-b)+\left(\frac{x-b_1}{x}\right)^{1/2}H(x-b_1)\right\},\quad 0<x<\infty \tag{43}$$

The distribution of the shear tractions for regime *II* is shown in Fig. 4. Loading from (P_1,Q_1) to the boundary $P = 0$, $-Q_1 - fP_1 < Q < Q_1 - fP_1$ of the regime *II* gives

$$a=0,\quad 2b=\frac{\kappa-1}{\pi(\kappa+1)}\frac{(-Q+fP_1+Q_1)}{fp^\infty} \tag{44}$$

$$S(x)=fp^\infty\left\{1-2\left(\frac{x-b}{x}\right)^{1/2}H(x-b)+\left(\frac{x-b_1}{x}\right)^{1/2}H(x-b_1)\right\},\ 0<x<\infty \tag{45}$$

Loading to the bottom boundary $Q = -Q_1 + f(P-P_1)$, $P < P_1 + Q_1/f$ leads to

$$b=b_1=\frac{\kappa-1}{\pi(\kappa+1)}\frac{fP_1+Q_1}{fp^\infty} \tag{46}$$

$$S(x)=fp^\infty\left\{\left(\frac{x-a}{x}\right)^{1/2}H(x-a)-\left(\frac{x-b}{x}\right)^{1/2}H(x-b)\right\},\quad 0<x<\infty \tag{47}$$

The shearing tractions then are the same as for loading from the origin to the point (P,Q) on the bottom boundary using a force with a constant direction. Consequently, consideration of the points below the bottom boundary can be omitted without loss of generality.

It is seen from (32) and (43) that the distributions of the shear tractions for regimes *I* and *II* are of the same general nature as for loading and weak friction unloading, respectively, using a force with a constant direction [1]. Moreover, the dependence on the load path for both regimes *I* and *II* is loose, in that the exact nature of the path does not matter as long as the inequalities (23) and (24) are satisfied at every point of the path. A rather interesting aspect of the problem arises also from the fact that the inequalities (23) and (24) partially overlap (see Fig. 2), and that the condition $-fdP < dQ < fdP$ on the forward tangent of the path is allowable for regime *I* ($V < 0$) as well as for regime *II* ($V > 0$). Comparing Figs. 2 and 3 it is seen that we can have $-fdP < dQ < fdP$ immediately upon departure from (P_1,Q_1) in regime *I*, but that some point under the restriction $dQ < fdP$ must be reached in regime *II* before $-fdP < dQ < fdP$ becomes compatible with the conditions under this regime.

Loading from (P_1,Q_1) along the common boundary $Q = Q_1 - f(P-P_1)$ between regimes *I* and *II* corresponds to $V(x) = 0$. Substituting this value of Q into (26) gives $b = b_1$ while (37) yields $b = a$. There is no contradiction, however, as (32) with $b = b_1$ and (43) with $b = a$ lead to the same result for the shear tractions:

$$S(x)=fp^\infty\left\{-\left(\frac{x-a}{x}\right)^{1/2}H(x-a)+\left(\frac{x-b_1}{x}\right)^{1/2}H(x-b_1)\right\},\quad 0<x<\infty \tag{48}$$

It is seen from (48) that, in this case, the separation zone simply propagates into the old slip zone $a_1 < x < b_1$ and there is no new slip.

No Slip Zone and Regime *III*

To complete the study of load paths originating from the point (P_1,Q_1) what remains is dealing with the triangle left between regimes *I* and *II* in the P,Q plane (see Fig. 3). The two possibilities are either a shrinking slip zone, which implies

$$\frac{db}{dt}<0 \tag{49}$$

or no slip zone.

Considering the possibility of a shrinking slip zone first, we assume that the initial extent of the slip zone is $a_1 < x < b_0$, and that the extent of the slip zone at some later time is a $a < x < b$, with $a < a_1$ because $P < P_1$ in the empty triangle. From (7) we have

$$\int_0^{b_0}\frac{B_x^c(\xi)d\xi}{\xi-x}=\frac{\pi(\kappa+1)}{2\mu}fp^\infty\left\{-\left(\frac{x-a}{x}\right)^{1/2}H(x-a)\operatorname{sgn}V-\left(\frac{x-a_1}{x}\right)^{1/2}H(x-a_1)H(b-a_1)+\left(\frac{x-b_1}{x}\right)^{1/2}H(x-b_1)H(b-b_1)\right\},\quad 0<x<b<b_0 \tag{50}$$

Since the interval $b < x < b_0$ is a stick zone, the dislocations in this interval are locked in and do not change with time. Consequently, differentiating (50) with respect to time we obtain

$$\int_0^{b}\frac{\dot{B}_x^c(\xi)d\xi}{\xi-x}=\frac{\pi(\kappa+1)}{4\mu}fp^\infty\frac{\dot{a}}{(x^2-ax)^{1/2}}H(x-a)\operatorname{sgn}V,\quad 0<x<b \tag{51}$$

The most general result is obtained by taking the solution of (51) that is unbounded at $x = b$. Thus

$$\dot{B}_x^c(x) = \frac{c}{(bx-x^2)^{1/2}} + \frac{\kappa+1}{4\mu} f p^\infty \dot{a} \operatorname{sgn} V \left\{ -\frac{1}{(ax-x^2)^{1/2}} H(a-x) + \frac{1}{(bx-x^2)^{1/2}} \right\}, \quad 0<x<b \tag{52}$$

The slip velocity is computed from (17), and imposing the condition that $V(b) = 0$, the result is identical with (21). Consequently (22)–(24) must be satisfied, and the empty region cannot be entered from (P_1, Q_1) with a shrinking slip zone.

Consider next the possibility of no slip zone for which (7) yields upon differentiation with respect to time

$$\int_0^a \frac{\dot{B}_x^c(\xi) d\xi}{\xi - x} = 0, \quad 0<x<a \tag{53}$$

Hence

$$\dot{B}_x^c(x) = \frac{C}{(ax-x^2)^{1/2}}, \quad 0<x<a \tag{54}$$

Evaluating C from the requirement that $V(a) = 0$ with the aid of (17),

$$\dot{B}_x^c(x) = -\frac{\kappa-1}{4\pi\mu} \frac{\dot{Q}}{(ax-x^2)^{1/2}} \tag{55}$$

Since $\dot{Q} = \dot{P}(dQ/dP)$ and $\dot{P}$ is related to $\dot{a}$ through (1),

$$\dot{B}_x^c(x) = -\frac{p^\infty(\kappa+1)}{4\mu} \frac{\dot{a}(dQ/dP)}{(ax-x^2)^{1/2}}, \quad 0<x<a \tag{56}$$

It is seen from (56) that, because of the term dQ/dP, the solution cannot be carried further without specifying the actual path in the P,Q plane. This means that the distribution of shear tractions left on the cut as the separation zone closes without slip depends on the full details of the load path.

As an example we consider loading along a straight line, or

$$\frac{dQ}{dP} = kf = \text{const}, \quad -1<k<1 \tag{57}$$

which is called regime *III*. The problem then is identical to unloading with strong friction in [2], and

$$B_x^c(x) = \frac{\kappa+1}{2\mu} k f p^\infty \left\{ -\left(\frac{a-x}{x}\right)^{1/2} H(a-x) + \left(\frac{a_1-x}{x}\right)^{1/2} \right\}, \quad 0<x<a \tag{58}$$

$$S(x) = fp^\infty \left\{ k\left(\frac{x-a}{x}\right)^{1/2} H(x-a) - (1+k)\left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0<x<\infty \tag{59}$$

The distribution of the shear tractions for positive and negative k are shown schematically in Fig. 4.

Loading along the common boundary $Q = Q_1 - f(P-P_1)$ between the regimes *I* and *III* corresponds to $V(x) = 0$, and the distribution of the shear tractions is given by (48). In contrast, loading from (P_1, Q_1) along the common boundary $Q = Q_1 + f(P-P_1)$ between regimes *II* and *III* corresponds to $b = a_1 =$ const and $V(x) > 0$, and

$$S(x) = fp^\infty \left\{ \left(\frac{x-a}{x}\right)^{1/2} H(x-a) - 2\left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0<x<\infty \tag{60}$$

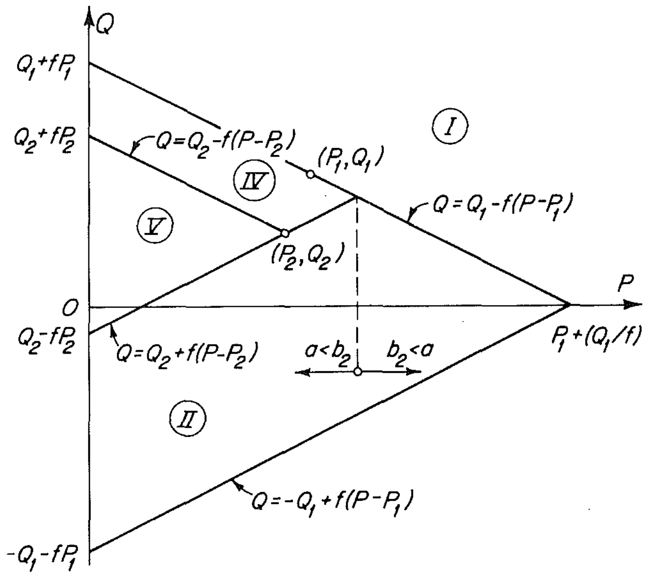

Fig. 5 Regimes achieved by loading from the point (P_2, Q_2) that belongs to regime *II*

Loading From Regimes *I* and *III*

Suppose that loading is carried on from (P_1, Q_1) to some point (P_2, Q_2) under the conditions of regime *I*, but that beyond (P_2, Q_2) condition (24) is violated. Comparing (32) with (11) is it clear that the shear tractions at (P_2, Q_2) are of the same nature as those at (P_1, Q_1). Consequently the transition to new regimes beyond (P_2, Q_2) is the same as that analyzed at (P_1, Q_1). New transitions must be studied, however, if the point (P_2, Q_2) is reached by loading under the conditions of regime *II*. From (1), (37), and (43)

$$a_2 = \frac{\kappa-1}{\pi(\kappa+1)} \frac{P_2}{p^\infty} \tag{61}$$

$$b_2 = \frac{\kappa-1}{\pi(\kappa+1)} \frac{f(P_1+P_2)+Q_1-Q_2}{2fp^\infty} \tag{62}$$

$$S_2(x) = fp^\infty \left\{ \left(\frac{x-a_2}{x}\right)^{1/2} H(x-a_2) - 2\left(\frac{x-b_2}{x}\right)^{1/2} H(x-b_2) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0<x<\infty \tag{63}$$

Taking $S_2(x)$ as $S_i(x)$ in (7), the investigation of the different cases is quite similar to that for loading from (P_1, Q_1) and, consequently, the derivations are omitted and only the results presented.

The general situation is shown in Fig. 5. Regime *II* is of course continued at (P_2, Q_2) if (23) is not violated. Otherwise the two regimes *IV* and *V* are encountered.

Regime *IV* is characterized by $db/dt > 0$, $V(x) > 0$ (so that $dQ > -fdP$ must be satisfied at all points), and $a_2 < b < b_2$. The extent of the slip zone is given by

$$b = \frac{\kappa-1}{\pi(\kappa+1)} \frac{f(P+P_2)+Q-Q_2}{2fp^\infty} \tag{64}$$

The shear tractions

$$S(x) = fp^\infty \left\{ -\left(\frac{x-a}{x}\right)^{1/2} H(x-a) + 2\left(\frac{x-b}{x}\right)^{1/2} H(x-b) - 2\left(\frac{x-b_2}{x}\right)^{1/2} H(x-b_2) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0<x<\infty \tag{65}$$

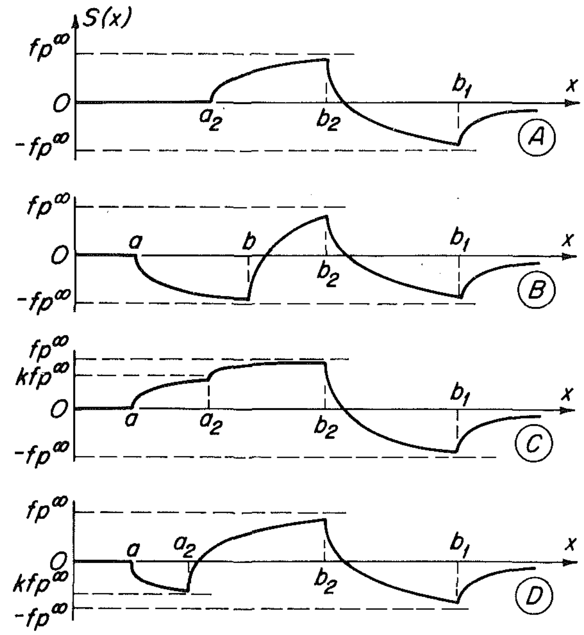

Fig. 6 Distributions of shear tractions for loading from (P_2, Q_2) that belong to regime *II*: (*A*)-initial distribution; (*B*)-regime *IV*; (*C*)-regime *V* for $k > 0$; (*D*)-regime *V* for $k < 0$

are shown in Fig. 6. Loading under regime *IV* to the boundary $Q = Q_1 - f(P-P_1)$ leads to

$$b = b_2 \tag{66}$$

$$S(x) = fp^\infty \left\{ -\left(\frac{x-a}{x}\right)^{1/2} H(x-a) + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\},$$

$$0 < x < \infty \tag{67}$$

and loading slightly beyond this boundary returns the slip process to regime *I*, while involving a sudden jump in the slip zone from b_2 to b_1.

Regime *V* is similar to regime *III* in that the separation zone recedes, and there is no new slip. Again a strict dependence on the load path is involved as

$$\dot{B}_x^c(x) = -\frac{p^\infty(\kappa+1)}{4\mu} \frac{\dot{a}(dQ/dP)}{(ax-x^2)^{1/2}}, \quad 0 < x < a \tag{68}$$

For the example considered before, or $dQ/dP = k$,

$$S(x) = fp^\infty \left\{ k\left(\frac{x-a}{x}\right)^{1/2} H(x-a) \right.$$

$$+ (1-k)\left(\frac{x-a_2}{x}\right)^{1/2} H(x-a_2) - 2\left(\frac{x-b_2}{x}\right)^{1/2} H(x-b_2)$$

$$\left. + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0 < x < \infty \tag{69}$$

The shear tractions are shown schematically in Fig. 6 for positive and negative values of k.

Loading From Regime *III*

We can only continue to consider the example $dQ/dP = k$ because the shear tractions at (P_2, Q_2) need to be known to do a full analysis. Using

$$a_2 = b_2 = \frac{\kappa-1}{\pi(\kappa+1)} \frac{P_2}{p^\infty} \tag{70}$$

$$S_2(x) = fp^\infty \left\{ k\left(\frac{x-a_2}{x}\right)^{1/2} H(x-a_2) \right.$$

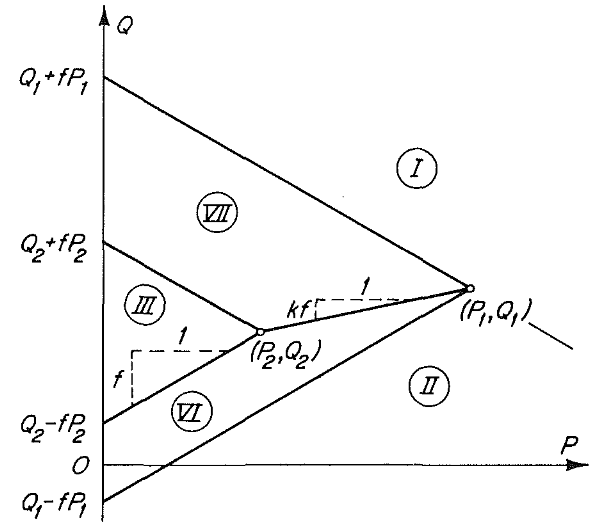

Fig. 7 Regimes achieved by loading from the point (P_2, Q_2) that belongs to regime *III*

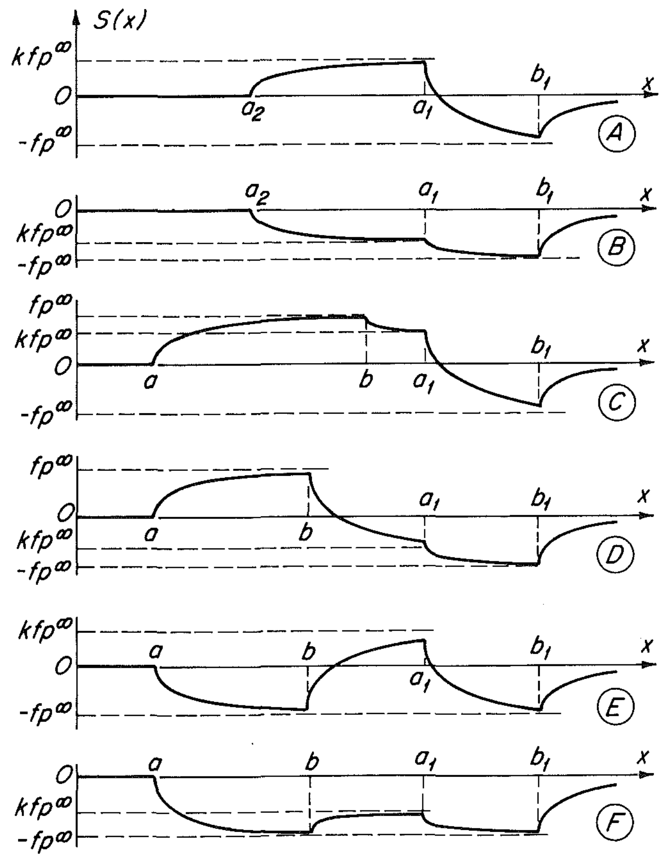

Fig. 8 Distributions of shear tractions for loading from (P_2, Q_2) that belongs to regime *III*: (*A*)-initial distribution for $k > 0$; (*B*)-initial distribution for $k < 0$; (*C*)-regime *VI* for $k > 0$; (*D*)-regime *VI* for $k<0$; (*E*)-regime *VII* for $k > 0$; (*F*)-regime *VII* for $k < 0$.

$$- (1+k)\left(\frac{x-a_1}{x}\right)^{1/2} H(x-a_1)$$

$$\left. + \left(\frac{x-b_1}{x}\right)^{1/2} H(x-b_1) \right\}, \quad 0 < x < \infty \tag{71}$$

The analysis leads to the general situation shown in Fig. 7, which indicates the possibility of continuing under regime *III* if $dQ < -fdP$ and $dQ > fdP$, and also shows two new regimes.

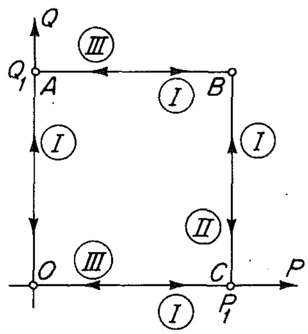

Fig. 9 Load path used in the example

Regime *VI* involves $db/dt > 0$, $V(x) > 0$, and $\max(a,a_2) < b < a_1$. The extent of the slip zone is given by

$$b=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{f(P-kP_2)-Q+Q_2}{(1-k)fp^\infty} \tag{72}$$

and the shear tractions are

$$S(x)=fp^\infty\Big\{\Big(\frac{x-a}{x}\Big)^{1/2}H(x-a)-(1-k)\Big(\frac{x-b}{x}\Big)^{1/2}H(x-b)$$
$$-(1+k)\Big(\frac{x-a_1}{x}\Big)^{1/2}H(x-a_1)+\Big(\frac{x-b_1}{x}\Big)^{1/2}H(x-b_1)\Big\},$$
$$0<x<\infty \tag{73}$$

They are shown in Fig. 8. If the load path crosses the boundary $Q = Q_1 + f(P-P_1)$ the slip process returns to regime *II*.

Regime *VII* is characterized by $db/dt > 0$, $V(x) < 0$, and $\max(a,a_2) < b < a_1 < b_1$. The results are

$$b=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{f(P+kP_2)+Q-Q_2}{(1+k)fp^\infty} \tag{74}$$

$$S(x)=fp^\infty\Big\{-\Big(\frac{x-a}{x}\Big)^{1/2}H(x-a)$$
$$+(1+k)\Big(\frac{x-b}{x}\Big)^{1/2}H(x-b)$$
$$-(1+k)\Big(\frac{x-a_1}{x}\Big)^{1/2}H(x-a_1)$$
$$+\Big(\frac{x-b_1}{x}\Big)^{1/2}H(x-b_1)\Big\},$$
$$0<x<\infty \tag{75}$$

The shear tractions are shown schematically in Fig. 8. If the load path crosses the boundary $Q = Q_1 - f(P-P_1)$, the slip process returns to regime *I*.

Example

As an example consider a closed rectangular load path starting and terminating at the origin, as shown in Fig. 9. Regardless of whether the load path is followed in a clockwise or counterclockwise direction, the first two legs correspond to regime *I* and, consequently from (8), (26), and (32), we have at point *B*

$$a_B=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{P_1}{p^\infty} \tag{76}$$

$$b_B=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{fP_1+Q_1}{fp^\infty} \tag{77}$$

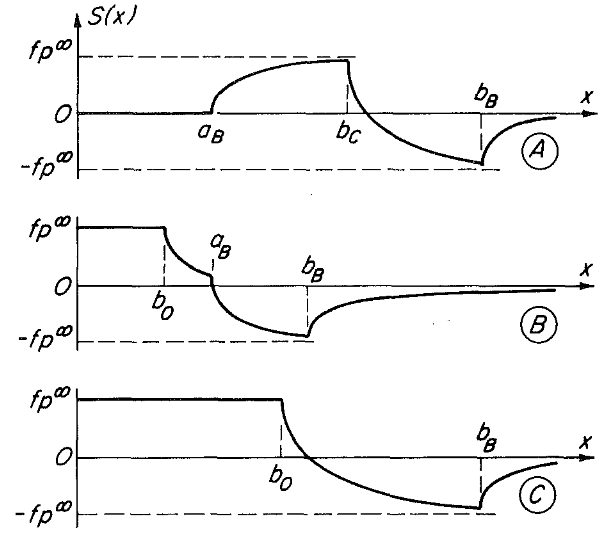

Fig. 10 Residual shear tractions left on the cut when the loads are removed: (*A*)-clockwise traverse; (*B*)-counterclockwise traverse for $f > Q_1/P_1$; (*C*)-counterclockwise traverse for $f < Q_1/P_1$.

$$S_B(x)=fp^\infty\Big\{-\Big(\frac{x-a_B}{x}\Big)^{1/2}H(x-a_B)$$
$$+\Big(\frac{x-b_B}{x}\Big)^{1/2}H(x-b_B)\Big\},\quad 0<x<\infty \tag{78}$$

Clockwise Traverse. The leg *BC* corresponds to regime *II*. It follows then from (37) and (43) that at point *C*

$$b_C=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{2fP_1+Q_1}{2fp^\infty} \tag{79}$$

$$S_C(x)=fp^\infty\Big\{\Big(\frac{x-a_B}{x}\Big)^{1/2}H(x-a_B)$$
$$-2\Big(\frac{x-b_C}{x}\Big)^{1/2}H(x-b_C)+\Big(\frac{x-b_B}{x}\Big)^{1/2}H(x-b_B)\Big\},$$
$$0<x<\infty \tag{80}$$

The segment *CO* belongs to regime *V* with $k = 0$. Consequently on the basis of (69), the residual shear stress left upon removal of the loads is

$$S_0(x)=fp^\infty\Big\{\Big(\frac{x-a_B}{x}\Big)^{1/2}H(x-a_B)-2\Big(\frac{x-b_C}{x}\Big)^{1/2}H(x-b_C)$$
$$+\Big(\frac{x-b_B}{x}\Big)^{1/2}H(x-b_B)\Big\},\quad 0<x<\infty \tag{81}$$

Counterclockwise Traverse. The part *BA* of the counterclockwise load path corresponds to regime *III* with $k = 0$ and $a_A = 0$. Thus from (59)

$$S_A(x)=fp^\infty\Big\{-\Big(\frac{x-a_B}{x}\Big)^{1/2}H(x-a_B)$$
$$+\Big(\frac{x-b_B}{x}\Big)^{1/2}H(x-b_B)\Big\},\quad 0<x<\infty \tag{82}$$

For loading along the last segment *AO* of the counterclockwise circuit, it is necessary to distinguish between two cases. If $f > Q_1/P_1$, the whole segment *AO* corresponds to regime *VI*. Then noting that $a_A = 0$, it follows from (72) and (73) with $k = 0$ that

$$b_0=\frac{\kappa-1}{\pi(\kappa+1)}\,\frac{Q_1}{fp^\infty} \tag{83}$$

$$S_0(x)=fp^\infty\left\{1-\left(\frac{x-b_0}{x}\right)^{1/2}H(x-b_0)\right.$$

$$\left.-\left(\frac{x-a_B}{x}\right)^{1/2}H(x-a_B)+\left(\frac{x-b_B}{x}\right)^{1/2}H(x-b_B)\right\},$$

$$0<x<\infty \qquad (84)$$

In case of $f < Q_1/P_1$, only the first part of the segment AO belongs to regime VI. It is seen from Fig. 7 that the point on AO at which a transition to a different regime takes place is $Q = Q_1 - fP_1$. At this point,

$$b=\frac{\kappa-1}{\pi(\kappa-1)}\frac{P_1}{p^\infty}=a_B \qquad (85)$$

$$S(x)=fp^\infty\left\{1-2\left(\frac{x-a_B}{x}\right)^{1/2}H(x-a_B)\right.$$

$$\left.+\left(\frac{x-b_B}{x}\right)^{1/2}H(x-b_B)\right\},\quad 0<x<\infty \qquad (86)$$

from (72) and (73). Although loading from regime VI has not been considered, there is no need to do new mathematics. Comparing (86) with (63) for $a_2 = 0$, $b_2 = a_B$ and $b_1 = b_B$, it is seen that loading along the bottom part of AO is the same as loading that starts and stays in regime II. Consequently

$$b_0=\frac{1}{2}b_B \qquad (87)$$

from (37), and (43) yields

$$S_0(x)=fp^\infty\left\{1-2\left(\frac{x-b_0}{x}\right)^{1/2}H(x-b_0)\right.$$

$$\left.+\left(\frac{x-b_B}{x}\right)^{1/2}H(x-b_B)\right\},\quad 0<x<\infty \qquad (88)$$

for the residual shear tractions left when the concentrated load is completely removed.

The residual shear tractions given by (81), (84), and (88) are shown schematically in Fig. 10. It is seen that the residual tractions are quite different not only for the two directions of traverse, but that also their qualitative features may be affected by the value of the friction coefficient.

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