Constraints on Moving Strong Discontinuity Surfaces in Dynamic Plane-Stress or Plane-Strain Deformations of Stable Elastic-Ideally Plastic Materials

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For dynamic deformations of compressible elastic-ideally plastic materials in the practically important cases of plane stress and plane strain, we investigate the possible existence of propagating surfaces of strong discontinuity (across which components of stress, strain, or material velocity jump) within a small-displacement-gradient formulation. For each case, an explicit proof of the impossibility of such a propagating surface (except at an elastic wave speed) is achieved for isotropic materials satisfying a Huber-Mises yield condition and associated flow rule, and we show that our method of proof can be generalized to a large class of anisotropic materials. Nevertheless, we demonstrate that moving surfaces of strong discontinuity cannot be ruled out for all stable (i.e., satisfying the maximum plastic work inequality) materials, as in the case of a material whose yield surface contains a linear portion. A clear knowledge of the conditions under which dynamically propagating strong discontinuity surfaces can and cannot exist is crucial to the attainment of correct and complete solutions to such practical elastic-plastic problems as dynamic crack propagation, impact and rapidly moving load problems, high-speed forming, cutting, and other manufacturing processes.

1 Introduction

Since the monumental work of Hadamard (1903) on discontinuity surfaces in continuum mechanics, this subject has attracted the attention of many researchers in mechanics and applied mathematics. Many of the early contributions are referenced in the fine book by Courant and Friedrichs (1948), which deals mainly with nonlinear wave propagation in gas dynamics, but much of their work is also important to the study of moving strong discontinuity surfaces (i.e., shocks) in solids. Thomas (1961) and Hill (1961) extended Hadamard's (1903) work on general invariant-form compatibility conditions for discontinuity surfaces, and Hill (1961) deduced restrictions on strong discontinuities in rigid-plastic materials under quasi-static conditions. Much of the ensuing work on moving strong discontinuities in elastic-plastic solids, which usually deals with special (often one-dimensional) situations, is reviewed in Ting's (1976) paper on shock and weaker waves in elastic-plastic materials.

The significance of this knowledge about possible discontinuities lies in its necessity for constructing the solutions of boundary value problems in solid mechanics and for demonstrating the possible existence and uniqueness of such solutions. Recent interest in the moving strong discontinuity problem is partly motivated by the analysis of stress and deformation fields near a growing crack tip in elastic-ideally plastic material (see, e.g., the quasi-static crack growth analysis of Drugan and Chen (1989), and the dynamic crack growth analysis of Leighton et al. (1987)).

Drugan and Rice (1984) and Drugan (1986) developed a new direct approach for analyzing moving strong discontinuity surfaces in general three-dimensional deformations of quasi-statically deforming elastic-plastic solids. They employed a small-displacement-gradient formulation of standard weak continuum mechanical assumptions coupled with merely skeletal constitutive assumptions which are believed to describe realistically a large class of elastic-plastic materials, and which subsume many specific elastic-plastic models that are frequently employed. Among the results, they showed that the complete stress tensor must be continuous across quasi-statically moving surfaces for the general class of solids analyzed, and that jumps

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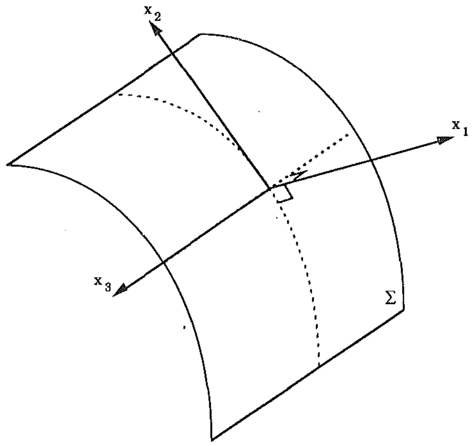

Fig. 1 Portion of a propagating strong discontinuity surface

in certain components of the strain tensor and the material velocity vector are possible only if specific conditions are met at the propagating surface. The special case of such discontinuities under quasi-static generalized plane stress was considered separately, by Pan (1982) and Narasimhan and Rosakis (1987).

Drugan and Shen (1987) deduced restrictions on *dynamically* propagating surfaces of strong discontinuity in very general elastic-plastic materials by generalizing the analyses of Drugan and Rice (1984) and Drugan (1986). In that paper, we derived one condition on jumps (termed here "Condition 1") by integrating the maximum plastic work inequality along an arbitrary portion of the deformation path across a hypothetical moving discontinuity surface, and another condition (termed here "Condition 2") by seeking nontrivial solutions of the incremental stress-strain equations across a moving discontinuity (valid for materials with smooth yield surfaces). By first analyzing shock waves that produce no plastic straining, we proved that if a shock is to propagate at a speed other than an elastic wave speed, the yield condition must be satisfied *throughout* the stress path across the shock (this requirement also applies to Condition 2). Thus, a necessary condition for the existence of an elastic-plastic shock wave (moving strong discontinuity accompanied by plastic strain variation) results by requiring that any path in stress space experienced by a material point during shock passage satisfy all the restrictions just summarized (i.e., Conditions 1 and 2, and the yield condition). In addition to deriving these general restrictions on discontinuities in arbitrary three-dimensional deformations, Drugan and Shen (1987) were able to employ Condition 1 (their Equation (3.30)) together with the yield condition to *rule out* moving strong discontinuities that propagate at speeds other than an elastic wave speed for the special cases of antiplane strain and incompressible plane-strain deformations of materials whose yield surfaces do not contain linear segments. Materials whose yield surfaces do have linear portions, such as ductile single crystals, were found to permit elastic-plastic shock waves under certain specific conditions. These results have already found practical application in the analysis of dynamic antiplane shear crack growth in elastic-ideally plastic single crystals by Nikolic and Rice (1988), who showed that a solution for the stress and deformation fields near a crack tip under such conditions must contain elastic-plastic shocks whose behavior is circumscribed by the Drugan and Shen (1987) conditions. Further confirmation of the aforementioned quasi-static and dynamic small-displacement-gradient discontinuity analyses is provided by the very recent, rigorous *finite deformation* discontinuity analysis of Drugan and Shen (1990).

Because Condition 2 (Equation (4.12) of Drugan and Shen, 1987) is not utilized in analyzing the special cases of antiplane-strain and incompressible plane-strain deformations, a natural question is whether that relation is independent of the other restrictions employed. If the answer is affirmative, that relation becomes yet another restriction on possible strong discontinuity waves, which should then allow us to rule out such waves in more general deformations.

In the present work, we apply the general discontinuity restrictions derived in Drugan and Shen (1987) to analyze the practically important cases of plane-stress and plane-strain deformations of compressible materials, within a small-displacement-gradient formulation. It is very difficult to assess the independence of these restrictions for a general material, but we do show that they are not naturally dependent on one another. We shall prove independence of these restrictions (thereby ruling out strong discontinuity surfaces moving at other than elastic wave speeds) first for isotropic materials with a Huber-Mises yield condition and associated flow rule, and then for certain classes of more general materials. We shall also further discuss the special cases of antiplane-strain and incompressible plane-strain analyzed by Drugan and Shen (1987).

With reference to Fig. 1, let Σ be a portion of a regular surface of strong discontinuity that moves with speed $V>0$ in its normal direction. A Cartesian coordinate system x_1, x_2, x_3 moves with the surface and is oriented so that x_1 is in the direction of the normal while x_3 is perpendicular to *the plane* of "plane stress" or "plane strain." $\mathbf{u}$, $\boldsymbol{\epsilon}$, and $\boldsymbol{\sigma}$ denote the tensors of displacement, strain and stress, respectively. Values of a tensorial field quantity, say $\mathbf{g}(x_1, x_2, x_3, t)$ where t is time, directly ahead of and directly behind the moving surface Σ will be denoted as $\mathbf{g}^{\pm} \equiv \lim_{\mu \downarrow 0} \mathbf{g}(x_1, x_2, x_3, t_a \mp \mu)$, respectively, where t_a is the time at which Σ arrives at a particular material point. The jump in such a field quantity across Σ will be denoted as $[\![\mathbf{g}]\!] \equiv \mathbf{g}^+ - \mathbf{g}^-$. In the sequel, components of tensors with respect to the Cartesian coordinate system of Fig. 1 are indicated either by the Roman indices i,j,k,l which have range 1,2,3 or by the Greek indices $\alpha,\beta,\gamma,\delta$ which have range 1,2 only; both types of index follow the summation convention.

2 Plane-Stress Case

The plane-stress case is very special. Because of the approximation implied in the concept of generalized plane stress, there is a possibility of "necking," i.e., a jump in u_3 and hence $[\![\epsilon_{33}]\!]=0$ may *not* be true (Hill, 1952; Pan, 1982). By "generalized plane stress" is meant the assumptions that the field quantities $\mathbf{u}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$ represent thickness averages, and that $\sigma_{3i} \equiv 0$. For a thin sheet of material suitable for this type of description, in-plane components of stress and strain are assumed to be related by the constitutive equation

$$d\epsilon_{\alpha\beta} = d\epsilon^e_{\alpha\beta} + d\epsilon^p_{\alpha\beta} = M_{\alpha\beta\gamma\delta} d\sigma_{\gamma\delta} + d\Lambda \frac{\partial f}{\partial \sigma_{\alpha\beta}}, \tag{1}$$

where $\mathbf{M}$ is the (constant) elastic modulus tensor, assumed to be positive definite and to possess the usual symmetries, $f(\boldsymbol{\sigma})=0$ is the yield condition, $d\Lambda \geq 0$ is an unspecified parameter for the nonhardening materials to be considered, and superscripts e and p denote elastic and plastic part, respectively. Of course, we have assumed a smooth yield surface.

The physical requirement of finite total plastic work in any finite subregion of the body traversed by Σ demands continuity of the in-plane components of displacement, u_α. Assuming that $\partial u_\alpha / \partial x_\beta$ exist in a neighborhood of Σ and tend to finite limits as Σ is approached, a special result of Hadamard's (1903) lemma, known as Maxwell's (1873) theorem, is the compatibility requirement

$$[\![\partial u_\alpha / \partial x_2]\!] = 0. \tag{2}$$

(Recall that x_2 is the in-plane coordinate directed parallel to

Σ.) Since Σ is moving, continuity of u_α leads also to Hadamard's (1903) kinematical condition of compatibility

$$[\![v_\alpha]\!] = -V[\![\partial u_\alpha/\partial x_1]\!]. \tag{3}$$

In terms of the infinitesimal strain tensor, $\epsilon_{ij} \equiv \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, equation (2) and equation (3) require

$$[\![v_1]\!] = -V[\![\epsilon_{11}]\!],\ [\![v_2]\!] = -2V[\![\epsilon_{12}]\!],\ [\![\epsilon_{22}]\!] = 0. \tag{4}$$

The conservation of linear and angular momentum requires (Kotchine, 1926)

$$[\![\sigma_{1\alpha}]\!] = -\rho V[\![v_\alpha]\!], \tag{5}$$

where ρ is the material density, which is treated as continuous within the small-displacement-gradient formulation, as is V, so that conservation of mass is identically satisfied. Since we employ a purely mechanical constitutive theory, the conservation of energy provides no useful additional jump restriction.

It will prove convenient to combine equations (4) and (5) as

$$[\![\epsilon_{11}]\!] = \frac{1}{\rho V^2}[\![\sigma_{11}]\!],\ [\![\epsilon_{12}]\!] = \frac{1}{2\rho V^2}[\![\sigma_{12}]\!],\ [\![\epsilon_{22}]\!] = 0. \tag{6}$$

As argued in Drugan and Shen (1987), a purely mechanical description of shock waves requires equation (6) to be true for any subdivision of the shock transition zone, which leads to

$$d\epsilon_{11} = \frac{1}{\rho V^2}d\sigma_{11},\ d\epsilon_{12} = \frac{1}{2\rho V^2}d\sigma_{12},\ d\epsilon_{22} = 0. \tag{7}$$

These equations must be satisfied throughout the stress and strain paths experienced by a material point during shock passage.

Before analyzing a shock that produces plastic straining, we first investigate the case in which the stresses remain below yield across the shock, and hence $d\Lambda = 0$. Substituting equation (7) into equation (1) in this case, we must have for nonzero $d\sigma_{\alpha\beta}$ (clearly a necessity for nonzero $[\![\sigma_{\alpha\beta}]\!]$)

$$\det\begin{bmatrix} \left(M_{1111} - \frac{1}{\rho V^2}\right) & M_{1122} & 2M_{1112} \\ M_{2211} & M_{2222} & 2M_{2212} \\ M_{1211} & M_{1222} & \left(2M_{1212} - \frac{1}{2\rho V^2}\right) \end{bmatrix} = 0 \tag{8}$$

which can be used to determine the elastic strong discontinuity wave speeds. We will show that these speeds may differ from those for the three-dimensional case, due to our assumption of plane stress.

When plastic deformation accrues during shock passage, Drugan and Shen (1987) proved that the entire stress path across the shock must remain at yield. (This is easily seen here by observing that equation (8) permits purely elastic stress changes only when V equals an elastic wave speed.) For plane stress of elastic-ideally plastic material, the yield condition has the form

$$f(\sigma_{11}, \sigma_{22}, \sigma_{12}) = 0 \tag{9}$$

(assumed to depend symmetrically on σ_{12} and σ_{21}). Combining equations (1), (7), and the yield condition (9) in differential form (Prager's consistency condition) gives

$$\begin{bmatrix} \left(M_{1111} - \frac{1}{\rho V^2}\right) & M_{1122} & 2M_{1112} & \frac{\partial f}{\partial \sigma_{11}} \\ M_{2211} & M_{2222} & 2M_{2212} & \frac{\partial f}{\partial \sigma_{22}} \\ M_{1211} & M_{1222} & \left(2M_{1212} - \frac{1}{2\rho V^2}\right) & \frac{\partial f}{\partial \sigma_{12}} \\ \frac{\partial f}{\partial \sigma_{11}} & \frac{\partial f}{\partial \sigma_{22}} & 2\frac{\partial f}{\partial \sigma_{12}} & 0 \end{bmatrix} \begin{bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{12} \\ d\Lambda \end{bmatrix} \equiv [\mathbf{K}][d\mathbf{S}] = \mathbf{0}, \tag{10}$$

where this equation serves to define the matrix $\mathbf{K}$ and the vector $d\mathbf{S}$.

Thus, a necessary condition for an elastic-plastic shock to exist takes the form

$$\det(\mathbf{K}) = 0. \tag{11}$$

This equation, although it might be identified as the aforementioned Condition 2, cannot be directly deduced from equation (4.12) of Drugan and Shen (1987) because $d\epsilon_{33}$ is always required to be zero across shocks in that (general three-dimensional) derivation. As mentioned in Section 1, like the yield condition, equation (11) is to be satisfied throughout the passage of the shock since we are discussing a shock propagating at a speed V different from the elastic wave speeds. Observe that since equation (11) effectively requires an elastic-plastic shock to propagate at a plastic wave speed, the usual shock stability condition is identically satisfied.

There are two other conditions restricting the existence of elastic-plastic shock waves: one is the undifferentiated form of the yield condition, equation (9); the other is the plane-stress version of the aforementioned Condition 1. This condition is obtained by integrating the maximum plastic work inequality along any part of any admissible deformation path (one along which all governing equations are satisfied) across a hypothetical moving strong discontinuity surface:

$$\int_{\epsilon^{p+}}^{\epsilon^p} (\sigma_{ij} - \sigma_{ij}^o)d\epsilon_{ij}^p \geq 0. \tag{12}$$

Here, $\boldsymbol{\epsilon}^p$ represents any value of the plastic strain on an admissible path between $\boldsymbol{\epsilon}^{p+}$ and $\boldsymbol{\epsilon}^{p-}$, and $\boldsymbol{\sigma}^o$ is any fixed stress state that is at or below yield. Equation (12) can be explicitly evaluated by employing $\sigma_{3i} \equiv \sigma_{3i}^o \equiv 0$, the first equation of equation (1), the elastic part of the second equation of equation (1), and equation (7), in the manner illustrated in Drugan and Shen (1987). Doing this for the choices $\boldsymbol{\sigma}^o \equiv \boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}^o \equiv \boldsymbol{\sigma}$ (the $\boldsymbol{\sigma}$ − value corresponding to $\boldsymbol{\epsilon}^p$, the endpoint of integration in equation (12)), the resulting restriction is

$$<\sigma_{1\alpha}><\sigma_{1\alpha}> - \rho V^2 <\sigma_{\alpha\beta}> M_{\alpha\beta\gamma\delta} <\sigma_{\delta\gamma}> = 0 \tag{13}$$

where $<\mathbf{g}> \equiv \mathbf{g} - \mathbf{g}^+$, and here $\mathbf{g}$ represents any $\mathbf{g}$-value on an admissible path between $\mathbf{g}^+$ and $\mathbf{g}^-$.

In most cases, the stresses under restrictions of equations (9) and (11) cannot satisfy equation (13) unless $<\boldsymbol{\sigma}> = \mathbf{0}$, i.e., when there are no stress jumps. That is to say, equations (9), (11), and (13) are mutually independent except on some isolated points in $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ space. Geometrically, the surfaces defined by these three equations in $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ space cannot share a common intersecting curve from $\boldsymbol{\sigma}^+$ to $\boldsymbol{\sigma}^-$ that would facilitate a strong discontinuity. This independence can be explained by noting that the plastic part of equation (11) corresponds to the normality rule resulting from the maximum plastic work inequality (increments of plastic strain are normal to the yield surface), a local property of the material, while equation (13) is a result of integration along a finite path in stress space, a global property of the material during shock transition. Another form of equation (13) is $<\boldsymbol{\sigma}> : <\boldsymbol{\epsilon}^p> = 0$ which can be easily derived. Although in deriving equation

(11) we have employed the differential form of equation (13) ($d\sigma{:}d\epsilon^p=0$) and that of equation (9) ($df=0$), equation (11) replaces neither (9) nor (13) because neither one can be simply reverted to its original form via (10).

Actually proving the independence of equations (9), (11), and (13) is very difficult (if possible at all) for a general material. Our approach will be to prove this first for isotropic elastic-ideally plastic materials with a Huber-Mises yield condition; although this seems to be the simplest case possible, it is also very important because most theoretical solutions for the plane-stress case are for this material model. We will then show how our proof can be generalized to include certain classes of material anisotropy.

When the material is isotropic, the elastic moduli are

$$M_{ijkl}=\frac{1}{4G}\,(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-\frac{\lambda}{2G(2G+3\lambda)}\delta_{ij}\delta_{kl} \tag{14}$$

where G and λ are the Lamé elastic constants and δ_{ij} is the Kronecker delta.

The Huber-Mises yield condition in the plane-stress case is

$$f\equiv\frac{1}{3}(\sigma_{11}^2+\sigma_{22}^2-\sigma_{11}\sigma_{22})+\frac{1}{2}\,(\sigma_{12}^2+\sigma_{21}^2)-k^2=0, \tag{15}$$

where k is the yield stress in shear.

When there is no plastic straining across a moving shock, equation (8) can be used to calculate the elastic shock wave speeds. For isotropic materials, two such speeds result:

$$V^2=\frac{G}{\rho},\ \text{or}\ V^2=\frac{4G(G+\lambda)}{\rho(2G+\lambda)}. \tag{16}$$

Here the dilatant shock wave speed differs from the value for general three-dimensional deformations $\left(\sqrt{(2G+\lambda)/\rho}\right)$ because of the aforementioned reason that $d\epsilon_{33}$ is allowed to be nonzero across the shock.

When the yield condition equation (15) is continuously satisfied across a moving shock (as Drugan and Shen (1987) showed it must be for a shock to propagate at any speed other than an elastic wave speed), equation (11) applies, and can be expanded to give:

$$\left(\frac{1}{G}-\frac{1}{\rho V^2}\right)\left\{\left[\frac{1}{\rho V^2}-\frac{(5G+3\lambda)}{G(2G+3\lambda)}\right]\sigma_{11}^2\right.$$
$$\left.+\left[\frac{4}{\rho V^2}-\frac{(5G+3\lambda)}{G(2G+3\lambda)}\right]\sigma_{22}^2+\left[-\frac{4}{\rho V^2}+\frac{(8G+3\lambda)}{G(2G+3\lambda)}\right]\sigma_{11}\sigma_{22}\right\}$$
$$-\frac{36}{G(2G+3\lambda)}\left(\frac{2G+\lambda}{4G}-\frac{G+\lambda}{\rho V^2}\right)\sigma_{12}^2=0. \tag{17}$$

Another condition is equation (13), which for an isotropic material becomes

$$\left[1-\frac{\rho V^2(G+\lambda)}{G(2G+3\lambda)}\right]<\sigma_{11}>^2+\frac{\rho V^2\lambda}{G(2G+3\lambda)}<\sigma_{11}><\sigma_{22}>$$
$$-\frac{\rho V^2(G+\lambda)}{G(2G+3\lambda)}<\sigma_{22}>^2+\left(1-\frac{\rho V^2}{G}\right)<\sigma_{12}>^2=0. \tag{18}$$

In the stress space of σ_{11}, σ_{22}, and σ_{12}, as shown in Fig. 2 (for simplicity, equation (18), which depicts a cone centered at a point on (15), is not drawn in the figure), (15) represents an ellipsoid, and (17) is a cone (unless V is one of the elastic wave speeds given in (16), in which case it degenerates into a pair of planes). It can be easily proved that when the propagation speed V is smaller than the elastic wave speeds, the axis of the cone lies in the $\sigma_{11}-\sigma_{22}$ plane. (In fact, this is true as long as V is smaller than the dilatant elastic wave speed; if V is greater than that speed, equation (18) can only be satisfied by zero stress jumps.) The intersection of (15) and (17) is two closed curves; we can see from these equations (or Fig. 2) that

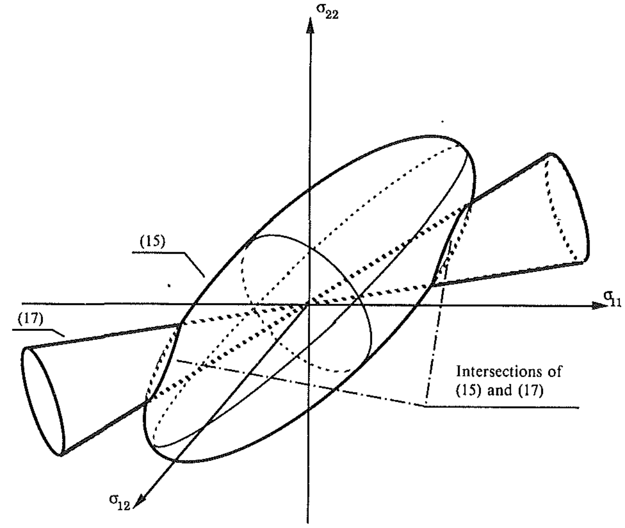

Fig. 2 Intersection of the surfaces representing two of the trans-shock stress path restrictions for the plane-stress case

if (σ_{11}, σ_{22}, σ_{12}) is on one of the curves, then (σ_{11}, σ_{22}, $-\sigma_{12}$) must fall on the same curve. Thus, if one of the curves falls on the surface described by (18), the curve should include (σ_{11}, σ_{22}, σ_{12}) and (σ_{11}, σ_{22}, $-\sigma_{12}$) simultaneously. This means that if equation (18) passes through one of these critical curves then (18), with $-\sigma_{12}$ substituted for σ_{12}, should pass through the same curve. Since both equations from (18) (with σ_{12} or $-\sigma_{12}$) pass through that curve, any linear combination of these two different versions of (18) passes through the same curve. Subtracting these two versions gives

$$\left(1-\frac{\rho V^2}{G}\right)\sigma_{12}^+\sigma_{12}=0, \tag{19}$$

which applies for all states σ_{12} (including $\sigma_{12}=\sigma_{12}^+$) on an admissible path from σ_{12}^+ to σ_{12}^-.

We have already assumed $\rho V^2\neq G$; thus, for a shock to exist equation (19) requires $\sigma_{12}=0$. (If $\sigma_{12}^+=0$ and there is some $\sigma_{12}\neq 0$ on the path from $\boldsymbol{\sigma}^+$ to $\boldsymbol{\sigma}^-$, we may choose that σ_{12} as a new σ_{12}^+ and start from there to get $\sigma_{12}=0$, then $\sigma_{12}^+=0$ by continuity, showing $\sigma_{12}=0$ in this case as well.) That is, the curve of intersection between equations (15) and (17) must lie on the $\sigma_{11}-\sigma_{22}$ plane, which is impossible unless either the curve degenerates to a point, which means no stress changes are possible; or $\sigma_{12}\equiv\sigma_{12}^+\equiv 0$, which also leads to no possible stress changes because then equations (15) and (17) can only be satisfied by constant σ_{11} and σ_{22}.

Therefore, we have proved that there is no common intersecting curve for the three surfaces of equations (15), (17), (18), and hence that there is no stress path that can be followed to produce an *elastic-plastic* stress jump across a propagating surface in the material.

One question that may arise is: could equation (18) pass through a *part* of a curve of intersection of (15) and (17)? This would require that at some points (at least those points at which the curve and the surface begin to separate) the tangential direction of the curve or the normal direction of the surface or their derivatives are discontinuous. This is impossible for the nice equations (15), (17), and (18) (with the exception of the vertices of the cones, but evidently the center of (17) does not lie on (15)).

We have proved that there can be no elastic-plastic shock waves in a material satisfying the Huber-Mises yield condition and associated flow rule, but the possibility of shock waves in the plane-stress case cannot generally be ruled out. In fact, for a yield condition with a linear portion (possible for a single crystal, for example), it is easy to show that an elastic-plastic shock wave is possible. When the yield condition equation (9) has a linear portion in (σ_{11}, σ_{22}, σ_{12}) space, (11) becomes an equation involving the coefficients of the stresses in equation (9), while equation (13) remains unchanged. In most situations, equations (9) and (13) intersect on a curve in (σ_{11}, σ_{22}, σ_{12}) space; equation (11) merely specifies the possible elastic-plastic shock speeds in terms of the coefficients in equation (9), the elastic moduli, and the material density.

It should be pointed out that for a shock wave to exist at one of the permissible speeds derived above, it is necessary that there be a jump in ϵ_{33}, i.e., necking must occur and move with the shock wave to be compatible with the plane stress requirement that $\sigma_{33} \equiv 0$. If there is no necking present, the problem becomes much easier.

When there is no necking present, in addition to the equations we have discussed ((9), (11), and (13) or their simpler forms when there is no plastic straining during shock passage), we can use restrictions involving ϵ_{33} to eliminate shock waves for most plane-stress cases. One such restriction is the plastic incompressibility condition which is true for most of the materials under discussion, and can be expressed as (Drugan and Shen, 1987)

$$<\sigma_{11}> - \rho V^2 M_{jj\gamma\delta} <\sigma_{\gamma\delta}> = 0 \tag{20}$$

where here we have only three nonzero stresses. This equation can be rewritten as

$$[1 - \rho V^2(M_{1111} + M_{2211} + M_{3311})] <\sigma_{11}> - \rho V^2 (M_{1122} + M_{2222} + M_{3322}) <\sigma_{22}> - 2\rho V^2 (M_{1112} + M_{2212} + M_{3312}) <\sigma_{12}> = 0 \tag{21}$$

which is a linear relation between σ_{11}, σ_{22}, and σ_{12}. When equation (21) is substituted into (13), the latter takes the form

$$A <\sigma_{22}>^2 + B <\sigma_{22}> <\sigma_{12}> + C <\sigma_{12}>^2 = 0 \tag{22}$$

where A, B, C are constants whose complicated forms are omitted here. Now, all the restrictions imposed by equations (13) and (21) are maintained by enforcing (22) and (21). Equation (22) requires σ_{22} to be linear in σ_{12}. As we reviewed earlier, the yield condition (equation (9)) must be satisfied throughout the shock transition. Thus, linearity of equations (21) and (22) requires that $< \boldsymbol{\sigma} > = \mathbf{0}$ unless (9) contains a straight-line segment in stress space that is representable by some combination of (21) and (22). *The conclusion is that except for this special situation, dynamic plane-stress deformation of elastic-ideally plastic solids cannot sustain propagating shocks unless they move at an elastic wave speed or they move with a neck for which equations (9), (11), and (13) happen to have a common intersection on a continuous set of points.* It should be pointed out that equation (11) does not provide an additional restriction if the yield condition is the linear surface defined by equations (21) and (22) in stress space.

3 Plane-Strain Case

The plane-strain case is more complicated. Drugan and Shen (1987) considered it for fully incompressible materials; here, we treat the more general situation involving elastic compressibility. All the restrictions derived in the general *three-dimensional* analysis of Drugan and Shen (1987) apply here, but the restrictions now involve only four nonzero stress components: σ_{11}, σ_{22}, σ_{33}, and σ_{12}, since $\sigma_{13} \equiv \sigma_{23} \equiv 0$ by assuming symmetry of the material and deformation about all planes x_3 = constant. These restrictions are:

(a) yield condition:

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}) = 0 \tag{23}$$

(assumed to depend symmetrically on σ_{12} and σ_{21});

(b) plastic incompressibility:

$$<\sigma_{11}> - \rho V^2 M_{jjkl} <\sigma_{kl}> = 0; \tag{24}$$

(c) Condition 1 (as described in Section 1):

$$<\sigma_{1i}> <\sigma_{1i}> - \rho V^2 <\sigma_{ij}> M_{ijkl} <\sigma_{kl}> = 0; \tag{25}$$

(d) Condition 2 (also described in Section 1):

$$\rho^3(c_1^2 - V^2)(c_2^2 - V^2)(c_3^2 - V^2) - \left(\frac{\partial f}{\partial \sigma_{ij}} g_{ij}\right)^{-1} \Big\{\eta_1(c_2^2 - V^2)(c_3^2 - V^2) + \eta_2(c_3^2 - V^2)(c_1^2 - V^2) + \eta_3(c_1^2 - V^2)(c_2^2 - V^2)\Big\} = 0, \tag{26}$$

where $g_{ij} \equiv M^{-1}_{ijkl} (\partial f / \partial \sigma_{kl}) = g_{ji}$, c_i are the elastic wave speeds, and $\eta_i \equiv (q_j^{(i)} g_{j1})^2$, *where* $\mathbf{q}^{(i)}$ are the eigenvectors of M^{-1}_{j1k1} corresponding to c_i.

In most cases, the four equations (23)–(26) should be independent of one another, with possible exceptions occurring at some isolated points in (σ_{11}, σ_{22}, σ_{33}, σ_{12}) (hyper) space, and hence there can be no admissible stress path to facilitate an elastic-plastic moving strong discontinuity. The independence of these equations can be reasoned by observing that each of them describes a different aspect of the mechanical properties of the material during shock wave passage. It is very difficult mathematically to prove the independence of these equations for a general material (especially if the yield condition form is unspecified). Thus, we intend first to approach the problem for an isotropic material with a Huber-Mises yield condition. This material model is very important for many theoretical investigations. Also, the proof for one specific type of material should make it clear that these four equations are not naturally mutually dependent.

When the material is isotropic and obeys the Huber-Mises yield condition, we have

$$M^{-1}_{ijkl} = G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda \delta_{ij}\delta_{kl} \tag{27}$$

and

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}) \equiv \frac{1}{3}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11}) + \frac{1}{2}(\sigma_{12}^2 + \sigma_{21}^2) - k^2 = 0. \tag{28}$$

Now, the elastic wave speeds are $c_1^2 = (2G + \lambda)/\rho$ and $c_2^2 = c_3^2 = G/\rho$, so that equations (24), (25) and (26) become

$$\left(1 - \frac{\rho V^2}{2G + 3\lambda}\right) <\sigma_{11}> - \frac{\rho V^2}{2G + 3\lambda} (<\sigma_{22}> + <\sigma_{33}>) = 0, \tag{29}$$

$$<\sigma_{11}>^2 + <\sigma_{12}>^2 - \rho V^2 \left\{\frac{1}{G(2G + 3\lambda)} [(G + \lambda)(<\sigma_{11}>^2 + <\sigma_{22}>^2 + <\sigma_{33}>^2) - \lambda(<\sigma_{11}><\sigma_{22}> + <\sigma_{22}><\sigma_{33}> + <\sigma_{33}><\sigma_{11}>)] + \frac{1}{G} <\sigma_{12}>^2\right\} = 0, \tag{30}$$

$$(G - \rho V^2)(2G + \lambda - \rho V^2)k^2 - \frac{1}{9} G(G - \rho V^2)(4\sigma_{11}^2 - 4\sigma_{11}\sigma_{22} - 4\sigma_{11}\sigma_{33} + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{22}\sigma_{33}) - G(2G + \lambda - \rho V^2)\sigma_{12}^2 = 0. \tag{31}$$

Here we have four unknowns in four equations (28)–(31). It is easier to deal with only three unknowns in three equations by eliminating one of the variables. Because we can do this by linear combination and direct substitution, the properties of the possible intersecting curves do not change.

Substitution of equation (29) into (30) serves to eliminate σ_{11}:

$$\frac{\rho V^2}{G(2G + 3\lambda - \rho V^2)^2} \{ -[(\rho V^2)^2 - 3(G + \lambda)\rho V^2 + (G + \lambda) \times (2G + 3\lambda)](<\sigma_{22}>^2 + <\sigma_{33}>^2) + [-(\rho V^2)^2 + 2G\rho V^2 + \lambda(2G + 3\lambda)] <\sigma_{22}><\sigma_{33}> \} + \left(1 - \frac{\rho V^2}{G}\right) <\sigma_{12}>^2 = 0. \tag{32}$$

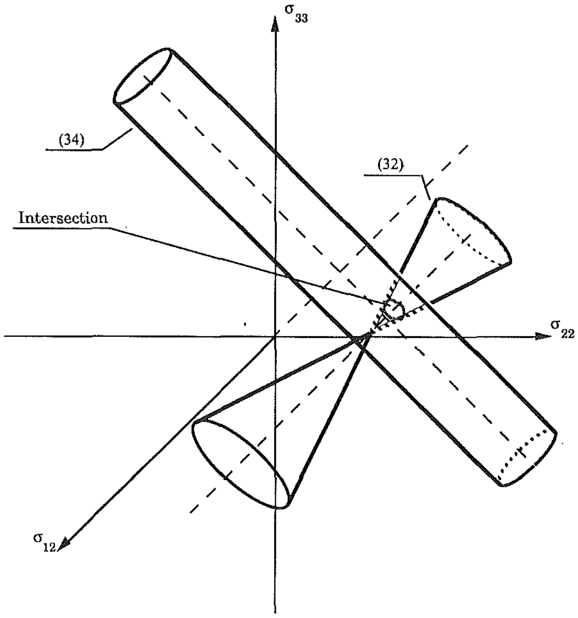

Fig. 3 Intersection of the surfaces representing two of the trans-shock stress path restrictions for the plane-strain case

Next, 3 × equation (31) + $4G(G-\rho V^2)$ × equation (28) gives another relation without σ_{11}:

$$G(G-\rho V^2)(\sigma_{22}^2+\sigma_{33}^2-2\sigma_{22}\sigma_{33})-G(2G+3\lambda+\rho V^2)\sigma_{12}^2$$
$$+(G-\rho V^2)(2G+3\lambda-3\rho V^2)k^2=0. \quad (33)$$

One more equation may be obtained by substituting equation (29) into (31) to eliminate σ_{11} and by rearranging:

$$(G-\rho V^2)(2G+\lambda-\rho V^2)k^2-\frac{1}{9}G(G-\rho V^2)\ \{(1-2h)(<\sigma_{22}>$$
$$+<\sigma_{33}>)-[2\sigma_{11}^+-(\sigma_{22}^+ +\sigma_{33}^+)]\}^2-G(2G+\lambda-\rho V^2)\sigma_{12}^2=0$$
$$(34)$$

where

$$h=\frac{\rho V^2}{2G+3\lambda-\rho V^2}\ . \quad (35)$$

Equations (32), (33) and (34) are the restrictions on the three variables σ_{22}, σ_{33}, and σ_{12}; by including any one of equations (28)–(31), we get four equations (e.g., (31)–(34)) that can replace equations (28)–(31) without changing the subspace defined by them. Thus, we can use equations (31)–(34) to identify the intersection problem.

Equation (32) is an elliptic cone centered at $(\sigma_{22}^+, \sigma_{33}^+, \sigma_{12}^+)$ in $(\sigma_{22}, \sigma_{33}, \sigma_{12})$ subspace; when $\rho V^2 < (2G/3)+\lambda$, the axis of the cone is parallel to the σ_{12} coordinate; when $(2G/3)+\lambda < \rho V^2 < G$ (which requires Poisson's ratio <0.125), the axis of the cone is perpendicular to the σ_{12}-coordinate and parallel to the line bisecting the σ_{22} and σ_{33} directions. Equation (34) is an elliptic cylinder whose axis lies in the $\sigma_{22}-\sigma_{33}$ plane, and perpendicular to the line bisecting the σ_{22} and σ_{33} directions (see Fig. 3, where equation (33) is not drawn).

When $\rho V^2 < (2G/3)+\lambda$, if we exchange the positions of $<\sigma_{22}>$ and $<\sigma_{33}>$, equations (32) and (34) remain unchanged. Therefore, exchanging $<\sigma_{22}>$ and $<\sigma_{33}>$ does not alter the intersecting curve of equation (32) and (34). Besides, such an exchange will remain on the same curve if there is more than one intersection, as can be seen from the geometry (also note that there are points on the intersection other than $(\sigma_{22}^+, \sigma_{33}^+, \sigma_{12}^+)$ that pass through the $\langle\sigma_{22}\rangle = \langle\sigma_{33}\rangle$ plane).

If equation (33) passes through one of the curves defined by (32) and (34), it must stay on the same curve when an exchange of $<\sigma_{22}>$ and $<\sigma_{33}>$ is made in (33).

Let us rewrite equation (33) as

$$G(G-\rho V^2)(\sigma_{22}-\sigma_{33})^2+p=0$$

where we use p to represent the last two terms of (33); the above equation can also be written as

$$G(G-\rho V^2)(<\sigma_{22}>-<\sigma_{33}>+\sigma_{22}^+-\sigma_{33}^+)^2+p=0. \quad (36)$$

When $<\sigma_{22}>$ and $<\sigma_{33}>$ are exchanged, we obtain

$$G(G-\rho V^2)(-<\sigma_{22}>+<\sigma_{33}>+\sigma_{22}^+-\sigma_{33}^+)^2+p=0. \quad (37)$$

Since both equations (36) and (37) pass through the same curve, their linear combination must include that curve as well. Subtraction of equation (37) from (36) gives

$$G(G-\rho V^2)(\sigma_{22}^+-\sigma_{33}^+)(<\sigma_{22}>-<\sigma_{33}>)=0. \quad (38)$$

Here we do not consider $\rho V^2=G$. Then equation (38) depicts a plane

$$<\sigma_{22}>-<\sigma_{33}>=0. \quad (39)$$

($\sigma_{22}^+ - \sigma_{33}^+ = 0$ will also lead to $\sigma_{22}-\sigma_{33}=0$, and hence $<\sigma_{22}>-<\sigma_{33}>=0$ by an argument similar to that adopted in the plane-stress case following equation (19)). As can be seen geometrically in Fig. 3, the curve produced by the intersection of equations (32) and (34) cannot be on such a plane. This can also be easily seen from equation (33) because $<\sigma_{22}>-<\sigma_{33}>=0$ leads to constant σ_{12}, which further leads to constant σ_{22} and σ_{33} through equations (34) and (39).

Thus, we have proved for $\rho V^2 < (2G/3)+\lambda$ that there is no common intersection curve of these equations. For $(2G/3)+\lambda \le \rho V^2 < G$, a similar scheme can be adopted by utilizing the symmetry of equations (33) and (34) with respect to the $\sigma_{22}-\sigma_{33}$ plane. Since now $\sigma_{22}=\sigma_{33}$ is the plane of symmetry of equation (33) that does not touch the hyperbolic cylinder, one can subtract versions of (32) with $\pm\sigma_{12}$ to eliminate the possibility of common intersection curves among the three equations. The overall conclusion is: The solution space consists only of discrete points, and hence (in the four-dimensional stress space) there can be no admissible stress path to produce a moving elastic-plastic stress jump.

The above proof may not be valid if a surface were to pass through only part of a curve of intersection between the other two surfaces. That would require at least two points on the curve at which either the curve or the surface has discontinuous derivatives. This can not happen for the well-behaved equations (28) through (31) or (31) through (34).

As observed earlier, the possibility of shock waves cannot generally be ruled out. With a yield condition possessing a linear portion, for example, it is easy to show that an elastic-plastic shock wave is possible. When yield condition equation (23) is linear in $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12})$ space, (26) becomes an equation of the coefficients of the stresses in equation (23), while (24) and (25) remain unchanged. We can then substitute equation (23) into (24) and (25) to eliminate σ_{11}. In most situations, an intersection between those two surfaces in $(\sigma_{22}, \sigma_{33}, \sigma_{12})$ subspace will exist, and equation (26) will merely specify possible elastic-plastic shock wave speeds in terms of coefficients of stresses in equation (23), elastic moduli, and the material density.

4 Generalization to Anisotropic Materials

In the previous sections we have proved that in the plane-stress and plane-strain cases, there can be no propagating jumps in stresses (except at elastic wave speeds) for isotropic materials with a Huber-Mises yield surface and associated flow rule. To obtain the same conclusion for a general material seems to be too difficult mathematically, but generalizing what we have done thus far is possible.

Plane Stress. For the case of plane stress, if the yield condition can be written as

$$f \equiv g(\sigma_{11}, \sigma_{22}) + C(\sigma_{12}^2 + \sigma_{21}^2) - k^2 = 0 \tag{40}$$

where C is a constant and the equation is a closed convex surface, the same proof we applied in Section 2 is still valid *as long as the intersection of the yield condition and the new equation (17) contains σ_{12} and $-\sigma_{12}$ on the same curve.* A typical example is a yield condition of the form

$$f \equiv A(\sigma_{11}^2 + \sigma_{22}^2) + B\sigma_{11}\sigma_{22} + C(\sigma_{12}^2 + \sigma_{21}^2) - k^2 = 0 \tag{41}$$

which forms a closed (convex) surface, where A, B, C are constants. An argument similar to that in Section 2 goes through because, in this case, if we cannot prove that σ_{12} and $-\sigma_{12}$ are on the same curve, we can still use the property that when $-\sigma_{11}$ and $-\sigma_{22}$ are substituted for σ_{11} and σ_{22} simultaneously, they stay on the same curve. This facilitates the proof because equation (18) cannot be symmetrical about the σ_{12}-coordinate (except when $\sigma_{12}^+ = 0$, in which case if a jump is assumed, a $\sigma_{12} \neq 0$ on the path from σ^+ to σ^- can be chosen as a new σ_{12}^+, as discussed earlier). This strategy may also be applied to problems with elastic anisotropy, having the property that $M_{1112} = M_{2212} = 0$ (so that coupling between σ_{12} and either σ_{11} or σ_{22} is precluded in the discontinuity restrictions).

Plane Strain. For the plane-strain case, it is a special situation that our proof can go through to reduce the four-dimensional problem of independence to three dimensions without losing certain symmetry properties (interchangeability of $<\sigma_{22}>$ and $<\sigma_{33}>$ in two of the restrictions) that are crucial to our method of proof. But we still have space for some generalization. The following seems to be the most general quadratic yield condition to which our method of proof applies:

$$f \equiv A(\sigma_{11}^2 + \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11} - \sigma_{11}\sigma_{22}) + B(\sigma_{22}^2 + \sigma_{33}^2 - 2\sigma_{22}\sigma_{33}) + C(\sigma_{12}^2 + \sigma_{21}^2) - k^2 = 0, \tag{42}$$

which forms a closed (convex) surface in stress space and satisfies plastic incompressibility, where A, B, C are constants; note that the Huber-Mises yield condition is a special case of equation (42) when $A = B = \frac{1}{3}$, $C = \frac{1}{2}$. For this form of yield condition, we can directly use equation (32) and the new equation (33); a new equation (34) is obtained by first linearly combining the new equations (28) and (31) so that the σ_{22}^2, σ_{33}^2 and $\sigma_{22}\sigma_{33}$ terms appear in the form $(\sigma_{22} + \sigma_{33})^2$, then eliminating σ_{11} through the substitution of equation (29). It can be shown that the same proof applied in Section 3 is still valid. The most obvious example is the case $A = B$, for which the proof of Section 3 applies directly.

Another possible way to prove the independence of the equations when the yield condition is quadratic (and the elastic moduli may be anisotropic) would be to reduce the problem to three quadratic equations. If two of them intersect on one curve which is not on a plane, the problem then becomes a linear independence one.

5 Discussion

Although we have not given a proof for general stable elastic-ideally plastic materials, our "plane-stress" and "plane-strain" analyses indicate that there can be no strong discontinuities traveling at speeds other than the elastic wave speeds unless there is an unusual combination of the elastic moduli and yield condition that make the aforementioned equations dependent on a continuous set of points.

In Drugan and Shen (1987), we discussed the special case of antiplane shear deformation, showing that there can be no strong discontinuity wave with speed other than one of the elastic wave speeds unless the yield condition has a linear portion in σ_{13}, σ_{23} space (equation (3.34) of Drugan and Shen, 1987). We can prove further than when the yield condition coincides with equation (3.34) of Drugan and Shen (1987), Condition 2 will be naturally satisfied. Therefore, for the antiplane shear case, Condition 2 does not provide an additional restriction when the yield surface is that linear one. But when the yield surface is nonlinear, the condition is meaningful (although redundant in the simple antiplane strain case for eliminating strong discontinuity waves); in fact, the derivation in the Appendix of Drugan and Shen (1987) makes use of all these equations.

For the incompressible plane-strain case, Drugan and Shen (1987) reached a similar conclusion that there can be no propagating (elastic-plastic) stress jumps unless the yield condition contains a special straight line segment (defined by equations (3.42) and (3.45) in Drugan and Shen, 1987). We can prove that if the yield condition is indeed a linear one that contains the line segment defined by those two equations, equation (26) here will be naturally satisfied. In fact, for all the special cases mentioned in this work, an elastic-plastic shock wave is possible whenever there is a linear portion on the yield surface. The difference is: for antiplane and incompressible plane-strain cases (and plane stress without necking), a linear portion on a yield surface is a must for existence of an elastic-plastic shock wave; whereas for plane stress and compressible plane strain, other possibilities for elastic-plastic shock waves, though not very likely, have not been eliminated.

We take this opportunity to note that equation (3.45) in Drugan and Shen (1987) was printed incorrectly; it should read:

$$\begin{aligned}(M_{1111} + M_{3333} - 2M_{1133}) <s_{11}>^2 + 2(M_{1122} + M_{3333} - M_{1133} \\ - M_{2233}) <s_{11}> <s_{22}> + 4(M_{1112} - M_{3312}) <s_{11}> <s_{12}> \\ + (M_{2222} + M_{3333} - 2M_{2233}) <s_{22}>^2 + 4(M_{2212} - M_{3312}) \\ \times <s_{22}> <s_{12}> + \left(4M_{1212} - \frac{1}{\rho V^2}\right) <s_{12}>^2 = 0,\end{aligned} \tag{43}$$

or in more concise form, having employed equation (3.41) of Drugan and Shen (1987):

$$\begin{aligned}(M_{1111} - M_{1133}) <s_{11}>^2 + (2M_{1122} - M_{1133} - M_{2233}) <s_{11}> \\ \times <s_{22}> + (4M_{1112} - 2M_{3312}) <s_{11}> <s_{12}> + (M_{2222} - M_{2233}) \\ \times <s_{22}>^2 + (4M_{2212} - 2M_{3312}) <s_{22}> <s_{12}> \\ + \left(4M_{1212} - \frac{1}{\rho V^2}\right) <s_{12}>^2 = 0.\end{aligned} \tag{44}$$

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