A Bayesian way to solve the pdf selection problem: an application in geotechnical analysis

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Single probability density functions (pdf) are often selected based on a best-fit approach or the minimum description length (MDL) theorem that finds its description by the theory of complexity by Kolmogorov. However, it often occurs that random data cannot be accurately described by any of the commonly used pdf's. In this paper a method is presented which solves the problem of selecting single pdf's by predicting the probability of contending pdf's true given the data. These probabilities are combined when drawing inferences. The new method is illustrated on the prediction of the ultimate limit state (bearing capacity) of a single foundation pile. The Bayesian statistical method for combining information on pile capacity with the results of full-scale tests has been applied to establish the probability of contending pdf's of the model uncertainty. The results obtained by the Level II method have been used to obtain partial factors. The application of the new method to a single pile case is shown to be successful and has indicated a number of possible further applications.

Key words: Bayesian analysis, multiparameter probability density function, pile foundation

1 Introduction

A number of researchers has been applying Bayes' theorem in civil engineering problems. For instance, Rackwitz [1] discusses the Bayesian approach to the spatial variation of soils. An attempt has been made to quantify prior information for the Bayesian updating process. Here single pdf's have been used. Franchin et al. [2] introduces a model correction factor method in order to push the idealized limit state function into the actual limit state function. A single pdf has been used. Random fields are defined for the cohesion along a single pile and for pile groups. Igusa et al [3] has also introduced the notion of model uncertainty. They use single Bayesian conjugate pdf's. Zong et al [4] introduces the concept of a single pdf not true given the data. Note that in the current paper this notion has been extended to multiple pdf's. Hansen et al [5] deals with the problem of single pdf selection based on MDL approach.

The aim of this paper is demonstrating the application of the Bayesian statistical method in combining statistical properties based of multiple pdf's. Waarts et al [6] have introduced the notion of the probability of a distribution being true given the data. However they have not attempted to combine these probabilities when drawing inferences.

The present paper is outlined as follows. First the application of Bayes' theorem will be discussed to solve the problem of single pdf selection, i.e. the probabilities of contending pdf's being true given the data. Then the individual probabilities of the contending pfd's will be combined to one multiple parameter pdf. Thereafter, the new method is illustrated by an example where three contending pdf's have been combined for the model factor of axial pile capacities, i.e. the ratio of measured to calculated values. Subsequently, Level II analyses are carried out using the multiple parameter pdf for the model uncertainty. These analyses yield results that are important for practical use, such as the assessment of the partial safety factors and the establishments of the cumulative distribution function for axial pile capacity. Finally, concluding remarks and directions for further research are given.

2 Bayesian way to evaluate probabilities of contending pfd's being true given the data

2.1 ***Introduction***

The fundamental question of this chapter is what are the probabilities of a number of contending distributions being true given the data *D*? For a single pdf the approach of Zong et al [4] can be adopted. Then the question may arise how to combine these single pdf's being true given the data? To answer this fundamental question first the well-known Bayesian approach will be discussed.

The Bayesian procedure consists mainly of the following steps [7]: The first step is to assign prior probabilities for contending distributions and associated parameters. In general there are three ways of obtaining prior probabilities based on:

(i) professional opinion,

(ii) prior 'ignorance',

(iii) previous Bayesian analysis using earlier data.

2.2 *General*

The essential idea underlying Bayesian statistics is that any uncertain quantity can be treated as if it were random. The uncertain quantity does not have to be of the "physically" random type that can be studied by carrying out a sufficient large number of experiments. Accordingly, if one adopts a Bayesian viewpoint, statistical uncertainties can be treated by assigning probabilities to various sets of parameters (parameter vectors) in pdf's. Furthermore, modelling uncertainties can be handled in a similar manner by assigning probabilities to various contending pdf's. As new data (information) becomes available, these parameter and pdf probabilities can be updated using Bayes' theorem. In Kerstens et al [8] some of the principles of the following procedure have been given.

2.3 *Pdf's and parameter sets*

Let a finite number of types of pdf's be postulated to describe the modelling uncertainty α being a random variable of the measured to calculated value. The pdf's are denoted by $f_j, j = 1,..,m$. Associated with the general pdf f_j is a parameter set θ_j, e.g. for a normal distribution $\theta_N = \{\mu, \sigma\}$. Let a particular (the i^{th}) choice of θ_j be denoted by θ_{ji}. The subscript *i* can take on a number of discrete values $i = 1,..n_j$. The symbol n_j denotes the number of different parameter sets considered in f_j. The way in which n_j is selected will be discussed later.

2.4 *Prior information*

It is assumed that, before any set of data is obtained, information is available to enable "prior" probabilities to be assigned to various models and parameter sets within each pdf f_j. Let the prior probabilities of the pdf's f_j be denoted by $\Pr[f_j]$. Let the prior probabilities of the parameter sets within each pdf f_j be denoted by $\Pr[\theta_{ji} | f_j]$.

To make the first estimate the prior probabilities of pdf's and parameter sets can be selected to represent prior 'ignorance':

$$\sum_{i=1}^{m} \Pr[f_j] = \frac{1}{m} \tag{1}$$

$$\sum_{i=1}^{n_j} \Pr[\theta_{ji}] = \frac{1}{n_j} \tag{2}$$

2.5 *Updating (revising) prior probabilities*

As new data become available, Bayes' theorem allows the prior probabilities of pdf's f_j and parameter vectors θ_{ji} to be revised in the light of new data to give "posterior" probabilities. Let H_{ji} denote the hypothesis that a particular pdf f_j and a particular set θ_{ji} within this pdf are correct. The prior probability of H_{ji} is given by

$$\Pr[H_{ji}] = \Pr[f_j].\Pr[\theta_{ji} \mid f_j] \tag{3}$$

Bayes' theorem then allows $\Pr[H_{ji}]$ to be revised in accordance with any new data D according to

$$\Pr[H_{ji} \mid D] = \frac{1}{C_1}.\Pr[D \mid H_{ji}].\Pr[H_{ji}] \tag{4}$$

where C_1 is a normalizing constant

$$C_1 = \sum_{H_{ji}} \Pr[D \mid H_{ji}].\Pr[H_{ji}]$$

Substituting equation (3) into equation (4) leads to:

$$\Pr[H_{ji} \mid D] = \frac{1}{C_2}.\Pr[D \mid H_{ji}].\Pr[f_j].\Pr[\theta_{ji} \mid f_j] \tag{5}$$

where C_2 is a normalizing constant

$$C_2 = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \Pr[D \mid H_{ji}].\Pr[f_j].\Pr[\theta_{ji} \mid f_j]$$

In equation (5) $\Pr[D \mid H_{ji}]$ is known as the likelihood function of the data set D. It expresses the probability of observing D conditional upon pdf f_j and θ_{ji}. Henceforth for brevity it will be denoted by L_{ji}. Suppose that we seek to select the pdf's for random variable α. Further, let the number of observations p for α be denoted by α_k, k = 1,.., p and the pdf of α based on pdf f_j and parameter set θ_{ji} by $f_j(\alpha_k \mid \theta_{ji})$, then the probability of observing the single data in data D is

$$L_{ij} = \prod_{k=1}^{p} f_j(\alpha_k \mid \theta_{ji}) \tag{6}$$

After some re-arrangement equation (5) becomes

$$\Pr[H_{ji} \mid D] = \frac{1}{C_3}.\Pr[f_j].L_{ji}.\Pr[\theta_{ji} \mid f_j] \tag{7}$$

where C_3 is a normalizing constant

$$C_3 = \sum_{j=1}^{m} \Pr[f_j].\sum_{i=1}^{n_j} L_{ji}.\Pr[\theta_{ji} \mid f_j]$$

The posterior probability that pdf f_j is correct is obtained from equation (7) by summing over all possible parameter vectors for that pdf

$$\Pr[f_j \mid D] = \frac{1}{C_3}.\Pr[f_j].\sum_{i=1}^{n_j} L_{ji}.\Pr[\theta_{ji} \mid f_j] \tag{8}$$

The posterior probability that parameter vector θ_{ji} in pdf f_j is correct is given by dividing the right-hand side of equation (7) by the right-hand side of equation (8)

$$\Pr[\theta_{ji} \mid D] = \frac{1}{C_4}.L_{ji}.\Pr[\theta_{ji} \mid f_j] \tag{9}$$

where C_4 is a normalizing constant

$$C_4 = \sum_{i=1}^{n_j} L_{ji} \Pr[\theta_{ji} \mid f_j]$$

Equation (9) allows a discrete joint probability mass function (pmf) of the parameters of each model to be developed. These are also used to assist in the practical application of the Bayesian analysis procedure, because the ranges for the parameters and the required degree of discretization (grid size; n_j is equal to the number of grid points) for performing the summation have to be established by trial and error. With the pmf it is possible to check whether the selected ranges and grid size are appropriate, see as examples Figures 1 and 2. Initially, the centre points of the ranges can be estimated using point estimates of the parameters and the ranges should cover parameter ranges for which the posterior function, see Equation (9), takes on significant values. From these figures it can be seen that when taking point estimates considerable information is lost. Further, applying the Bayesian updating scheme in view of new data the spread of these distributions will generally become increasingly narrow.

2.6 Special measures for Bayesian processing of large data sets

Central to the calculations associated with the Bayesian techniques is the evaluation of the likelihood function L_{ji}, see equation (6). When large data sets are being dealt with, the values p can be relatively large the product Π may lead to figures smaller than a computer can handle. To overcome this it is necessary to implement a scaling procedure. This is accomplished by applying a logarithmical transformation of L_{ji} in the form of

$\ln = [L_{ji}] = \ln[\prod_{k=1}^{p} f_j(\alpha_k|\theta_{ji})]$ or $\ln[L_{ji}] = \sum_{k=1}^{p} \ln[f_j(\alpha_k|\theta_{ji})]$ which reduces the product of very small numbers in the addition of large negative values. The scaling factor SF is chosen such that $\{\max \ln[L_{ji}]\} + SF = 0$. Therefore $SF = -$ maximum value. This effectively means that the scaling operation has to be performed for every parameter evaluation i and pdf j. Hereafter the back transformation e^x needs to be performed.

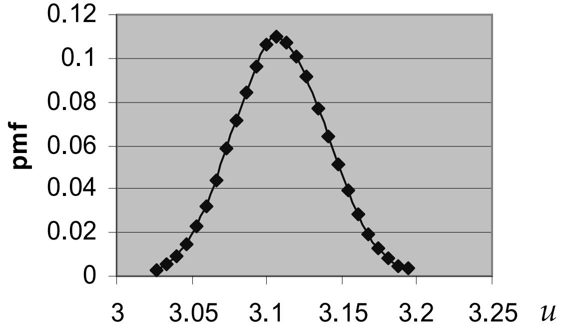

Figure 1: Discretisation parameter u, Weibull, the dots show the sampling density

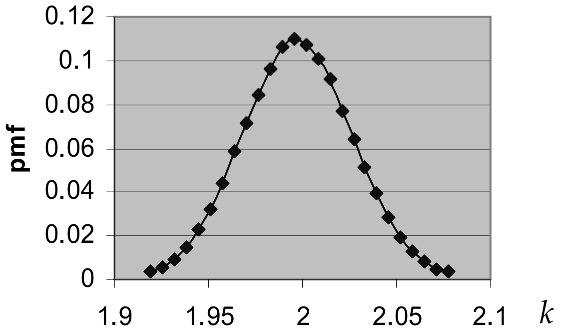

Figure 2: Discretisation parameter k, Weibull, the dots show the sampling density

2.7 *The development of a multi parameter pdf*

Because a single pdf will not fit the data D sufficiently accurately, the application of the Bayesian method in combining contending pdf's will be discussed here. Let there be a limit state function $Z = \alpha R - S$ where α is the model uncertainty for the resistance R and S the loading. The multi parameter pdf of α is as follow

$$f(\alpha) = \sum_{j=1} \Pr[f_j | D] f_j(\alpha) \tag{10}$$

The factors $\Pr[f_j | D]$ represent a degree of belief in f_j being correct in view of the data D. these parameters are estimated using the Bayesian method as explained in the preceding section.

3 Illustrative example: limit state reliability analysis of axial pile capacity

3.1 ***Introduction***

The capacity Q_s of the foundation pile is equal to the total friction along the pile:

$$Q_s = \pi d \int_0^l f_s(z) dz \tag{11}$$

Where d is the pile diameter, l is the length of the pile and $f_s(z)$ is the frictional stress at a distance z from the surface, see Figure 3. Many models exist for the frictional stress; in this paper the total stress method will be considered according to API-RP2A [9].

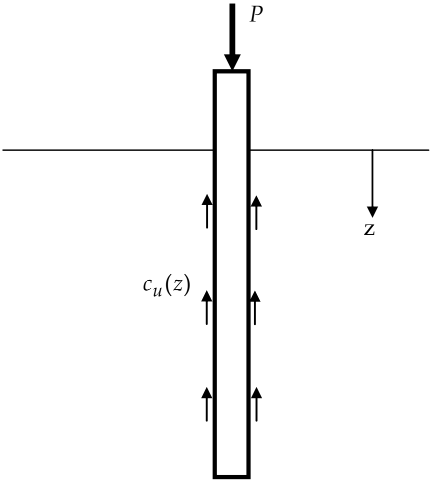

Figure 3: Axially loaded pile in clay

3.2 *The total stress method*

This method is based on the friction being proportional to the undrained shear strength $c_u(z)$:

$$f_s(z) = \alpha c_u(z), \tag{12}$$

where α is a model factor, i.e. the ratio of measured to calculated values, which is independent of the depth. The shear strength $c_u(z)$ can be written as $c_u(z) = c_{u0} + kz$, where *k* is the gradient of $c_u(z)$, see Figure 4. Combining equations (11) and (12) gives

$$Q_s = \alpha \pi d (c_{u0} l + \tfrac{1}{2} k l^2) \tag{13}$$

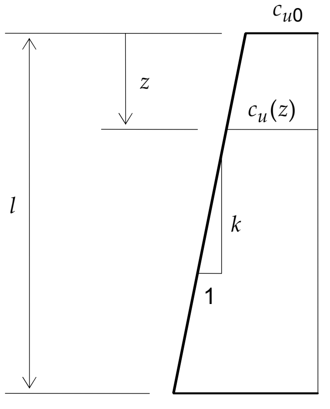

Figure 4: Shear strength distribution

3.3 *Types of uncertainty in the geostatical model*

In equation (13) the variables are known with acceptable accuracy. The uncertainties are:

1. the undrained shear strength at the top of the pile c_{u0} is not known exactly because of various measurement errors in the testing procedure,
2. the variable *k*, the mean strength increase, is not known exactly because of measurement errors and the limited number of measurements,
3. the ratio α is associated with an imperfect model resulting in model uncertainty.

The uncertainties in c_{u0} and *k* are the result of an imperfect knowledge. Repeated correct measurements of the variables will reduce the scatter of the results. The model factor α can only be reduced by an improved geostatical model.

3.4 *Axial ultimate limit state reliability analysis*

Based on the distribution of the random load P and the pile capacity or pile resistance derived earlier, the following limit state function can be defined:

$$g(Z) = g(\alpha, c_{u0}, k) = Q_s - P = \alpha\pi d(c_{u0}l + \tfrac{1}{2}kl^2) - P \tag{14}$$

It is noted that the model factor α is distributed according to equation (10). The g-function is formulated in such a way that g < 0 corresponds to failure, while g > 0 is associated with a safe situation, see Figure 5. The surface corresponding to g = 0 is called the failure surface.

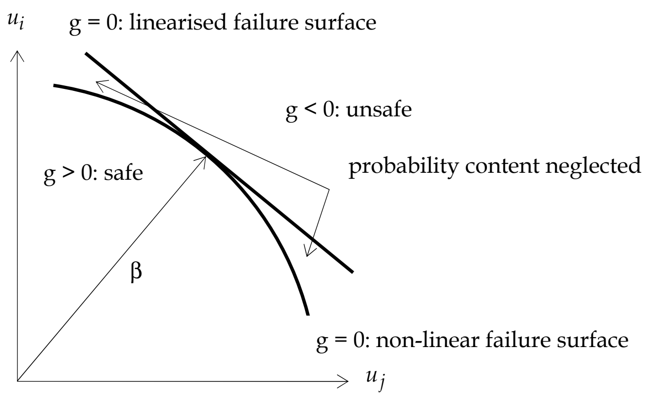

Figure 5: Level II analysis

In Table 1 the basic variables $Z(\alpha, c_{u0}, k)$ are given. They are assumed to be statistically independent. The probability that the pile strength Q_s is smaller than the applied load P, is:

$$\Pr[Q_s < P] = \Pr[Z < 0] \tag{15}$$

In most cases the calculation of the probability of failure by numerical integration or by Monte-Carlo simulation is too time-consuming, and as an alternative the Level II technique is applied. A transformation into a standardized normal space, the u-space, is carried out. The probability of failure given by equation (15) is illustrated in Figure 5; the probability of failure can often be sufficiently accurately approximated by the probability content in the half-space outside the tangent hyper plane. The plane is the tangent hyper plane at the 'design point' $\{\overline{u}\}$, which is the point at the failure surface closest to the origin and thus the point with the highest probability density. The design point therefore yields the values of the basic variables for the maximum probability of failure. The probability in the half-space outside the approximating hyper plane is simply $\phi(-\beta)$, where $\beta = \|\overline{u}\|$ is the reliability index [10]. Owing to the transformation into the u-space, the design point $\{\overline{u}\}$ is related to the point $\{\overline{z}\}$ in the original space of the basic variables. A set of partial factors γ_i, associated with reliability level β, are then obtained as treated in [10]

$$y_i = \frac{\overline{z}_{ic}}{z_i} \tag{16}$$

where $\overline{z}_{ic}$ is a representative value of z; e.g. the mean value or the 5% fractile.

3.5 ***Modelfactor***

The uncertainties in the prediction methods can be grouped into physical, statistical, measurement and model uncertainties. Uncertainties in the theoretical relationship used to describe the behaviour of a (sub) system should, if possible, be assessed by tests on a full-scale (sub) system or representative models thereof. This is needed because even in the best theoretical relationship derived to model a particular limit state, various effects are likely to be neglected either because they are not known or are difficult to take into account. This implies that errors will be present in the theoretical model so that α (= actual behaviour/predicted behaviour) tends to vary in a random way. The ratio α = Q_m/Q_c between measured and calculated pile capacity or model factor has been determined in [11] from about one thousand tests on driven piles in clay presented in the form of a histogram. The data indicate that the APIC formula is not biased, i.e. the mean value of the ratio Q_m/Q_c is approximately one. As expected, a considerable scatter exists, see Figure 6 for an indication of this scatter. As an example the calculated capacities Q_c using the APIC formula and the measured capacities Q_m are taken, see Figure 6. For the other contending pdf's the normal and Weibull distributions are selected. Using the Bayesian approach the following results are obtained, see equation (8):

$$\begin{aligned} \Pr[N|D] &= 0.150 \\ \Pr[LN|D] &= 0.685 \\ \Pr[W|D] &= 0.165 \end{aligned} \tag{17}$$

From the results given in equation (17), it can be concluded that the lognormal pdf alone will not fit the data of Figure 6 accurately. Therefore, equation (10) can be written as follows

$$f(\alpha) = 0.150 f_N(\alpha) + 0.685 f_{LN}(\alpha) + 0.165 f_W(\alpha) \quad (18)$$

Complications have not been encountered when performing Level II analyses as these analyses allow limit state functions to be combinations of linear and non-linear functions for various types of pfd's.

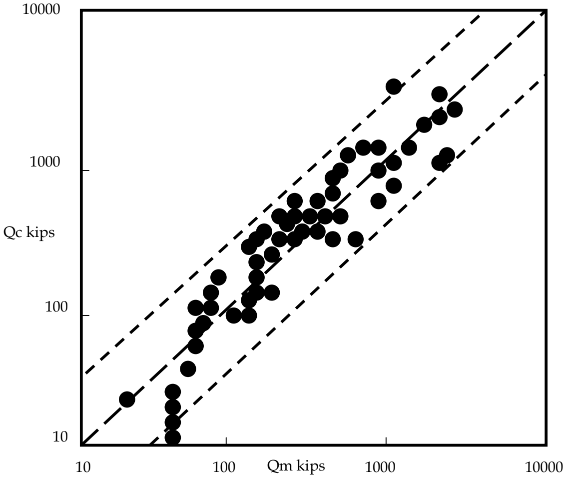

Figure 6: $Q_m - Q_c$ *data using APIC code (1 kip = 6.985 N/mm²)*

3.6 **Results from Level II reliability analysis**

Using individual contending pfd's for the gradient k and the undrained shear strength at the surface c_{u0} and a linear combination of contending pdf's for the model factor α Level II analyses have been carried out to illustrate the feasibility of this consistent Bayesian approach and the differences in the results obtained with the new method and the classical method. The values of the soil parameters are given in Table 1. The median values (β = 0) for the pile capacity are 6.50 MN, 5.40 MN and 6.70 MN for a normal, lognormal and Weibull pdf for the model uncertainty, respectively (pile: d = 0.75 m and l = 75 m). Note that for low probabilities of failure large differences are observed between the results obtained with a normal, lognormal and Weibull distribution for the model uncertainty. These differences are due to the well known 'tail problem'. This observation emphasizes the need for a multiple parameter pdf for the model factor α.

For various values of P the corresponding β for the three contending distributions are evaluated using the Level II method (AFDA). Also, based on equations (14) and (18) the reliability index β for the multi-parameter pdf $f(\alpha)$ for various values of P have been obtained.

Table 1: Definition of soil parameter properties

Description	Symbol	Mean Value	Standard deviation	Distribution
model uncertainty	α	to be determined		multi-parameter pdf
shear strength at z = 0	c_{u0}	10 kPa	10 kPa	lognormal
shear strength gradient	k	1.0 kPa/m	0.2 kPa/m	lognormal

Table 2: Results level II analyses: reliability index β for individual and multi-parameter pdf's

P MN	β_N	β_{LN}	β_W	β
1.0	2.91	4.68	3.12	3.42
2.0	2.35	2.97	2.28	2.57
3.0	1.79	1.80	1.67	1.73
4.0	1.25	0.91	1.15	0.91
5.0	0.72	0.20	0.69	0.32
6.0	0.22	-0.30	0.26	-0.11

3.7 *Calculation of partial safety factors*

Let the characteristic values of the multiple parameter distributed α be α_c , C and k be the mean value, 5% fractile and the mean value respectively or

$$(\alpha_c, c_{u0,c}, k_c) = (E[\alpha], c_{u0,0.05}, E[k]) = (0.912,\ 1.667\text{ kPa},\ 1.0\text{ kPa/m}) \tag{19}$$

Note that $E[\alpha]$ can be obtained by equation (18) because this equation is the linear combination of the random variables whose expected values are known. The design point for P = 2.0 MN, corresponding to β = 2.57 or $\Pr[g(Z) < 0] \approx 5 \cdot 10^{-3}$ is (see also Table 3):

$$(\alpha_c, \bar{c}_{u0}, \bar{k}) = (0.3406,\ 2.431\text{ kPa},\ 0.761\text{ kPa/m}) \tag{20}$$

Corresponding to β = 2.57 (corresponding to a $\Pr[g < 0] \approx 5 \cdot 10^{-3}$) the following set of equivalent partial safety factors is obtained:

$$\gamma_\alpha = \frac{\alpha_c}{\overline{\alpha}} = 0.912 / 0.3406 = 2.68 \tag{21}$$

$$\gamma_{c_{u0}} = \frac{c_{u0,0.05}}{\overline{c}_{u0}} = 1.667 / 2.431 = 0.69 \tag{22}$$

$$\gamma_k = \frac{k_c}{\overline{k}} = 1.0 / 0.761 = 1.31 \tag{23}$$

Note that multiplying the partial safety factors with the values of the associated variables and substituting these values in equation (13) gives a pile capacity Q_s , such that P = 2.0 MN associated with $\hat{\beta}$ = 2.57.

Table 3: Combined design point values for P = 2.0 MN, β = 2.57

Variable	Design point value
$\overline{\alpha}$	0.3406
$\overline{C}u0$	2.431
$\overline{k}$	0.7610

3.8 *Cumulative distribution of axial capacity for a single pile*

The probability of failure for a single pile is defined as the probability that the pile resistance Q_s is smaller than the applied axial load $P : \Pr[Q_s < P]$, see equation (15). In the present case the applied load is taken to be deterministic, so that the above definition can also be interpreted in terms of the probability of exceedance for a given resistance.
In this way the cumulative distribution function (cdf) in terms of β for Q_s can be constructed by replacing the load P by its exceeded resistance Q_s . Figure 7 shows the combined curve obtained with the Bayesian method (see previous section). With the aid of this figure a number of inferences can be drawn, e.g. the 5% fractile in the pdf for Q_s is 2.8 MN (i.e. there is a 5% probability of failure for an axial load of 2.8 MN).

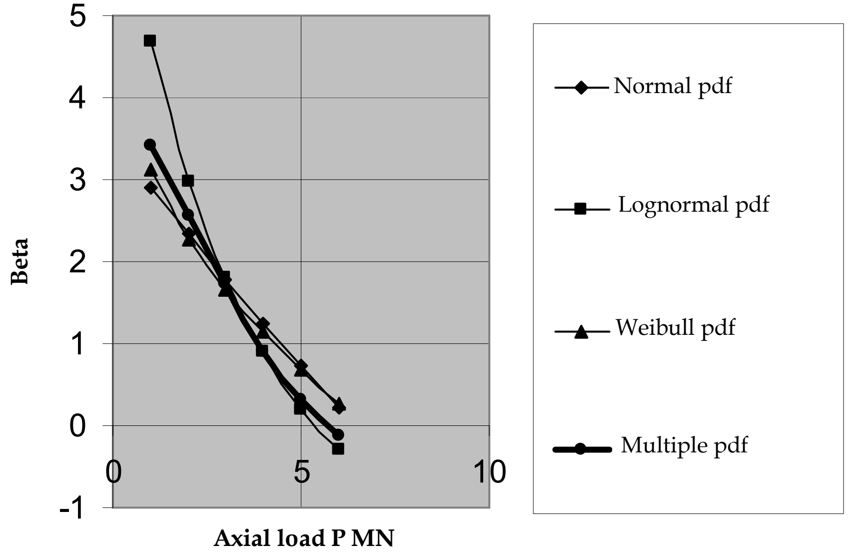

Figure 7: Cumulative probability of failure

4 Concluding remarks

The new method proposed to solve the problem of selecting pdf's in Level II analyses appears to be successful. This method is based on a linear combination of the probabilities of the contending pfs's being true given the data and the contending pdf's of the stochastic variable in question, i.e. the multiple parameter pdf. The probabilities of contending pdf's being true given the data of the stochastic variable in question are evaluated using the Bayesian approach. However, when large data sets are processed special measures have to be taken to avoid numerical overflow. The estimation of reliability indices for the case considered showed that the combined reliability index in the range between 3.0 and 5.0 is considerably lower than the reliability index based on a single best fitted distribution (i.e. lognormal), see Figure 7. This shows that inferences drawn on single best-fit pdf may be potentially dangerous. In this paper a method is proposed to solve the problem of selecting pdf's in Level II analyses. When new data becomes available the established inferences could be updated in a similar manner as proposed in this paper. It is pointed out that the new method may also be applied to other problems and multiple stochastic variables.

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Nomenclature

c_{u0}	shear strength at $z = 0$
$c_u(z)$	undrained shear strength as function of z
d	pile diameter
$f(.)$	pdf
$f_j(.)$	pdf for $j = N$ or $j = LN$ or $j = W$
$f_s(z)$	shear strength as function of z
Z	reliability function
k	gradient of shear strength
l	length of a pile
m	number of models (pdf,s)
u_i	normal variate of z_i
n_j	number of discrete points for parameter evaluation
p	number of data
z	soil depth variable measured from sea bed
z_i	basic variable in $g(Z)$
cdf	cumulative distribution function
C_i	normalising constant i
D	data set
$F(.)$	cdf
H_{ji}	hypothesis pdf j given parameter vector i is correct
LN	log-normal
L_{ji}	likelihood function of pdf j and parameter i
N	normal
pmf	probability mass function
P	axial pile load
R	stochastic resistance
S	stochastic load
$\Pr[.]$	probability
Q_c	calculated pile capacity

Q_m	measured pile capacity
Q_s	shear capacity or pile resistance
W	Weibull
Z	failure function
α	model uncertainty
α_k	k^{th} sample of model uncertainty
α_j	model uncertainty distributed according pdf j for $j = N$, $j = LN$ and $j = W$
β	reliability index
γ_i	partial safety factor basic variable i
θ_{ji}	parameter set model j parameter i
$\phi(.)$	normal pdf
$(\bar{.})$	design point value
$\{.\}$	set of variables
$.c$	characteristic value