Cut-off point of linear discriminant rule for large dimension

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Abstract

This paper is concerned with the problem of classifying a observation vector into one of two populations $\Pi_1 : N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $\Pi_2 : N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$. Anderson (1973, Ann. Statist.) gave an asymptotic expansion of the studentized statistic, and derived cut-off point to achieve a specified probability of misclassification. But the dimension p gets large, the precision becomes worse. So in this paper, we proposed studentized statistic in terms of (n, p) asymptotic. An asymptotic expansion of the statistic is derived up to the order O_1, where O_1 is a term with respect to $\{p^{-1/2}, N_1^{-1/2}, N_2^{-1/2}, m^{-1/2}\}$ for each sample size N_i and $m = N_1 + N_2 - 2 - p$. Using the expansion, we gave cut-off point to achieve a specified probability of misclassification.
Keywords: Linear discriminant rule, cut-off point, (n, p) asymptotic
AMS 2010 subject classification: Primary 62H30, Secondary 62E20

1 Introduction

This paper is concerned with the problem of classifying a observation vector into one of two populations $\Pi_1 : N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $\Pi_2 : N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$. The observation $\boldsymbol{x}$ is classified as coming from either Π_1 or Π_2 based on the samples

$$\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{iN_i} \sim N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}) \quad (i = 1, 2),$$

which are independent. For this problem, linear discriminant analysis is used. Let

$$W = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)' \boldsymbol{S}^{-1} \left\{ \boldsymbol{x} - \frac{1}{2}(\bar{\boldsymbol{x}}_1 + \bar{\boldsymbol{x}}_2) \right\},$$

where $\bar{\boldsymbol{x}}_1$, $\bar{\boldsymbol{x}}_2$ and $\boldsymbol{S}$ are the sample mean vectors and the pooled sample covariance matrix defined by

$$\bar{\boldsymbol{x}}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \boldsymbol{x}_{ij}, \quad i = 1, 2, \quad \boldsymbol{S} = \frac{1}{n} \sum_{i=1}^{2} \sum_{j=1}^{N_i} (\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)',$$
$$n = N - 2 = N_1 + N_2 - 2.$$

Linear discriminant rule classifies $\boldsymbol{x}$ as Π_1 if $W(\boldsymbol{x}) > c$ and Π_2 if $W(\boldsymbol{x}) < c$ for a constant c. Inference concerning linear discriminant analysis is studied under large sample asymptotic framework A0:

$$\text{A0} : N_1 \to \infty, \quad N_2 \to \infty, \quad N_1/N_2 \to c \in (0, \infty).$$

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For a review of results under A0, see, e.g., Fujikoshi et al. [4]. When p becomes large, accuracy of results proposed using A0 gets worse. As a way to improve the poorness, it is studied under the asymptotic framework A1:

$$\begin{aligned}\text{A1}: p\to\infty,\quad N_1\to\infty,\quad N_2\to\infty,\quad p/N_1\to\gamma_1\in(0,1),\quad p/N_2\to\gamma_2\in(0,1)\\ \text{and}\quad N_1/N_2\to\gamma\in(0,\infty).\end{aligned}$$

As results under A1, Fujikoshi and Seo [3] gave an asymptotic approximation of the probabilities of misclassification, Fujikoshi [2] gave its error bound. These results are reviewed in Fujikoshi et al. [4]. Following Lachenbruch [5], for $\boldsymbol{x}\in\Pi_i$,

$$W=(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}\left\{\boldsymbol{x}-\frac{1}{2}(\bar{\boldsymbol{x}}_1+\bar{\boldsymbol{x}}_2)\right\}=V^{1/2}Z_i+(-1)^iU_i, \tag{1}$$

where

$$\begin{aligned}V&=(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}\boldsymbol{\Sigma}\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2),\\ Z_i&=V^{-1/2}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i),\\ U_i&=(-1)^{i+1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_i-\boldsymbol{\mu}_i)-\frac{1}{2}D^2,\end{aligned}$$

and D^2 is the squared sample Maharanobius distance defined by $D^2=(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)$. From the normality of $\boldsymbol{x}$, Z_i is distributed as the standard normal distribution, which we denote it as $Z_i\sim N(0,1)$. Since it does not depend on $\{\bar{\boldsymbol{x}}_1,\bar{\boldsymbol{x}}_2,\boldsymbol{S}\}$, Z_i is independent to the set, and so Z_i is independent from $\{U_i,V\}$. The limiting distribution of W under the asymptotic framework A1 is normal with mean $u_{i,0}=(-1)^i\lim_{\text{A1}}E[U_i]$ and variance $v_0=\lim_{\text{A1}}\{E[V]+\text{Var}(U_i)\}$ if $\boldsymbol{x}\in N_p(\boldsymbol{\mu}_i,\boldsymbol{\Sigma})$. Under the assumption that the Mahalanobis distance $\Delta=\sqrt{(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)}$ converges to a positive constant under A1, Fujikoshi and Seo [3] and Fujikoshi [2] showed that $\text{Var}(U_i)\to 0$. So, we can abbreviate as $v_0=\lim_{\text{A1}}E[V]$.

One may want to determine the cut-off point c to adjust the probabilities of misclassification. Results under A0 is written in Anderson [1]. On the other hand, from Fujikoshi [2], under the assumption that $\boldsymbol{x}\in\Pi_i$, the limiting distribution $(W-u_i)/\sqrt{v}$ is $N(0,1)$, where u_i and v are constants such that $\lim_{\text{A1}}(u_i-u_{i,0})=0$ and $\lim_{\text{A1}}(v-v_0)=0$. Using this result, we find that the approximation of the misclassification probability that $\boldsymbol{x}$ is allocated to Π_j even though $\boldsymbol{x}\in\Pi_i$ is given as $\Phi((-1)^{i-1}(c-u_i)/\sqrt{v})$ for $i,j=1,2$ with $i\neq j$, where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution. Since u_i and v contain Δ^2, which need to be estimated. The unbiased estimator is given as

$$\widehat{\Delta^2}=\frac{n-p-1}{n}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)-\frac{pN}{N_1N_2}.$$

Consistency under the asymptotic framework A1 holds. One can choose c from the fact that the limiting distribution of $(W-\widehat{u_i})/\sqrt{\hat{v}}$ is $N(0,1)$ if $\boldsymbol{x}\in N_p(\boldsymbol{\mu}_i,\boldsymbol{\Sigma})$, where $(\widehat{u_i},\hat{v})$ is (u_i,v) with replacing Δ^2 by $\widehat{\Delta^2}$.

This paper is organized as follows. In Section 2, we give an asymptotic expansion of the $(W-\widehat{u_i})/\sqrt{\hat{v}}$ for the case that $\boldsymbol{x}\in\Pi_i$. We propose a cut-off point which the misclassification probability becomes presetting level, asymptotically. Section 3 presents simulation results for misclassification probability. Proof of lemma and derivation of expectations are given in Appendix.

2 Asymptotic expansion under A1

For technical reason, set

$$\begin{aligned}u_i&=\frac{n}{2(m-1)}\left\{(-1)^{i+1}\Delta^2+\left(\frac{p}{N_2}-\frac{p}{N_1}\right)\right\},\\ v&=\frac{n^2(n+1)}{(m-1)(m+1)(m+2)}\left(\Delta^2+\frac{Np}{N_1N_2}\right),\end{aligned}$$

where $m = n - p$. Note that $u_i = (-1)^i E[U_i]$, but v equals $E[V]$, asymptotically, under A1. Then $\lim_{\mathrm{A1}} u_i = u_{i,0}$ and $\lim_{\mathrm{A1}} v = v_0$. Using unbiased estimator of (u_i, v), we have

$$P\left(\frac{W - \widehat{u_1}}{\sqrt{\hat{v}}} < x \middle| \boldsymbol{x} \in \Pi_1\right) = E\left[\Phi\left(\frac{\sqrt{\hat{v}}x + U_1 + \widehat{u_1}}{\sqrt{V}}\right)\right],$$
$$P\left(\frac{W - \widehat{u_2}}{\sqrt{\hat{v}}} > x \middle| \boldsymbol{x} \in \Pi_2\right) = E\left[\Phi\left(\frac{-\sqrt{\hat{v}}x + U_2 - \widehat{u_2}}{\sqrt{V}}\right)\right].$$

Let

$$\boldsymbol{u}_1 = \left(\frac{1}{N_1} + \frac{1}{N_2}\right)^{-1/2} \boldsymbol{\Sigma}^{-1/2}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2),$$
$$\boldsymbol{u}_2 = \frac{1}{\sqrt{N}} \boldsymbol{\Sigma}^{-1/2}(N_1\bar{\boldsymbol{x}}_1 + N_2\bar{\boldsymbol{x}}_2 - N_1\boldsymbol{\mu}_1 - N_2\boldsymbol{\mu}_2),$$
$$\boldsymbol{B} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{S}\boldsymbol{\Sigma}^{-1/2},$$

where $N = N_1 + N_2$. Then $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{B}$ are independent. In addition, $\boldsymbol{u}_1 \sim N_p((1/N_1 + 1/N_2)^{-1/2}\boldsymbol{\delta}, \boldsymbol{I}_p)$ and $\boldsymbol{u}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, where $\boldsymbol{\delta} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$. It also holds that $n\boldsymbol{B}$ is distributed as a Wishart distribution with degrees of freedom $n = N - 2$ and covariance matrix $\boldsymbol{I}_p$, which is denoted as $W_p(n, \boldsymbol{I}_p)$. Substituting them,

$$U_i = -\frac{(-1)^{i+1}}{2}\left(\frac{p}{N_2} - \frac{p}{N_1}\right)\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} + \frac{(-1)^{i+1}p}{\sqrt{N_1N_2}}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_2}{p} - \tau_i\frac{\boldsymbol{\delta}'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{\sqrt{p}},$$
$$V = \frac{Np}{N_1N_2}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-2}\boldsymbol{u}_1}{p},$$
$$\tau_i = \sqrt{\frac{pN_{3/2+(-1)^{i+1}/2}}{NN_{3/2-(-1)^{i+1}/2}}}.$$

In addition,

$$\widehat{\Delta^2} = \frac{Np}{N_1N_2}\left\{\frac{m-1}{n}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} - 1\right\}.$$

Then,

$$\widehat{u_i} = (-1)^{i+1}\frac{n}{2(m-1)}\left\{\frac{Np}{N_1N_2}\frac{m-1}{n}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p} - \frac{Np}{N_1N_2} + (-1)^{i+1}\left(\frac{p}{N_2} - \frac{p}{N_1}\right)\right\},$$
$$\hat{v} = \frac{n(n+1)}{(m+1)(m+2)}\frac{Np}{N_1N_2}\frac{\boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1}{p}.$$

The following lemma gives that these random variables can be expressed as functions of the independent standard normal and chi-squared variables, simultaneously.

Lemma 1. *Let $\boldsymbol{v}_1 \sim N_p(\boldsymbol{\delta}, \boldsymbol{I}_p)$, $\boldsymbol{v}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, $\boldsymbol{A} \sim W_p(n, \boldsymbol{I}_p)$, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{A}$ are independent. Then the following equalities in distribution hold, simultaneously:*

$$S \equiv \boldsymbol{\delta}'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \overset{\mathcal{D}}{=} \frac{\Delta}{Y_1}\left(Z_1 + \Delta - \sqrt{\frac{Y_2}{Y_3}}Z_2\right),$$
$$T \equiv \boldsymbol{v}_2'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \overset{\mathcal{D}}{=} \sqrt{\frac{1}{Y_1^2}\left(1 + \frac{Y_2}{Y_3}\right)\{(Z_1+\Delta)^2 + Z_2^2 + Y_4\}}Z_3$$
$$U \equiv \boldsymbol{v}_1'\boldsymbol{A}^{-1}\boldsymbol{v}_1 \overset{\mathcal{D}}{=} \frac{1}{Y_1}\{(Z_1+\Delta)^2 + Z_2^2 + Y_4\},$$
$$V \equiv \boldsymbol{v}_1'\boldsymbol{A}^{-2}\boldsymbol{v}_1 \overset{\mathcal{D}}{=} \frac{1}{Y_1^2}\left(1 + \frac{Y_2}{Y_3}\right)\{(Z_1+\Delta)^2 + Z_2^2 + Y_4\},$$

where $\Delta = \sqrt{\boldsymbol{\delta}'\boldsymbol{\delta}}$*;* $Z_1,\ Z_2,\ Z_3,\ Y_1,\ldots,Y_4$ *are independent,* $Z_i \sim N(0,1),\ i = 1,2,3,\ Y_i \sim \chi^2_{f_i},\ i = 1,\ldots,4,$

$$f_1 = n-p+1,\ f_2 = p-1,\ f_3 = n-p+2,\ f_4 = p-2.$$

Note that Fujikoshi and Seo [3] has also given similar results, but their results are individual ones, so cannot treat simultaneously. The proof of Lemma 1 is given in Appendix.

It can be described that

$$\begin{aligned}
&(-1)^{i+1}\sqrt{\widehat{v}}x + U_i - (-1)^i\widehat{u_i}\\
&= (-1)^{i+1}\sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\omega^{-1}\sqrt{\frac{Q_1}{p}}x - \frac{(-1)^{i+1}}{2}\left(\frac{p}{N_2}-\frac{p}{N_1}\right)\frac{Q_1}{p} + \frac{(-1)^{i+1}p}{\sqrt{N_1N_2}}\frac{B_2}{p} - \tau_i\frac{B_1}{\sqrt{p}}\\
&\quad + \frac{\omega^{-2}}{2}\frac{Q_1}{p} - \frac{n}{2(m-1)}\omega^{-2} + \frac{(-1)^{i+1}}{2}\frac{n}{m-1}\left(\frac{p}{N_2}-\frac{p}{N_1}\right), \qquad (2)\\
&V = \omega^{-2}\frac{Q_2}{p}, \qquad (3)
\end{aligned}$$

where $Q_1 = \boldsymbol{u}_1'\boldsymbol{B}^{-1}\boldsymbol{u}_1$, $Q_2 = \boldsymbol{u}_1'\boldsymbol{B}^{-2}\boldsymbol{u}_1$, $B_1 = \boldsymbol{\delta}'\boldsymbol{B}^{-1}\boldsymbol{u}_1$ and $B_2 = \boldsymbol{u}_2'\boldsymbol{B}^{-1}\boldsymbol{u}_1$, $\omega^2 = N_1N_2/(Np)$. From Lemma 1, we have

$$\begin{aligned}
\frac{Q_1}{p} &\overset{\mathcal{D}}{=} \frac{n}{f_1}\frac{1}{1+\sqrt{2/f_1}W_1}S,\\
\frac{B_1}{\sqrt{p}} &\overset{\mathcal{D}}{=} \frac{n}{f_1}\frac{\Delta}{1+\sqrt{2/f_1}W_1}\left(\frac{Z_1}{\sqrt{p}} + \omega\Delta - \sqrt{\frac{f_2}{f_3}}\sqrt{T}\frac{Z_2}{\sqrt{p}}\right),\\
\frac{B_2}{p} &\overset{\mathcal{D}}{=} \frac{n}{f_1}\frac{1}{1+\sqrt{2/f_1}W_1}\sqrt{\left(1+\frac{f_2}{f_3}T\right)S}\frac{Z_3}{\sqrt{p}},\\
\frac{Q_2}{p} &\overset{\mathcal{D}}{=} \frac{n^2}{f_1^2}\frac{1}{(1+\sqrt{2/f_1}W_1)^2}\left(1+\frac{f_2}{f_3}T\right)S,
\end{aligned}$$

where $W_i = \sqrt{f_i/2}(Y_i/f_i - 1)$ for $i = 1,\ldots,4$,

$$\begin{aligned}
S &= \left(\frac{Z_1}{\sqrt{p}} + \omega\Delta\right)^2 + \left(\frac{Z_2}{\sqrt{p}}\right)^2 + \frac{p-2}{p}\left(1+\sqrt{\frac{2}{f_4}}W_4\right),\\
T &= \frac{1+\sqrt{2/f_2}W_2}{1+\sqrt{2/f_3}W_3}.
\end{aligned}$$

Sorting S in descending order, it can be expressed as

$$S = s_0 + S_{1/2} + S_1 + O_{3/2},$$

where

$$\begin{aligned}
s_0 &= 1 + \omega^2\Delta^2,\\
S_{1/2} &= \frac{2\omega\Delta}{\sqrt{p}}Z_1 + \sqrt{\frac{2}{f_4}}W_4,\\
S_1 &= \frac{Z_1^2}{p} + \frac{Z_2^2}{p} - \frac{2}{p},
\end{aligned}$$

and $O_{j/2}$ is a term of j-th order with respect to $\{p^{-1/2}, N_1^{-1/2}, N_2^{-1/2}, m^{-1/2}\}$. By Maclaurin expansion of $(1+\sqrt{2/f_3}W_3)^{-1}$ up to the term with order of f_3^{-1},

$$T = \left(1+\sqrt{\frac{2}{f_2}}W_2\right)\left(1-\sqrt{\frac{2}{f_3}}W_3 + \frac{2}{f_3}W_3^2\right) + O_{3/2},$$

which can be sorted in descending order as

$$T = 1 + T_{1/2} + T_1 + O_{3/2}$$

with

$$T_{1/2} = \sqrt{\frac{2}{f_2}} W_2 - \sqrt{\frac{2}{f_3}} W_3,$$
$$T_1 = \frac{2}{f_3} W_3^2 - \frac{2}{\sqrt{f_2 f_3}} W_2 W_3.$$

Doing Maclaurin expansion of $(1 + \sqrt{2/f_1} W_1)^{-1}$ in Q_1/p up to the term with order of f_1^{-1}, and sort it in descending order, it is written as

$$\frac{Q_1}{p} = q_{1,0} + Q_{1,1/2} + Q_{1,1} + O_{3/2},$$

where

$$q_{1,0} = \frac{n}{f_1} s_0,$$
$$Q_{1,1/2} = \frac{n}{f_1} \left(S_{1/2} - \sqrt{\frac{2}{f_1}} s_0 W_1 \right),$$
$$Q_{1,1} = \frac{n}{f_1} \left(S_1 - \sqrt{\frac{2}{f_1}} S_{1/2} W_1 + \frac{2}{f_1} s_0 W_1^2 \right),$$

and using this expansion,

$$\sqrt{\frac{Q_1}{p}} = \sqrt{q_{1,0}} \left\{ 1 + \frac{1}{2q_{1,0}} (Q_{1,1/2} + Q_{1,1}) - \frac{1}{8q_{1,0}^2} Q_{1,1/2}^2 \right\} + O_{3/2}.$$

Using similar way, we can express Q_2/p as

$$\frac{Q_2}{p} = q_{2,0} + Q_{2,1/2} + Q_{2,1} + O_{3/2},$$

where

$$q_{2,0} = \left(\frac{n}{f_1}\right)^2 \left(1 + \frac{f_2}{f_3}\right) s_0,$$
$$Q_{2,1/2} = \left(\frac{n}{f_1}\right)^2 \left[\left\{ \left(1 + \frac{f_2}{f_3}\right) S_{1/2} + \frac{f_2}{f_3} s_0 T_{1/2} \right\} - 2\sqrt{\frac{2}{f_1}} \left(1 + \frac{f_2}{f_3}\right) s_0 W_1 \right],$$
$$Q_{2,1} = \left(\frac{n}{f_1}\right)^2 \left[\left\{ \left(1 + \frac{f_2}{f_3}\right) S_1 + \frac{f_2}{f_3} S_{1/2} T_{1/2} + \frac{f_2}{f_3} s_0 T_1 \right\} - 2\sqrt{\frac{2}{f_1}} \left\{ \left(1 + \frac{f_2}{f_3}\right) S_{1/2} + \frac{f_2}{f_3} s_0 T_{1/2} \right\} W_1 + \frac{6}{f_1} \left(1 + \frac{f_2}{f_3}\right) s_0 W_1^2 \right].$$

In addition, it is also expanded that

$$\frac{B_1}{\sqrt{p}} = b_{1,0} + B_{1,1/2} + B_{1,1} + O_{3/2},$$

where

$$b_{1,0} = \frac{n}{f_1} \omega \Delta^2,$$
$$B_{1,1/2} = \frac{n}{f_1} \left\{ \left(\frac{Z_1}{\sqrt{p}} - \sqrt{\frac{f_2}{f_3}} \frac{Z_2}{\sqrt{p}} \right) \Delta - \sqrt{\frac{2}{f_1}} \omega \Delta^2 W_1 \right\},$$
$$B_{1,1} = \frac{n}{f_1} \left\{ -\frac{\Delta}{2} \sqrt{\frac{f_2}{f_3}} \frac{Z_2}{\sqrt{p}} T_{1/2} - \sqrt{\frac{2}{f_1}} \left(\frac{Z_1}{\sqrt{p}} - \sqrt{\frac{f_2}{f_3}} \frac{Z_2}{\sqrt{p}} \right) \Delta W_1 + \frac{2}{f_1} \omega \Delta^2 W_1^2 \right\}.$$

Sorting in descending order, it can be described that

$$\left(1+\frac{f_2}{f_3}T\right)S = s_0\left(1+\frac{f_2}{f_3}\right)(1+\tilde{T}_{1/2}+\tilde{S}_{1/2})+O_1,$$

where $\tilde{S}_{1/2} = S_{1/2}/s_0$ and $\tilde{T}_{1/2} = \{(f_2/f_3)/(1+f_2/f_3)\}T_{1/2}$, and so

$$\sqrt{\left(1+\frac{f_2}{f_3}T\right)S} = \sqrt{s_0\left(1+\frac{f_2}{f_3}\right)}\left(1+\frac{1}{2}(\tilde{T}_{1/2}+\tilde{S}_{1/2})\right)+O_1.$$

Substituting this expansion into B_2/p, and using Maclaurin expansion of $(1+\sqrt{2/f_1}W_1)^{-1}$ up to the term with order of $f_1^{-1/2}$, we have

$$\frac{B_2}{p} = B_{2,1/2}+B_{2,1}+O_{3/2},$$

where

$$\begin{aligned}
B_{2,1/2} &= \frac{n}{f_1}\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\frac{Z_3}{\sqrt{p}},\\
B_{2,1} &= \frac{n}{f_1}\sqrt{\left(1+\frac{f_2}{f_3}\right)s_0}\left\{\frac{1}{2}(\tilde{T}_{1/2}+\tilde{S}_{1/2})-\sqrt{\frac{2}{f_1}}W_1\right\}\frac{Z_3}{\sqrt{p}}.
\end{aligned}$$

Substituting these expansions in (2) and (3), and coordinating it in order, (2) and (3) can be represented as the sum of terms with descending order, respectively, which are

$$\begin{aligned}
&(-1)^{i+1}\sqrt{\hat{v}}x+U_i-(-1)^i\widehat{u_i} = (-1)^{i+1}\sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0}\omega^{-1}x+U_{i,1/2}+U_{i,1}+O_{3/2},\\
&V = \omega^{-2}\frac{n^2(n+1)}{(m+1)^2(m+2)}s_0+\omega^{-2}Q_{2,1/2}+\omega^{-2}Q_{2,1}+O_{3/2},
\end{aligned}$$

where

$$\begin{aligned}
U_{i,1/2} &= A_iQ_{1,1/2}-\tau_iB_{1,1/2}+\frac{(-1)^{i+1}p}{\sqrt{N_1N_2}}B_{2,1/2},\\
U_{i,1} &= A_iQ_{1,1}-(-1)^{i+1}\sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}x}{8q_{1,0}^{3/2}}Q_{1,1/2}^2+\frac{(-1)^{i+1}p}{\sqrt{N_1N_2}}B_{2,1}-\tau_iB_{1,1}\\
&\quad+\frac{n}{(m-1)(m+1)}\left[(-1)^{i+1}\left(\frac{p}{N_2}-\frac{p}{N_1}\right)-\omega^{-2}\right],\\
A_i &= (-1)^{i+1}\sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}}{2\sqrt{q_{1,0}}}x-\frac{(-1)^{i+1}}{2}\left(\frac{p}{N_2}-\frac{p}{N_1}\right)+\frac{\omega^{-2}}{2}.
\end{aligned}$$

These expansions lead that

$$R_i = \frac{(-1)^{i+1}\sqrt{\hat{v}}x+U_i+(-1)^{i+1}\widehat{u_i}}{\sqrt{V}} = (-1)^{i+1}x+R_{i,1/2}+R_{i,1}+O_{3/2},$$

where

$$\begin{aligned}
R_{i,1/2} &= -\frac{1}{2}(-1)^{i+1}x\tilde{Q}_{2,1/2}+\tilde{U}_{i,1/2},\\
R_{i,1} &= (-1)^{i+1}x\left(\frac{3}{8}\tilde{Q}_{2,1/2}^2-\frac{1}{2}\tilde{Q}_{2,1}\right)-\frac{1}{2}\tilde{U}_{i,1/2}\tilde{Q}_{2,1/2}+\tilde{U}_{i,1}
\end{aligned}$$

with

$$\tilde{U}_{i,j/2} = U_{i,j/2} \Bigg/ \left\{ \sqrt{\frac{n^2(n+1)}{(m+1)^2(m+2)} s_0} \omega^{-1} \right\},$$

$$\tilde{Q}_{i,j/2} = Q_{i,j/2} \Bigg/ \left\{ \frac{n^2(n+1)}{(m+1)^2(m+2)} s_0 \right\}.$$

By Taylor expansion,

$$\Phi(R_i) = \Phi((-1)^{i+1}x) + \phi((-1)^{i+1}x) \left[R_{i,1/2} + R_{i,1}\right] - \frac{(-1)^{i+1}x}{2} \phi((-1)^{i+1}x) R_{i,1/2}^2 + O_{3/2}.$$

Since $R_{i,1/2}$ is represented as the linear combination of $\{Z_1, Z_2, Z_3, W_1, \dots, W_4\}$, $E[R_{i,1/2}] = 0$. As a result, we give the following theorem.

Theorem 1. *Let*

$$\tilde{x}_i = x - \left\{ (-1)^{i+1} \widehat{E[R_{i,1}]} - \frac{x}{2} \widehat{E[R_{i,1/2}^2]} \right\},$$

where $\widehat{E[R_{i,j}^k]}$ *is* $E[R_{i,j}^k]$ *with replacing* Δ^2 *by* $\widehat{\Delta^2}$*. For* $i = 1, 2$*,*

$$P\left((-1)^{i+1} \frac{W - \widehat{u_i}}{\sqrt{\hat{v}}} < (-1)^{i+1} \tilde{x}_i \middle| \boldsymbol{x} \in \Pi_i \right) = \Phi((-1)^{i+1}x) + O_{3/2}.$$

Explicit formula of $E[R_{i,1}]$ and $E[R_{i,1/2}^2]$ can be derived, which are given in Appendix.

Based on the expansion, we set cutoff point c_i as

$$c_i = \sqrt{\hat{v}} \left[(-1)^{i+1} z_\alpha - \left\{ (-1)^{i+1} \widehat{E[R_{i,1}]} - \frac{(-1)^{i+1} z_\alpha}{2} \widehat{E[R_{i,1/2}^2]} \right\} \right] + \hat{u}_i \quad (i = 1, 2),$$

where z_α is the α percentile point of the standard normal distribution. The cutoff point $c_1(c_2)$ makes the desired misclassification probability to be α within the error $O_{3/2}$. The other misclassification probabilities can be described as

$$P(W > c_1 | \boldsymbol{x} \in \Pi_2) = E\left[\Phi\left(\frac{-(c_1 - \widehat{u_2}) + U_2 - \widehat{u_2}}{\sqrt{V}} \right) \right],$$

$$P(W < c_2 | \boldsymbol{x} \in \Pi_1) = E\left[\Phi\left(\frac{(c_2 - \widehat{u_1}) + U_1 + \widehat{u_1}}{\sqrt{V}} \right) \right],$$

respectively. Note that

$$\hat{v}/V \xrightarrow{p} 1,$$

$$U_1 + \hat{u}_1 \xrightarrow{p} 0, \quad U_2 - \hat{u}_2 \xrightarrow{p} 0,$$

$$E[R_{i,j/2}] = O_{j/2} \quad (i, j = 1, 2).$$

From the expansion, we have

$$\widehat{u_1} - \widehat{u_2} = \frac{n}{m-1} \left\{ \frac{Np}{N_1 N_2} \frac{m-1}{n} \frac{\boldsymbol{u}_1' \boldsymbol{B}^{-1} \boldsymbol{u}_1}{p} - \frac{Np}{N_1 N_2} \right\} = \omega^{-2}(1 + \omega^2 \Delta^2) \frac{n}{m+1} - \frac{n}{m-1} \omega^{-2} + O_{1/2}.$$

It will be found that

$$\widehat{u_1} - \widehat{u_2} - \frac{n}{m} \Delta^2 \xrightarrow{p} 0$$

under the asymptotic framework A1. In addition,

$$\hat{v} - \frac{n^3}{m^3} \omega^{-2} (1 + \omega^2 \Delta^2) \xrightarrow{p} 0.$$

Combining these results,

$$\lim_{\mathrm{A1}} P(W > c_1 | \boldsymbol{x} \in \Pi_2) = \lim_{\mathrm{A1}} P(W < c_2 | \boldsymbol{x} \in \Pi_1) = \Phi\left(z_{1-\alpha} - \lim_{\mathrm{A1}} \sqrt{\frac{m}{n}} \frac{\Delta^2}{\sqrt{\Delta^2 + \omega^{-2}}} \right).$$

A Proof of Lemma 1

Proof of Lemma 1. Let $\boldsymbol{\Gamma}$ be a orthogonal matrix of order p which the first row is proportional to $\boldsymbol{\delta}'$, and let $\boldsymbol{B} = \boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}'$ and $\boldsymbol{w}_i = \boldsymbol{\Gamma}\boldsymbol{v}_i$, $i = 1, 2$. Then $\boldsymbol{B} \sim W_p(n, \boldsymbol{I}_p)$, $\boldsymbol{w}_1 \sim N_p(\Delta\boldsymbol{e}_1, \boldsymbol{I}_p)$, $\boldsymbol{w}_2 \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$ and $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{B}$ are independent;

$$
\begin{aligned}
S &= (\boldsymbol{\Gamma}\boldsymbol{\delta})'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \overset{\mathcal{D}}{=} \Delta\boldsymbol{e}_1'\boldsymbol{B}^{-1}\boldsymbol{w}_1,\\
T &= (\boldsymbol{\Gamma}\boldsymbol{v}_2)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \overset{\mathcal{D}}{=} \boldsymbol{w}_2'\boldsymbol{B}^{-1}\boldsymbol{w}_1 \overset{\mathcal{D}}{=} \sqrt{\boldsymbol{w}_1'\boldsymbol{B}^{-2}\boldsymbol{w}_1}Z\\
U &= (\boldsymbol{\Gamma}\boldsymbol{v}_1)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-1}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{B}^{-1}\boldsymbol{w}_1,\\
V &= (\boldsymbol{\Gamma}\boldsymbol{v}_1)'(\boldsymbol{\Gamma}\boldsymbol{A}\boldsymbol{\Gamma}')^{-2}(\boldsymbol{\Gamma}\boldsymbol{v}_1) \overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{B}^{-2}\boldsymbol{w}_1,
\end{aligned}
$$

where $\boldsymbol{e}_i$ denotes fundamental vector with 1 in i-th position, $Z \sim N(0, 1)$, and Z and $\{\boldsymbol{B}, \boldsymbol{w}_1\}$ are independent. By using reflection matrix(Householder matrix) $\boldsymbol{H}$ between $\boldsymbol{e}_1$ and $(1/\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1})\boldsymbol{w}_1$,

$$
S \overset{\mathcal{D}}{=} \Delta\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1}(\boldsymbol{H}\boldsymbol{e}_1)'(\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{-1}\{\boldsymbol{H}(1/\sqrt{\boldsymbol{w}_1'\boldsymbol{w}_1})\boldsymbol{w}_1\} = \Delta\boldsymbol{w}_1'(\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{-1}\boldsymbol{e}_1.
$$

Besides,

$$
\begin{aligned}
U &\overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{B}^{-1}\boldsymbol{w}_1 = \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'(\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{-1}\boldsymbol{e}_1,\\
V &\overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{B}^{-2}\boldsymbol{w}_1 = \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'(\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{-2}\boldsymbol{e}_1.
\end{aligned}
$$

Given $\boldsymbol{w}_1$, $\boldsymbol{C} \equiv \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}' \sim W_p(n, \boldsymbol{I}_p)$, so $\boldsymbol{C}$ and $\boldsymbol{w}_1$ are independent. Partition

$$
\boldsymbol{C} = \begin{pmatrix} c_{11} & \boldsymbol{c}_{21}' \\ \boldsymbol{c}_{21} & \boldsymbol{C}_{22} \end{pmatrix} \quad \text{and} \quad \boldsymbol{w}_1 = \begin{pmatrix} w_{11} \\ \boldsymbol{w}_{21} \end{pmatrix}.
$$

It can be expressed that

$$
S \overset{\mathcal{D}}{=} \Delta\boldsymbol{w}_1'\boldsymbol{C}^{-1}\boldsymbol{e}_1 = \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \boldsymbol{w}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}),
$$

where $c_{11\cdot 2} = c_{11} - \boldsymbol{c}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}$. In addition,

$$
\begin{aligned}
U &\overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'\boldsymbol{C}^{-1}\boldsymbol{e}_1 = \frac{\boldsymbol{w}_1'\boldsymbol{w}_1}{c_{11\cdot 2}},\\
V &\overset{\mathcal{D}}{=} \boldsymbol{w}_1'\boldsymbol{w}_1 \cdot \boldsymbol{e}_1'\boldsymbol{C}^{-2}\boldsymbol{e}_1 = \frac{\boldsymbol{w}_1'\boldsymbol{w}_1}{c_{11\cdot 2}^2}(1 + \boldsymbol{c}_{21}'\boldsymbol{C}_{22}^{-1}\boldsymbol{c}_{21}).
\end{aligned}
$$

It is noted that $\boldsymbol{x} \equiv \boldsymbol{C}_{22}^{-1/2}\boldsymbol{c}_{21} \sim N_{p-1}(\boldsymbol{0}, \boldsymbol{I}_p)$, $\boldsymbol{D} \equiv \boldsymbol{C}_{22} \sim W_{p-1}(n, \boldsymbol{I}_{p-1})$, and $\boldsymbol{x}$ and $\boldsymbol{D}$ are independent, thus w_{11}, $\boldsymbol{w}_{21}$, $\boldsymbol{x}$, $\boldsymbol{D}$ and $c_{11\cdot 2}$ are independent. Using these results, we have

$$
S \overset{\mathcal{D}}{=} \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \boldsymbol{w}_{21}'\boldsymbol{D}^{-1/2}\boldsymbol{x}) \quad \text{and} \quad V \overset{\mathcal{D}}{=} \frac{\boldsymbol{w}_{11}'\boldsymbol{w}_{11}}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}).
$$

Let $\boldsymbol{G}$ be orthogonal matrix of order $p - 1$ which the first row is proportional to $\boldsymbol{x}'\boldsymbol{D}^{-1/2}$. Given $\boldsymbol{x}$ and $\boldsymbol{D}$, $\boldsymbol{y} \equiv \boldsymbol{G}\boldsymbol{w}_{21} \sim N_{p-1}(\boldsymbol{0}, \boldsymbol{I}_{p-1})$, and it is found that w_{11}, $c_{11\cdot 2}$, $\boldsymbol{x}$, $\boldsymbol{D}$ and $\boldsymbol{y}$ are independent. Partitioning $\boldsymbol{y} = (y_1\ \boldsymbol{y}_2')'$, we have

$$
\begin{aligned}
S &\overset{\mathcal{D}}{=} \frac{\Delta}{c_{11\cdot 2}}\{w_{11} - (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{D}^{-1/2}\boldsymbol{x})\} \overset{\mathcal{D}}{=} \frac{\Delta}{c_{11\cdot 2}}(w_{11} - \sqrt{\boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}}y_1),\\
U &\overset{\mathcal{D}}{=} \frac{w_{11}^2 + (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{w}_{21})}{c_{11\cdot 2}} \overset{\mathcal{D}}{=} \frac{1}{c_{11\cdot 2}}(w_{11}^2 + y_1^2 + \boldsymbol{y}_2'\boldsymbol{y}_2),\\
V &\overset{\mathcal{D}}{=} \frac{w_{11}^2 + (\boldsymbol{G}\boldsymbol{w}_{21})'(\boldsymbol{G}\boldsymbol{w}_{21})}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x}) \overset{\mathcal{D}}{=} \frac{1}{c_{11\cdot 2}^2}(1 + \boldsymbol{x}'\boldsymbol{D}^{-1}\boldsymbol{x})(w_{11}^2 + y_1^2 + \boldsymbol{y}_2'\boldsymbol{y}_2).
\end{aligned}
$$

These show the conclusion of the lemma. □

B Expectations

Firstly, we calculate $E[R_{i,1/2}^2]$. It is that

$$E[R_{i,1/2}^2] = \frac{x^2}{4} E[\tilde{Q}_{2,1/2}^2] + E[\tilde{U}_{i,1/2}^2] - (-1)^{i+1} x E[\tilde{Q}_{2,1/2} \cdot \tilde{U}_{i,1/2}].$$

Since $S_{1/2} \perp\!\!\!\perp T_{1/2} \perp\!\!\!\perp W_1$,

$$E[Q_{2,1/2}^2] = \frac{n^4}{f_1^4} \left[\left(1 + \frac{f_2}{f_3}\right)^2 E[S_{1/2}^2] + \left(\frac{f_2}{f_3}\right)^2 s_0^2 E[T_{1/2}^2] + 4 \cdot \frac{2}{f_1} \left(1 + \frac{f_2}{f_3}\right)^2 s_0^2 E[W_1^2] \right].$$

Noting that

$$S_{1/2}^2 = \frac{4\omega^2\Delta^2}{p} Z_1^2 + \frac{2}{f_4} W_4^2 + 2\frac{2\omega\Delta}{\sqrt{p}} \sqrt{\frac{2}{f_4}} Z_1 W_4,$$

it holds that

$$E[S_{1/2}^2] = \frac{4\omega^2\Delta^2}{p} + \frac{2}{f_4}.$$

It also holds that

$$E[T_{1/2}^2] = \frac{2}{f_2} E[W_2^2] + \frac{2}{f_3} E[W_3^2] = \frac{2}{f_2} + \frac{2}{f_3}.$$

Thus,

$$\begin{aligned} E[Q_{2,1/2}^2] = \frac{n^4}{f_1^4} \Bigg[& \left(1 + \frac{f_2}{f_3}\right)^2 \left(\frac{4\omega^2\Delta^2}{p} + \frac{2}{f_4}\right) + \left(\frac{f_2}{f_3}\right)^2 (1 + \omega^2\Delta^2)^2 \left(\frac{2}{f_2} + \frac{2}{f_3}\right) \\ & + \frac{8}{f_1} \left(1 + \frac{f_2}{f_3}\right)^2 (1 + \omega^2\Delta^2)^2 \Bigg]. \end{aligned}$$

For evaluating $E[U_{i,1/2}^2]$, note that

$$Q_{1,1/2}^2 = \frac{n^2}{f_1^2} \left(S_{1/2}^2 + \frac{2}{f_1} s_0^2 W_1^2 - 2\sqrt{\frac{2}{f_1}} s_0 S_{1/2} W_1 \right).$$

Thus,

$$\begin{aligned} E[Q_{1,1/2}^2] &= \frac{n^2}{f_1^2} \left[\left(\frac{4\omega^2\Delta^2}{p} + \frac{2}{f_4} \right) + \frac{2}{f_1} s_0^2 \right] \\ &= \frac{n^2}{f_1^2} \left[\frac{2}{f_4} + \frac{2}{f_1} + \left(\frac{4}{p} + \frac{2}{f_1} \right) \omega^2 \Delta^2 \right]. \end{aligned}$$

Moreover, the following equalities hold.

$$\begin{aligned} E[B_{2,1/2}^2] &= \frac{n^2}{f_1^2} s_0 \left(1 + \frac{f_2}{f_3}\right) \frac{1}{p} = \frac{n^2}{f_1^2} \left(1 + \frac{f_2}{f_3}\right) (1 + \omega^2\Delta^2) \frac{1}{p}, \\ E[B_{1,1/2}^2] &= \frac{n^2}{f_1^2} \left\{ \left(\frac{1}{p} + \frac{f_2}{f_3} \frac{1}{p} \right) \omega\Delta^2 + \frac{2}{f_1} \omega^2 \Delta^4 \right\}. \end{aligned}$$

From independence,

$$\begin{aligned} E[B_{1,1/2} \cdot B_{2,1/2}] &= E[B_{1,1/2}] \cdot E[B_{2,1/2}] = 0, \\ E[Q_{1,1/2} \cdot B_{2,1/2}] &= E[Q_{1,1/2}] \cdot E[B_{2,1/2}] = 0. \end{aligned}$$

In addition,

$$E[Q_{1,1/2}\cdot B_{1,1/2}] = \frac{n^2}{f_1^2}\left[\frac{2\omega\Delta^2}{p}E[Z_1^2] + \frac{2}{f_1}s_0\omega\Delta^2 E[W_1^2]\right]$$
$$= \frac{n^2}{f_1^2}\left[\left(\frac{2}{p}+\frac{2}{f_1}\right)\omega\Delta^2 + \frac{2}{f_1}\omega^3\Delta^4\right].$$

On the other hand,

$$E[U_{i,1/2}^2] = A_i^2 E[Q_{1,1/2}^2] - 2\tau_i A_i E[Q_{1,1/2}\cdot B_{1,1/2}]$$
$$+ \frac{p^2}{N_1N_2}E[B_{2,1/2}^2] + \tau_i^2 E[B_{1,1/2}^2].$$

From independence,

$$E[Q_{2,1/2}\cdot U_{i,1/2}]$$
$$= \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)A_iE[S_{1/2}^2] - \tau_i\left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\frac{2\omega\Delta^2}{p}E[Z_1^2]$$
$$+ 2\frac{2}{f_1}\left(1+\frac{f_2}{f_3}\right)\left(\frac{n}{f_1}\right)^3 s_0^2A_iE[W_1^2] - 2\frac{2}{f_1}\left(1+\frac{f_2}{f_3}\right)\left(\frac{n}{f_1}\right)^3 s_0\tau_i\omega\Delta^2E[W_1^2]$$
$$= \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\left[\left(\frac{4\omega^2\Delta^2}{p}+\frac{2}{f_4}\right)+\frac{4}{f_1}(1+\omega^2\Delta^2)^2\right]A_i$$
$$- \left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)\frac{N_{3/2+(-1)^{i+1}/2}}{N}\frac{2\Delta^2}{p} - \frac{4}{f_1}\left(\frac{n}{f_1}\right)^3\left(1+\frac{f_2}{f_3}\right)(1+\omega^2\Delta^2)\frac{N_{3/2+(-1)^{i+1}/2}}{N}\Delta^2.$$

Next, we calculate $E[R_{i,1}]$. It can be expressed that

$$E[R_{i,1}] = (-1)^{i+1}x\left(\frac{3}{8}E[\tilde{Q}_{2,1/2}^2] - \frac{1}{2}E[\tilde{Q}_{2,1}]\right) - \frac{1}{2}E[\tilde{U}_{i,1/2}\tilde{Q}_{2,1/2}] + E[\tilde{U}_{i,1}].$$

Since $S_{1/2} \perp\!\!\!\perp T_{1/2} \perp\!\!\!\perp W_1$,

$$E[Q_{2,1}] = \frac{n^2}{f_1^2}\left[\left(1+\frac{f_2}{f_3}\right)E[S_1] + \frac{f_2}{f_3}s_0E[T_1] + \frac{6}{f_1}\left(1+\frac{f_2}{f_3}\right)s_0E[W_1^2]\right].$$

Noting that $E[S_1]=0$ and $E[T_1]=2/f_3$,

$$E[Q_{2,1}] = \frac{n^2}{f_1^2}\left[\frac{2f_2}{f_3^2}(1+\omega^2\Delta^2) + \frac{6}{f_1}\left(1+\frac{f_2}{f_3}\right)(1+\omega^2\Delta^2)\right].$$

It is described that

$$E[U_{i,1}] = A_iE[Q_{1,1}] - (-1)^{i+1}\sqrt{\frac{n(n+1)}{(m+1)(m+2)}}\frac{\omega^{-1}x}{8q_{1,0}^{3/2}}E[Q_{1,1/2}^2]$$
$$+ \frac{(-1)^{i+1}p}{\sqrt{N_1N_2}}E[B_{2,1}] - \tau_iE[B_{1,1}] + \frac{n}{(m-1)(m+1)}\left[(-1)^{i+1}\left(\frac{p}{N_2}-\frac{p}{N_1}\right)-\omega^{-2}\right],$$

where the following equalities hold.

$$E[Q_{1,1}] = \frac{n}{f_1}\left[\frac{2}{f_1}(1+\omega^2\Delta^2)\right],$$
$$E[B_{2,1}] = 0,$$
$$E[B_{1,1}] = \frac{2n}{f_1^2}\omega\Delta^2.$$

References

[1] Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed. Wiley, Hoboken.

[2] Fujikoshi, Y. (2000). Error bounds for asymptotic approximations of the linear discriminant function when the sample sizes and dimensionality are large. *J. Multivariate Anal.*, **73**, 1–17.

[3] Fujikoshi, Y. and Seo, T. (1998). Asymptotic approximations for EPMC's of the linear and the quadratic discriminant functions when the sample sizes and the dimension are large. *Random Oper. Stochastic Equations*, **6**, 269–280.

[4] Fujikoshi, Y., Ulyanov, V.V. and Shimizu, R. (2010). *Multivariate Statistics High-Dimensional and Large-Sample Approximations*. Wiley, Hoboken, NJ.

[5] Lachenbruch, P.A. (1968). On expected probabilities of misclassification in discriminant analysis, necessary sample size, and a relation with the multiple correlation coefficient. *Biometrics*, **24**, 823–834.