ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.3,pp.329-338

Bifurcational Behavior of a Cohen-Grossberg Neural Network of Two Neurons with Impulsive Effects

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(Received 3 May 2009, accepted 8 September 2009)

Abstract: In this paper, a Cohen–Grossberg neural network composed of two neurons with nonisochronous impulsive effects is proposed and investigated. By employing Mawhin's coincidence theorem, we first show that the existence of semi-trivial periodic solutions. Under this situation, sufficient conditions assuring the asymptotic stability of semi-trivial periodic solutions are derived by using Floquet theory of the impulsive differential equation. Finally, we extend the method in [17] and then obtain the bifurcation of nontrivial periodic solutions.

Keywords: neural network; non-isochronous impulsive effect; semi-trivial periodic solution; stability; bifurcation

1 Introduction

Mathematical modelling in neural networks has been based on neurons that are different both from real biological neurons and from the realistic functioning of simple electronic circuits. In the past three decades, neural networks architectures have been extensively researched and developed. There have been many nice works regarding the continuous or piecewise continuous, discrete and impulsive neural networks with and without delays [1-13] and the references cited therein.
In this paper, we consider the following impulsive differential equations of two neurons network:

$$\begin{cases} \left.\begin{array}{l} x_1'(t) = a_1x_1(t)\big[b_1(x_1(t)) + h_{11}f(x_1(t)) + h_{12}f(x_2(t)) + I_1\big], \\ x_2'(t) = a_2x_2(t)\big[b_2(x_2(t)) + h_{21}f(x_1(t)) + h_{22}f(x_2(t)) + I_2\big], \end{array}\right\} t \neq (n+\widetilde{l}-1)T,\ t \neq nT, \\ \left.\begin{array}{l} \triangle x_1(t) = 0, \\ x_2(t^+) = p_2x_2(t), \end{array}\right\} \qquad t = (n+\widetilde{l}-1)T, \\ \left.\begin{array}{l} x_1(t^+) = p_1x_1(t), \\ \triangle x_2(t) = 0, \end{array}\right\} \qquad t = nT. \end{cases} \tag{1}$$

Here, $n \in \mathbf{N}$, $0 < \widetilde{l} < 1$ indicates the intervals of time between the pulsed use of controls, of length $\widetilde{l}T$ and $(1-\widetilde{l})T$. Also, $\Delta x_i(t) = x_i(t^+) - x_i(t)(i = 1, 2)$. The impulsive conditions include the proportional perturbations. The proportional parts may be dependent on external sudden input current to the ith neuron$(i = 1, 2)$. It is also assumed that each neuron is activated in a periodic fashion with same period, but at different moments. The coefficients $a_i(\neq 0)$ and $p_i(> 0)$, $i = 1, 2$, are real constants. The appropriately behaved function $b_i(i = 1, 2) : \mathbf{R} \to \mathbf{R}$ is a PC^1-smooth constant parameter function and satisfies the following assumption.

(H_1) There exist four positive numbers d_i and $c_i(i = 1, 2)$ such that

$$c_i \le \frac{b_i(u)}{u} \le d_i,\ \forall u \in \mathbf{R} \setminus \{0\},\ i = 1, 2.$$

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The general activation function $f:\mathbf{R}\to\mathbf{R}$ is also a PC^1-smooth constant parameter functions. Assume that f possesses the following property

(H_2) There exist two real numbers $k\geq 0$ and $d\geq 0$ such that

$$|f(x)|\leq k|x|+d.$$

Note that the assumption (H_2) does not imply that the function f is monotonous and globally Lipschitz continuous. Without loss of generality, we also assume that $f'(0)=1$ [9].

This paper is organized as follows. In Section 2, we state some notations and definitions and state some preliminary results which attest the well-posedness of the model. It is then shown in Section 3 that once a threshold condition is reached, the semitrivial solution loses its stability and a nontrivial periodic solution appears via a bifurcation. Finally, a brief discussion is given in Section 4.

2 Preliminaries

2.1 Some preparations

In this subsection, we shall introduce some notations and definitions and state a preliminary lemma which will be useful for establishing our main results.

Let $\mathbf{R}_+=[0,\infty)$ and $\mathbf{J}\subset\mathbf{R}$. We introduce the following spaces of functions:
$PC(\mathbf{J},\mathbf{R})\doteq\big\{u:\mathbf{J}\to\mathbf{R}:u$ is continuous for $t\in\mathbf{J}$, $t\neq\tau_k$, continuous from the left for $t\in\mathbf{J}$, and has discontinuities of the first kind at the points $\tau_k\in\mathbf{J}$, $k\in\mathbf{N}\big\}$,
and
$PC^1(\mathbf{J},\mathbf{R})\doteq\big\{u\in PC(\mathbf{J},\mathbf{R}):u$ is continuously differentiable for $t\in\mathbf{J}$, $t\neq\tau_k$; $u'(\tau_k^+)$ and $u'(\tau_k^-)$ exist, $k\in\mathbf{N}\big\}$. Denote by $F=(F_1,F_2)^T$ the map defined by the right hand side of the first two equations in the system (1).

Let $V:\mathbf{R}_+\times\mathbf{R}^2\to\mathbf{R}$. Then V is said to belong to class V_0[14] if
(i) V is continuous on $((n-1)T,(n+\widetilde{l}-1)T]\times\mathbf{R}^2$ and $\big((n+\widetilde{l}-1)T,nT\big]\times\mathbf{R}^2$ and for each $X\in\mathbf{R}^2$ and $n\in\mathbf{N}$, $\lim_{(t,Y)\to\left((n+\widetilde{l}-1)T^+,X\right)}V(t,Y)=V\big((n+\widetilde{l}-1)T^+,X\big)$ and $\lim_{(t,Y)\to(nT^+,X)}V(t,Y)=V(nT^+,X)$ exist and are finite. Here, $0<\widetilde{l}<1$.
(ii) V is locally Lipschitzian in the second variable.

Definition 2.1 *Let $V\in V_0$. Then for $(t,X)\in\big((n-1)T,(n+\widetilde{l}-1)T\big]\times\mathbf{R}^2$ and $\big((n+\widetilde{l}-1)T,nT\big]\times\mathbf{R}^2$($0<\widetilde{l}<1$), the upper right derivative of $V(t,x)$ with respect to the impulsive differential system (1) is defined as*

$$D^+V(t,X)=\lim_{h\to 0^+}\sup\frac{1}{h}\big[V(t+h,X+hF(t,X))-V(t,X)\big].$$

Lemma 2.1 *[15] Let $\Omega\subset\mathbb{X}$ be an open bounded set and let $N:\mathbb{X}\to\mathbb{Y}$ be a continuous operator which is L-compact on $\bar{\Omega}$ (i.e., $QN:\bar{\Omega}\to\mathbb{Y}$ and $K_P(I-Q)N:\bar{\Omega}\to\mathbb{Y}$ are compact). Assume*

(i) for each $\lambda\in(0,1)$, $x\in\partial\Omega\bigcap\operatorname{Dom}L Lx\neq\lambda Nx$,

(ii) for each $x\in\partial\Omega\bigcap\ker L$. $QNx\neq 0$,

(iii) $\deg(JNQ,\Omega\bigcap\ker L,0)\neq 0$.

Then $Lx=Nx$ has at least one solution in $\bar{\Omega}\bigcap\operatorname{Dom}L$.

2.2 The existence of semi-trivial periodic solution of system (1)

In this subsection, we shall consider the following subsystem of system (1)

$$\begin{cases} x_1'(t) = a_1 x_1(t)\big[b_1(x_1(t)) + h_{11} f(x_1(t)) + h_{12} f(0) + I_1\big], & t \neq nT, \\ x_1(t^+) = p_1 x_1(t), & t = nT. \end{cases} \tag{2}$$

For the system (2), making the change of variable $\ln x_1 = x$, then (2) is transformed as

$$\begin{cases} x'(t) = a_1\big[b_1(e^{x(t)}) + h_{11} f(e^{x(t)}) + h_{12} f(0) + I_1\big], & t \neq nT, \\ x(t^+) = \ln p_1 + x(t), & t = nT. \end{cases} \tag{3}$$

Let

$$DomL = PC_T' = \{x \in PC^1([0,T], \mathbf{R}) | x(0) = x(T)\},$$

$$Z = \{(\mathfrak{f}, a) | x' = \mathfrak{f}, x \in PC_T', a = x(T^+) - x(T)\}$$

and

$$L:\ DomL \to Z,\ x \to (a_1\big[b_1(e^{x(t)}) + h_{11} f(e^{x(t)}) + h_{12} f(0) + I_1\big], \ln p_1).$$

Obviously, $KerL = \mathbf{R}$,

$$ImL = \{Z = (\mathfrak{f}, a) \in Z | \int_0^T \mathfrak{f}(s)ds + a = 0\},$$

and $dimKerL = 1 = codimImL$. Then ImL is closed in Z, and L is a Fredholm mapping of index zero. Define

$$Px = \frac{1}{T}\int_0^T \mathfrak{f}(s)ds,$$

$$Qz = Q(\mathfrak{f}, a) = (\frac{1}{T}[\int_0^T \mathfrak{f}(s)ds + a], 0).$$

It is easy to show that P and Q are continuous projectors satisfying

$$ImP = KerL,\ \ ImL = KerQ = Im(I - Q).$$

Furthermore, through an easy computation, we can find that the inverse K_p: ImL→ KerP∪DomL of L_p has the form

$$K_p z = \int_0^t \mathfrak{f}(s)ds + [\frac{t}{T}]a - \frac{1}{T}\int_0^T\int_0^t \mathfrak{f}(s)dsdt - a,\ t \in (0, T], \tag{4}$$

in which $[\frac{t}{T}]$ denotes the integer part of $\frac{t}{T}$. Then

$$\begin{aligned} QNx &= \Big(\ \tfrac{1}{T}\Big(\ \textstyle\int_0^T a_1\big[b_1(e^{x(t)}) + h_{11} f(e^{x(t)}) + h_{12} f(0) + I_1\big]dt \quad + \ln p_1\ \Big), 0\Big), \\ K_p(I - Q)Nx &= \int_0^t a_1\big[b_1(e^{x(s)}) + h_{11} f(e^{x(s)}) + h_{12} f(0) + I_1\big]ds + [\frac{t}{T}]\ln p_1 \\ &\quad -\frac{1}{T}\int_0^T\int_0^t a_1\big[b_1(e^{x(s)}) + h_{11} f(e^{x(s)}) + h_{12} f(0) + I_1\big]dsdt + \ln p_1 \\ &\quad -(\frac{t}{T} - \frac{1}{2})\Big(\int_0^T a_1\big[b_1(e^{x(s)}) + h_{11} f(e^{x(s)}) + h_{12} f(0) + I_1\big]ds + \ln p_1\Big). \end{aligned}$$

Clearly, QN and $K_p(I-Q)N$ are continuous, using Arzela-Ascoli theorem. It is easy to show, $\overline{K_p(I - Q)N(\overline{\Omega})}$ is compact for any open bounded subset $\Omega \subset \mathbb{X}$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on Ω for any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx,\ \lambda \in (0, 1)$, we have

$$\begin{cases} x'(t) = \lambda a_1\big[b_1(e^{x(t)}) + h_{11} f(e^{x(t)}) + h_{12} f(0) + I_1\big], & t \neq nT, \\ \Delta x(t) = \lambda \ln p_1, & t = nT. \end{cases} \tag{5}$$

Suppose that $x(t) \in \mathbb{X}$ is a solution of (5) for some $\lambda \in (0,1)$. Integrating (5) over the interval $[0,T]$, we obtain

$$\int_0^T \big[b_1(e^{x(t)}) + h_{11}f(e^{x(t)})\big]dt = -\frac{\ln p_1}{a_1} - (h_{12}f(0) + I_1)T. \tag{6}$$

It follows from (5) and (6) that

$$\int_0^\omega |x'(t)|dt \le 2\big[|\ln p_1| + |a_1(h_{12}f(0) + I_1)T|\big], \tag{7}$$

Since $x \in \mathbb{X}$, there exist $\xi, \tau \in (0,T]$ such that

$$x(\xi) = \min_{t\in(0,T]} x(t),\ x(\tau) = \min_{t\in(0,T]} x(t), \tag{8}$$

Assume that

$$\begin{cases} c_1 - |h_{11}|k > 0; \\ |\frac{\ln p_1}{a_1 T} + h_{12}(f(0) + I_1)| - d|h_{11}| > 0 \end{cases} \tag{9}$$

hold. Then from (6) and (8) together with (H_1) and (H_2) we can derive that

$$\begin{aligned} x(t) \ \le\ & \ln \frac{|\frac{\ln p_1}{a_1 T}| + |h_{12}(f(0) + I_1)| + d|h_{11}|}{c_1 - |h_{11}|k} \\ & +3|\ln p_1| + 2|a_1(h_{12}f(0) + I_1)T| \doteq \widetilde{H}_1. \end{aligned} \tag{10}$$

On the other hand, from (6) and (8) together with (H_1) and (H_2) we can get that

$$\begin{aligned} x(t) \ \ge\ & \frac{|\frac{\ln p_1}{a_1 T} + (h_{12}f(0) + I_1)| - d|h_{11}|}{d_1 + |h_{11}|k} \\ & -3|\ln p_1| - 2|a_1(h_{12}f(0) + I_1)T| \doteq \widetilde{H}_2. \end{aligned} \tag{11}$$

Clearly, $\widetilde{H}_i,\ i = 1,2$ are independent of λ. Take $\widetilde{H} = |\widetilde{H}_1| + |\widetilde{H}_2| + 1$, then $|x|_T = \max_{t\in(0,T]}\{x(t)\} < \widetilde{H}$ whenever $x \in \mathbb{X}$ is a solution to (5) for any $\lambda \in (0,1)$.

Define $\Omega = \{x \in \mathbb{X} : |x|_T < \widetilde{H}\}$. Then there are no $\lambda \in (0,1)$ and $x \in \partial\Omega$ such that $Lx = \lambda Nx$. Note that $QNx = JQNx$ and $x \in KerL$, it must be

$$QNx = \Bigg(\Big(a_1\big[b_1(e^x) + h_{11}f(e^x) + h_{12}f(0) + I_1\big] \quad +\tfrac{1}{T}\ln p_1 \ \Big), 0\Bigg).$$

Then for any $x \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbf{R}$, $x = \alpha$ and $|\alpha|_T = \widetilde{H}$. We have $QN\alpha \ne 0$, since $QN\alpha = 0$ only if $|h| < \max\{|\widetilde{H}_1|, |\widetilde{H}_2|\}$. Then $|\alpha|_T < \widetilde{H}$. It is easily seen that $(JQN)^{-1}(0) \cap (\Omega \cap KerL) \ne \emptyset$. Consider the homotopy $\mathfrak{F} : (\Omega \cap KerL) \times [0,1] \to \Omega \cap KerL$, defined by

$$\mathfrak{F}(x,\mu) = a_1(x + h_{12}f(0) + I_1) + \frac{1}{T}\ln p_1 + \mu a_1(b_1(e^x) + h_{11}f(e^x) - x).$$

Note that $\mathfrak{F}(x,1) = JQNx$. If $\mathfrak{F}(x,\mu) = 0$, then we get $|x|_T < \widetilde{H}$. Hence, $\mathfrak{F}(x,\mu) \ne 0$ for $(x,\mu) \in (\Omega \cap KerL) \times [0,1]$. It follows from the property of invariance under a homotopy that

$$deg\{JQN, \Omega \cap KerL, 0\} = sgn\{a_1\} \ne 0.$$

From the above-mentioned analysis together with Lemma 2.1, one notes the following result.

Theorem 2.1 *Suppose that (H_1), (H_2) and (9) hold. Then system (2) has a $T-$periodic solution $x_1^*(t)$.*

Remark 2.1 *If considering another subsystem of system (1)*

$$\begin{cases} x_2'(t) = a_2x_2(t)\big[b_2(x_2(t)) + h_{21}f(0) + h_{22}f(x_2(t)) + I_2\big], \ \} & t \neq (n+\widetilde{l}-1)T, \\ x_2(t^+) = p_2x_2(t), \ \} & t = (n+\widetilde{l}-1)T. \end{cases} \tag{12}$$

By using the similar method above, we can also observe that if (H_1), (H_2) *and*

$$\begin{cases} c_2 - |h_{22}|k > 0; \\ |\frac{\ln p_2}{a_2T} + h_{21}(f(0) + I_2)| - d|h_{22}| > 0 \end{cases} \tag{13}$$

hold, then system (12) has a $T-$*periodic solution* $x_2^*(t)$.

Remark 2.2 *Theorem 2.1 still holds if condition (9) is replaced by the condition*

(9') $|\frac{\ln p_1}{a_1T} + h_{12}(f(0) + I_1)| > \frac{dk}{c_1}$.

Remark 2.3 *Theorem 2.1 implies that system (1) has a semi-trivial* $T-$*periodic solution* $(x_1^*(t), 0)$.

2.3 The stability of the semi-trivial $T-$periodic solution

We consider the asymptotic stability of the semi-trivial periodic solution above by using small amplitude perturbation methods. Let $\big(x_1(t), x_2(t)\big)$ be a solution of (1) and let

$$x_1(t) = v(t) + x_1^*(t),\ x_2(t) = u(t)$$

where $u,\ v$ are understood to be small perturbations. The right-hand sides of the first two equations in (1) can be expanded using Taylor series. After neglecting higher-order terms, the linearized equations together with the corresponding impulsive perturbation conditions read as

$$\begin{cases} \left.\begin{array}{l} v'(t) = \theta_1(x_1^*(t))v + \theta_2(x_1^*(t))u, \\ u'(t) = \theta_3(x_1^*(t))u, \end{array}\right\} & t \neq (n+\widetilde{l}-1)T,\ t \neq nT, \\ \left.\begin{array}{l} \triangle v(t) = 0, \\ u(t^+) = p_2u(t), \end{array}\right\} & t = (n+\widetilde{l}-1)T, \\ \left.\begin{array}{l} v(t^+) = p_1v(t), \\ \triangle u(t) = 0, \end{array}\right\} & t = nT, \end{cases} \tag{14}$$

where

$$\begin{cases} \theta_1(x_1^*) = a_1\big(b_1(x_1^*) + x_1^*b_1'(x_1^*) + h_{11}f(x_1^*) + h_{11}x_1^*f'(x_1^*) + h_{12}f(0) + I_1\big); \\ \theta_2(x_1^*) = a_1h_{12}f'(0)x_1^* = a_1h_{12}x_1^*; \\ \theta_3(x_1^*) = a_2\big(b_2(0) + h_{21}f(x_1^*) + h_{22}f(0) + I_2\big). \end{cases} \tag{15}$$

Let $M(t)$ be the fundamental matrix of the subsystem formed with the first two equations in (14) . Then $M(t)$ must satisfies

$$\frac{dM(t)}{dt} = \begin{pmatrix} \theta_1(x_1^*(t)) & \theta_2(x_1^*(t)) \\ 0 & \theta_3(x_1^*(t)) \end{pmatrix} M(t)$$

and $M(0) = E_2$ (unit 2×2 matrix). Hence,

$$M^* = \begin{pmatrix} p_1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_2 \end{pmatrix} M(T).$$

According to the Floquet theory of impulsive differential equation [14], the semi-trivial periodic solution $(x_1^*(t), 0)$ is then asymptotic stable if and only if $|\lambda_i| < 1 (i = 1, 2)$, that is,

$$p_1 < e^{-\int_0^T \big[\theta_1(x_1^*(s))\big]ds} \ and\ p_2 < e^{-\int_0^T \big[\theta_3(x_1^*(s))\big]ds}. \tag{16}$$

Theorem 2.2 *Assume that conditions* (H_1), (H_2), *(9) and (16) hold. Then the semi-trivial* $T-$*periodic solution* $(x_1^*(t), 0)$ *is asymptotic stable.*

Corollary 2.1 *Assume that conditions* (H_1), (H_2), *(9) and*

$$p_1 > e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds} \text{ or } p_2 > e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds} \tag{17}$$

holds. Then the semi-trivial $T-$*periodic solution* $(x_1^*(t), 0)$ *is unstable.*

Remark 2.4 *Under the circumstance of the existence of the semi-trivial periodic solution, it is interesting to note that, as far as the effect of the proportional perturbations* $p_i(i = 1, 2)$ *are concerned, a large* $p_i(i = 1, 2)$ *may always distabilize the the semi-trivial* $T-$*periodic solution* $(x_1^*(t), 0)$ *by bring* p_1(*or* p_2) *above* $e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds}$(*or* $e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds}$).

Remark 2.5 *The inequalities* $p_1 < e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds}$ *and* $p_2 < e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds}$ *mean that the rate of destruction of the* $2th$ *neuron is sufficiently large.*

In the case when the product $p_1 e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds}$(or $p_2 e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds}$) is close to 1, the impulsive perturbation p_1(or p_2) has a significant effect on the $2th$ neuron. In the following we want to investigate the following three cases.

Case 1 $p_1 e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds} \neq 1, p_2 e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds} = 1;$

Case 2 $p_1 e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds} = 1, p_2 e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds} \neq 1;$

Case 3 $p_1 e^{-\int_0^T \left[\theta_1(x_1^*(s))\right]ds} = 1, p_2 e^{-\int_0^T \left[\theta_3(x_1^*(s))\right]ds} = 1.$

3 The bifurcation of a nontrivial periodic solution

First, we shall denote by $\Phi(t; U_0)$ the solution of the pulses-free system corresponding to system (1) with the initial data $U_0 = (u_0^1, u_0^2)$; also $\Phi = (\Phi_1, \Phi_2)$. We also define two maps $I_1, I_2 : R^2 \to R^2$ by

$$I_1(x_1, x_2) = (x_1, p_2 x_2),\ I_2(x_1, x_2) = (p_1 x_1, x_2).$$

and the map $F : R^2 \to R^2$ by

$$\begin{aligned} F(x_1, x_2) \quad = \quad & \Big(a_1 x_1(t)\big[b_1(x_1(t)) + h_{11} f(x_1(t)) + h_{12} f(x_2(t)) + I_1\big], \\ & a_2 x_2(t)\big[b_2(x_2(t)) + h_{21} f(x_1(t)) + h_{22} f(x_2(t)) + I_2\big]\Big). \end{aligned}$$

Next, we shall reduce the problem of finding a periodic solution of (1) to a certain fixed point problem. To this purpose, define $\Psi : [0, \infty) \times R^2 \to R^2$ by

$$\Psi(t, U_0) = I_2(\Phi((1 - \widetilde{l})T; I_1(\Phi(\widetilde{l}T, U_0))));$$

also

$$\Psi(t, U_0) = (\Psi_1(t, U_0), \Psi_2(t, U_0)).$$

Then U is a T-periodic solution of system (1) if and only if its initial data $U(0) = U_0$ is a fixed point for the operator Ψ. One easily obtains that

$$D_X \Psi(T, X_0) = \begin{pmatrix} d_{11} & d_{12} \\ 0 & d_{22} \end{pmatrix},$$

in which

$$\begin{aligned} d_{11} &= p_1 e^{\int_0^t \left[\theta_1(x_1^*(s))\right] ds}; \\ d_{12} &= p_1 \bigg(p_2 \int_{\widetilde{l}T}^{T} \theta_2(x_1^*(s)) e^{\int_0^s \theta_3(x_1^*(\xi))d\xi + \int_s^T \theta_1(x_1^*(\xi))d\xi} ds + \\ & \int_0^{\widetilde{l}T} \theta_2(x_1^*(s)) e^{\int_0^s \theta_3(x_1^*(\xi))d\xi + \int_s^T \theta_1(x_1^*(\xi))d\xi} ds \bigg); \\ d_{22} &= p_2 e^{\int_0^t \left[\theta_3(x_1^*(s))\right] ds}. \end{aligned}$$

To find a nontrivial periodic solution of period τ with initial data X, we need to solve the fixed point problem $X = \Psi(\tau, X)$, denoting $\tau = T + \overline{\tau}$ and $X = X_0 + \overline{X}$, that is,

$$X_0 + \overline{X} = \Psi(T + \overline{\tau}, X_0 + \overline{X}).$$

Let

$$N(\overline{\tau}, \overline{X}) = X_0 + \overline{X} - \Psi(T + \overline{\tau}, X_0 + \overline{X}) \tag{18}$$

and

$$N(\overline{\tau}, \overline{X}) = (N_1(\overline{\tau}, \overline{X}), N_2(\overline{\tau}, \overline{X})).$$

We are then led to solve the equation $N(\overline{\tau}, \overline{X}) = 0$. One notes that

$$D_X N(0, (0,0)) = E_2 - D_X \Psi(T, X_0) = \begin{pmatrix} 1 - d_{11} & -d_{12} \\ 0 & 1 - d_{22} \end{pmatrix} = \begin{pmatrix} a_0' & b_0' \\ 0 & d_0' \end{pmatrix}. \tag{19}$$

A necessary condition for the bifurcation of the nontrivial periodic solutions near the trivial periodic solution $(x_1^*(t), 0)$ is

$$\det\left[D_X N(0, (0,0))\right] = 0.$$

We first consider **Case 1**. It is easily seen that

$$\dim(Ker[D_X(0, (0,0))]) = 1,$$

and a basis in $Ker[D_X(0, (0,0))]$ is $(-b_0'/a_0', 1)$. Then the equation $N(\overline{\tau}, \overline{X}) = 0$ is equivalent to

$$\begin{cases} N_1(\overline{\tau}, \alpha Y_0 + z E_0) = 0, \\ N_2(\overline{\tau}, \alpha Y_0 + z E_0) = 0, \end{cases}$$

where

$$E_0 = (1, 0),\ Y_0 = (-b_0'/a_0', 1), \tag{20}$$

and $\overline{X} = \alpha Y_0 + z E_0 = (\alpha(-b_0'/a_0') + z, \alpha)$ represents the the direct sum decomposition of X using the projections onto $Ker[D_X N(0, (0,0))]$ and $Im[D_X N(0, (0,0))]$ ([16]).

Let

$$f_1(\overline{\tau}, \alpha, z) = N_1(\overline{\tau}, \alpha Y_0 + z E_0), \tag{21}$$

$$f_2(\overline{\tau}, \alpha, z) = N_2(\overline{\tau}, \alpha Y_0 + z E_0). \tag{22}$$

We need now solve the following system

$$\begin{cases} f_1(\overline{\tau}, \alpha, z) = 0, \\ f_2(\overline{\tau}, \alpha, z) = 0. \end{cases}$$

Since

$$\frac{\partial f_1}{\partial z}(0,0,0) = \frac{\partial N_1}{\partial x_1}(0, (0,0)) = a_0' \neq 0,$$

by applying the implicit function theorem, one may locally solve the equation $f_1(\overline{\tau}, \alpha, z) = 0$ near $(0, 0, 0)$ with respect to z as a function of $\overline{\tau}$ and α and find $z = z(\overline{\tau}, \alpha)$ such that $z(0, 0) = 0$ and

$$f_1(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)) = N_1(\overline{\tau}, \alpha Y_0 + z(\overline{\tau}, \alpha)E_0) = 0.$$

Derivating the implicit function above with respect to α at $(0, 0)$, we may then deduce that

$$\frac{\partial N_1}{\partial x_1}(0,(0,0))(\frac{\partial x_1}{\partial \alpha}(0,0) + \frac{\partial x_1}{\partial z}\frac{\partial z}{\partial \alpha}(0,0)) + \frac{\partial N_1}{\partial x_2}(0,(0,0))\frac{\partial x_2}{\partial \alpha}(0,0) = 0.$$

It follows from (19) that

$$\frac{\partial z}{\partial \alpha}(0,0) = 0. \tag{23}$$

In view of (18), we get that

$$\frac{\partial z}{\partial \tau}(0,0) = \frac{p_1}{a_0'}\big[(1-\widetilde{l})x_1^{*'}(T) + \widetilde{l}e^{\int_{\widetilde{l}T}^{T}\left[\theta_1(x_1^*(s))\right]ds}x_1^{*'}(\widetilde{l}T)\big].$$

It now remains to study the solvability of the equation

$$f_2(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)) = N_2(\overline{\tau}, \alpha Y_0 + z(\overline{\tau}, \alpha)E_0) = 0. \tag{24}$$

The equation (24) is called the determining equation and the number of its solutions equals the number of periodic solutions of (24) ([16]). In the following we shall proceed to solving (24) by using Taylor expansions. We denote

$$f(\overline{\tau}, \alpha) = f_2(\overline{\tau}, \alpha, z(\overline{\tau}, \alpha)). \tag{25}$$

First, we observe that

$$f(0, 0) = N_2(0,(0, 0)) = 0.$$

Second, we focus on the first order partial derivatives of f at $(0, 0)$. By (18) and (24), together with [19], it is easily seen that

$$\begin{aligned} \frac{\partial f}{\partial \alpha}(0,0) &= 1 - p_2\frac{\partial \Phi_2}{\partial x_2}\big((1-\widetilde{l})T; I_1(\Phi(\widetilde{l}T;X_0))\big)\frac{\partial \Phi_2}{\partial x_2}(\widetilde{l}T;X_0) \\ &= d_0' = 0. \end{aligned} \tag{26}$$

Since (23) holds and

$$\frac{\partial \Phi_2}{\partial x_1}((1-\widetilde{l})T; I_1(\Phi(\widetilde{l}T;X_0))) = 0, \tag{27}$$

$$\frac{\partial \Phi_2}{\partial \overline{\tau}}((1-\widetilde{l})T; I_1(\Phi(\widetilde{l}T;X_0))) = 0, \tag{28}$$

It easily follows that

$$\frac{\partial f}{\partial \overline{\tau}}(0,0) = 0. \tag{29}$$

Third, we compute the second order partial derivatives $\frac{\partial^2 f}{\partial \alpha^2}$, $\frac{\partial^2 f}{\partial \overline{\tau}^2}$ and $\frac{\partial^2 f}{\partial \alpha \partial \overline{\tau}}$.
After a few computations, we only determine that

$$\frac{\partial^2 f}{\partial \overline{\tau}^2} = 0, \tag{30}$$

while the sign of $\frac{\partial^2 f}{\partial \alpha^2}$ and $\frac{\partial^2 f}{\partial \alpha \partial \overline{\tau}}$ can not be determined. By constructing the second order Taylor expansion of f near $(0, 0)$, one obtains from (26)-(30) that

$$f(\overline{\tau}, \alpha) = \underbrace{\frac{\partial^2 f}{\partial \alpha \partial \overline{\tau}}(0,0)}_{\doteq A}\alpha\overline{\tau} + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial \alpha^2}(0,0)}_{\doteq B}\alpha^2 + o(\overline{\tau}, \alpha)(\overline{\tau}^2 + \alpha^2).$$

Let $\alpha = k\overline{\tau};\ k = k(\overline{\tau})$. It follows from the equation above that

$$f(\overline{\tau}) = \overline{\tau}^2\big[Ak + \frac{1}{2}Bk^2 + o(\overline{\tau}, k\overline{\tau})(1 + k^2)\big].$$

In conclusion, the above analysis may be summarized in the following result.

Theorem 3.1 *Assume that conditions* (H_1), (H_2), *(9) and* ***Case 1*** $(p_1 e^{-\int_0^T \left[\theta_1(x_1^*(s))\right] ds} \neq 1)$ *hold. Then system (1) has a supercritical bifurcation of nontrivial solutions if* $AB < 0$, *and if* $AB > 0$ *it has a subcritical case. If* $AB = 0$, *it has an undetermined case.*

Remark 3.1 *The final part of the existence argument can also be obtained by using the substitution* $\overline{\tau} = k\alpha(or - k\alpha);\ k = k(\alpha)$.

Remark 3.2 *Assume that conditions* (H_1), (H_2), *(9) and* ***Case 2*** *(or* ***Case 3****) hold. We only replace (20) by* $E_0 = (0, 1)$ *and* $Y_0 = (1, 0)$. *The remaining details of analysis are similar to those of the previous theorem and hence are omitted to avoid repetition.*

4 Conclusion

In this paper we proposed and analysed a Cohen–Grossberg neural network composed of two neurons with nonisochronous impulsive effects. By using Mawhin's continuation theorem, we first obtained the sufficient conditions for the existence of semi-trivial periodic solutions. Subsequently, the asymptotic stability of semi-trivial periodic solutions was investigated. Finally, we extended the method in [17] and then studied the bifurcation of nontrivial periodic solutions.

Acknowledgements

The work was supported by the high-level talent project of Jiangsu University(08JDG047).

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