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Relationship between certain classes of life distributions and some stochastic orderings

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Abstract. We consider distribution functions (dfs) in NBU, NBUE, NBUC and NBUT classes of life distributions and study the stochastic orderings of the associated random variables (rvs), their equilibrium and residual life rvs at fixed and at random times. These results offer more insight into the structure of these classes.

1. Introduction

We start with a few definitions which may be found in Barlow and Proschan [1], Shaked and Shanthikumar [2], Szekli [3] and Yue and Cao [4]. Let X and Y be two nonnegative independent random variables (rvs) with respective distribution functions (dfs) F and G, survival functions (sfs) $\overline{F}$ and $\overline{G}$. For $t \geq 0$, let X_t denote the excess or residual or remaining life rv at time t with df $F_t(x) = P(X_t \leq x) = 0,\ x \leq 0$, and $F_t(x) = \frac{F(t+x)-F(t)}{\overline{F}(t)} = 1 - \frac{\overline{F}(t+x)}{\overline{F}(t)},\ x > 0,\ t > 0$, and sf $\overline{F}_t(x) = 1,\ x \leq 0$ and $\overline{F}_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)},\ x > 0,\ t > 0$.

A few definitions:

1. X is said to be stochastically smaller than Y, denoted by $X \leq_{st} Y$, if $\overline{F}(t) \leq \overline{G}(t),\ t \geq 0$.
2. X is smaller than Y in convex order, denoted by $X \leq_c Y$, if $\int_t^\infty \overline{F}(u)du \leq \int_t^\infty \overline{G}(u)du,\ t \geq 0$.
3. X is said to be smaller than Y in the total time on test transform (ttt) order, denoted by $X \leq_{ttt} Y$, if $\int_0^{F^{-1}(p)} \overline{F}(u)du \leq \int_0^{G^{-1}(p)} \overline{G}(u)du,\ p \in (0,1)$, where $F^{-1}(p)$ denotes the p-th quantile of F.
4. X is said to be be smaller than Y with respect to the Laplace-Stieltjes transform order, denoted by $X <_L Y$, if

$$E(e^{-sX}) \geq E(e^{-sY}),\ s \geq 0 \quad \Leftrightarrow \quad \int_0^\infty e^{-su} dF(u) \geq \int_0^\infty e^{-su} dG(u),\ s \geq 0$$
$$\Leftrightarrow \quad \int_0^\infty e^{-su}\overline{F}(u)du \leq \int_0^\infty e^{-su}\overline{G}(u)du,\ s \geq 0.$$

5. F is said to be New Better than used or NBU if

$$\overline{F}(s+t) \leq \overline{F}(s)\overline{F}(t),\ s \geq 0,\ t \geq 0.$$

6. F is said to be New better than used in expectation or NBUE if
   (a) F has finite mean μ_F,

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(b) $\overline{F}(t) \geq \frac{1}{\mu_F} \int_t^\infty \overline{F}(u)du$, $t \geq 0$.

7. F is said to be New Better than used in Laplace order or NBUL if $X_t \leq_L X$, $t \geq 0$, or equivalently, F is NBUL iff

$$\int_0^\infty e^{-su}\overline{F}(t+u)du \leq \overline{F}(t) \int_0^\infty e^{-su}\overline{F}(u)du, \ t \geq 0.$$

8. F is said to be New Better than used in convex order or NBUC if $X_t \leq_{icx} X$, $t \geq 0$, where $Y \leq_{icx} Z$ means that Y is bounded by Z in increasing convex order. Equivalently, F is NBUC iff

$$\int_x^\infty \overline{F}(t+u)du \leq \overline{F}(t) \int_x^\infty \overline{F}(u)du, \ x \geq 0, \ t \geq 0.$$

9. F is said to be New Better than Used in total time on test order or NBUT if

$$\int_0^{F_t^{-1}(p)} \overline{F}(t+u)du \leq \overline{F}(t) \int_0^{F^{-1}(p)} \overline{F}(u)du, \ 0 < p < 1, \ t \geq 0.$$

The dual stochastic orders/classes are defined by reversing the inequalities.

2. Main Results

Let X_Y denote the residual life rv at random time Y. Then X_Y has sf $\overline{F}_{X_Y}(t) = P(X - Y > t | X > Y) = 0$, $t \leq 0$, and

$$\overline{F}_{X_Y}(t) = \frac{\int_0^\infty \overline{F}(t+y)dG(y)}{\int_0^\infty \overline{F}(y)dG(y)}, \ t > 0.$$

Yue and Cao [4] prove that F is NBUL (NWUL) iff $X_Y \leq_L (\geq_L) X$ for any rv Y and also that F is NBUL (NWUL) iff $X_Y \leq_{st} (\geq_{st}) X$ for any exponential rv Y. We look at some ordering relations satisfied by these rvs.

Theorem 2.1. *F is NBU (NWU) iff $X_Y \leq_{st} (\geq_{st}) X$ where Y is any nonnegative rv.*

Proof. Let Y be an arbitrary nonnegative rv with df G. Then

$$\begin{aligned} F \text{ is NBU} \ &\Rightarrow \ \overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y), \ x \geq 0, \ y \geq 0 \\ &\Rightarrow \ \int_0^\infty \overline{F}(x+y)dG(y) \leq \int_0^\infty \overline{F}(x)\overline{F}(y)dG(y), \ x \geq 0 \\ &\Rightarrow \ \frac{\int_0^\infty \overline{F}(x+y)dG(y)}{\int_0^\infty \overline{F}(y)dG(y)} \leq \overline{F}(x), \ x \geq 0 \\ &\Rightarrow \ \overline{F}_{X_Y}(x) \leq \overline{F}(x), \ x \geq 0 \ \Rightarrow X_Y \leq_{st} X. \end{aligned}$$

Conversely, if $X_Y \leq_{st} X$ for all nonnegative rvs Y, taking $Y \equiv t$, $t \geq 0$, we get

$$X_t \leq_{st} X \Rightarrow \overline{F}_t(x) \leq \overline{F}(x), \ x \geq 0 \ \Rightarrow \frac{\overline{F}(t+x)}{\overline{F}(t)} \leq \overline{F}(x), \ x \geq 0 \ \Rightarrow F \text{ is NBU}.$$

Hence the proof. □

The following result gives relationship between some of these stochastic orders.

Theorem 2.2. *If $X_Y \leq_{st} X$, then: (i) $X_Y \leq_c X$, (ii) $X_Y \leq_L X$, and (iii) $X_Y \leq_{ttt} X$.*

Proof.

$$(i)\ X_Y \le_{st} X \Rightarrow \overline{F}_{X_Y}(x) \le \overline{F}(x),\ x \ge 0 \Rightarrow \int_t^\infty \overline{F}_{X_Y}(x)dx \le \int_t^\infty \overline{F}(x)dx,\ t \ge 0 \Rightarrow X_Y \le_c X.$$

$$\begin{aligned}(ii)\ X_Y \le_{st} X \quad &\Rightarrow \quad e^{-sx}\overline{F}_{X_Y}(x) \le e^{-sx}\overline{F}(x),\ x \ge 0,\\ &\Rightarrow \quad \int_0^\infty e^{-sx}\overline{F}_{X_Y}(x)dx \le \int_0^\infty e^{-sx}\overline{F}(x)dx,\ s \ge 0\\ &\Rightarrow \quad X_Y \le_L X.\end{aligned}$$

$$\begin{aligned}(iii)\ X_Y \le_{st} X \quad &\Rightarrow \quad \int_0^{F^{-1}_{X_Y}(p)} \overline{F}_{X_Y}(x)dx \le \int_0^{F^{-1}_{X_Y}(p)} \overline{F}(x)dx,\ 0 < p < 1\\ &\Rightarrow \quad \int_0^{F^{-1}_{X_Y}(p)} \overline{F}_{X_Y}(x)dx \le \int_0^{F^{-1}(p)} \overline{F}(x)dx \text{ since } F^{-1}_{X_Y}(p) \le F^{-1}(p), 0 < p < 1,\\ &\Rightarrow \quad X_Y \le_{ttt} X.\end{aligned}$$

□

Theorem 2.3. *If F is NBU, then: (i) $X_Y \le_{st} X$, (ii) $X_Y \le_c X$, (iii) $X_Y \le_L X$, and (iv) $X_Y \le_{ttt} X$.*

Proof. (i) follows from Theorem 2.1 and the rest follow from Theorem 2.1 and Theorem 2.2. □

Theorem 2.4. *F is $NBUC(NWUC)$ iff $X_Y \le_c (\ge_c) X$ where Y is any nonnegative rv.*

Proof.

$$\begin{aligned}F \text{ is NBUC} \quad &\Rightarrow \quad \int_x^\infty \overline{F}(u+y)du \le \overline{F}(y)\int_x^\infty \overline{F}(u)du,\ x \ge 0,\\ &\Rightarrow \quad \int_0^\infty \int_x^\infty \overline{F}(u+y)dudG(y) \le \int_0^\infty \overline{F}(y)dG(y)\int_x^\infty \overline{F}(u)du,\ x \ge 0,\\ &\Rightarrow \quad \int_x^\infty \frac{\int_0^\infty \overline{F}(u+y)dG(y)}{\int_0^\infty \overline{F}(y)dG(y)}du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0,\\ &\Rightarrow \quad \int_x^\infty \overline{F}_{X_Y}(u)du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0, \Rightarrow X_Y \le_c X.\end{aligned}$$

Conversely, if $X_Y \le_c X$ holds for all nonnegative rvs Y, taking $Y \equiv t$, $t \ge 0$,

$$\begin{aligned}X_t \le_c X \quad &\Leftrightarrow \quad \int_x^\infty \overline{F}_t(u)du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0,\\ &\Leftrightarrow \quad \int_x^\infty \frac{\overline{F}(t+u)}{\overline{F}(t)}du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0,\\ &\Leftrightarrow \quad \int_x^\infty \overline{F}(t+u)du \le \overline{F}(t)\int_x^\infty \overline{F}(u)du,\ x \ge 0, \Leftrightarrow F \text{ is NBUC.}\end{aligned}$$

Hence the proof. □

Theorem 2.5. *(i) $X_t \le_{st} X \Leftrightarrow F$ is NBU.*

(ii) $X_t \le_c X \Leftrightarrow F$ is NBUC.

(iii) $X_t \le_L X \Leftrightarrow F$ is NBUL.

(iv) $X_t \le_{ttt} X \Leftrightarrow F$ is NBUT.

Proof.

$$
\begin{aligned}
(i)\ X_t \le_{st} X &\Leftrightarrow \overline{F}_t(x) \le \overline{F}(x),\ x \ge 0,\ t \ge 0\\
&\Leftrightarrow \frac{\overline{F}(t+x)}{\overline{F}(t)} \le \overline{F}(x),\ x \ge 0,\ t \ge 0\\
&\Leftrightarrow \overline{F}(t+x) \le \overline{F}(t)\overline{F}(x),\ x \ge 0,\ t \ge 0 \Leftrightarrow F \text{ is NBU.}
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ X_t \le_c X &\Leftrightarrow \int_x^\infty \overline{F}_t(u)du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0,\ t \ge 0,\\
&\Leftrightarrow \int_x^\infty \frac{\overline{F}(t+u)}{\overline{F}(t)}du \le \int_x^\infty \overline{F}(u)du,\ x \ge 0, t \ge 0,\\
&\Leftrightarrow \int_x^\infty \overline{F}(t+u)du \le \overline{F}(t)\int_x^\infty \overline{F}(u)du,\ x \ge 0, t \ge 0, \Leftrightarrow F \text{ is NBUC.}
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ X_t \le_L X &\Leftrightarrow \int_0^\infty e^{-sx}\overline{F}_t(x)dx \le \int_0^\infty e^{-sx}\overline{F}(x)dx,\ s \ge 0\\
&\Leftrightarrow \int_0^\infty e^{-sx}\frac{\overline{F}(t+x)}{\overline{F}(t)}dx \le \int_0^\infty e^{-sx}\overline{F}(x)dx,\ s \ge 0\\
&\Leftrightarrow \int_0^\infty e^{-sx}\overline{F}(t+x)dx \le \overline{F}(t)\int_0^\infty e^{-sx}\overline{F}(x)dx,\ s \ge 0 \Leftrightarrow F \text{ is NBUL.}
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ X_t \le_{ttt} X,\ t \ge 0 &\Leftrightarrow \int_0^{F_t^{-1}(p)} \overline{F}_t(x)dx \le \int_0^{F^{-1}(p)} \overline{F}(x)dx,\ t \ge 0,\ 0 < p < 1,\\
&\Leftrightarrow \int_0^{F_t^{-1}(p)} \frac{\overline{F}(t+x)}{\overline{F}(t)}dx \le \int_0^{F^{-1}(p)} \overline{F}(x)dx,\ t \ge 0,\ 0 < p < 1,\\
&\Leftrightarrow \int_0^{F_t^{-1}(p)} \overline{F}(t+x)dx \le \overline{F}(t). \int_0^{F^{-1}(p)} \overline{F}(x)dx,\ t \ge 0,\ 0 < p < 1,\\
&\Leftrightarrow F \text{ is NBUT.}
\end{aligned}
$$

□

The next few results are concerned about equilibrium distributions. Associated with the rv X with df F and mean μ_F, let X_1 denote the equilibrium rv with df $F_1(x) = \frac{1}{\mu_F}\int_0^x \overline{F}(t)dt$, $x \ge 0$, and $F_1(x) = 0$ otherwise. Then $\overline{F}_1(x) = \frac{1}{\mu_F}\int_x^\infty \overline{F}(t)dt$, $x \ge 0$.

Theorem 2.6. *If F is NBUE, then: (i) $X_1 \le_c X$, (ii) $X_1 \le_{ttt} X$, (iii) $X_1 \le_L X$.*

Proof.

$$
\begin{aligned}
(i)\ F \text{ is NBUE} &\Leftrightarrow \frac{1}{\mu_F}\int_t^\infty \overline{F}(x)dx \le \overline{F}(t),\ t \ge 0,\\
&\Leftrightarrow \overline{F}_1(t) \le \overline{F}(t),\ t \ge 0, \qquad (1)\\
&\Rightarrow \int_x^\infty \overline{F}_1(t)dt \le \int_x^\infty \overline{F}(t)dt,\ x \ge 0, \Rightarrow X_1 \le_c X.
\end{aligned}
$$

$$\begin{aligned}(ii)\ F \text{ is NBUE} &\Rightarrow \int_0^{F_1^{-1}(p)} \overline{F}_1(t)dt \le \int_0^{F_1^{-1}(p)} \overline{F}(t)dt,\ 0<p<1,\ \text{using (1)}\\ &\Rightarrow \int_0^{F_1^{-1}(p)} \overline{F}_1(t)dt \le \int_0^{F^{-1}(p)} \overline{F}(t)dt,\ \text{since } F_1^{-1}(p) \le F^{-1}(p)\\ &\Rightarrow X_1 \le_{ttt} X.\end{aligned}$$

$$\begin{aligned}(iii)\ F \text{ is NBUE} &\Rightarrow e^{-st}\overline{F}_1(t) \le e^{-st}\overline{F}(t),\ s\ge 0,\ t\ge 0,\\ &\Rightarrow \int_0^\infty e^{-st}\overline{F}_1(t)dt \le \int_0^\infty e^{-st}\overline{F}(t)dt, s\ge 0, \Rightarrow X_1 \le_L X.\end{aligned}$$

□

Theorem 2.7. *$X_1 \le_{st} X \Leftrightarrow F$ is NBUE.*

Proof.

$$X_1 \le_{st} X \Leftrightarrow \overline{F_1}(t) \le \overline{F}(t) \Leftrightarrow \frac{1}{\mu}\int_t^\infty \overline{F}(x)dx \le \overline{F}(t),\ t\ge 0, \Leftrightarrow F \text{ is NBUE.}$$

□

We next obtain a few results using the equilibrium distribution of the equilibrium distribution. These results show that the properties hold even if we iterate formation of equilibrium distributions. Let X_2 denote the equilibrium rv of X_1, with df $F_2(x) = \frac{1}{\mu(F_1)}\int_0^x \overline{F}_1(t)dt$, $x \ge 0$, and $F_2(x) = 0$ otherwise, where $\mu(F_1)$ is the mean of X_1 and F_1 is the df of X_1. Then $\overline{F}_2(x) = \frac{1}{\mu(F_1)}\int_x^\infty \overline{F}_1(t)dt$, $x \ge 0$.

Theorem 2.8. *$X_2 \le_{st} X_1 \Leftrightarrow F_1$ is NBUE.*

Proof.

$$X_2 \le_{st} X_1 \Leftrightarrow \overline{F}_2(t) \le \overline{F}_1(t) \Leftrightarrow \frac{1}{\mu(F_1)}\int_t^\infty \overline{F}_1(x)dx \le \overline{F}_1(t),\ t\ge 0, \Leftrightarrow F_1 \text{ is NBUE.}$$

□

Theorem 2.9. *Let F_1 be NBUE. Then: (i) $X_2 \le_c X_1$, (ii) $X_2 \le_{ttt} X_1$, (iii) $X_2 \le_L X_1$.*

Proof.

$$\begin{aligned}(i)\ F_1 \text{ is NBUE} &\Leftrightarrow \frac{1}{\mu(F_1)}\int_t^\infty \overline{F}_1(x)dx \le \overline{F}_1(t),\ t\ge 0,\\ &\Leftrightarrow \overline{F}_2(t) \le \overline{F}_1(t),\ t \ge 0, \qquad (2)\\ &\Rightarrow \int_x^\infty \overline{F}_2(t)dt \le \int_x^\infty \overline{F}_1(t)dt,\ x\ge 0,\\ &\Rightarrow X_2 \le_c X_1.\end{aligned}$$

$$\begin{aligned}(ii)\ F_1 \text{ is NBUE} &\Rightarrow \int_0^{F_2^{-1}(p)} \overline{F}_2(t)dt \le \int_0^{F_2^{-1}(p)} \overline{F}_1(t)dt,\ 0<p<1, \text{using (2)}\\ &\Rightarrow \int_0^{F_2^{-1}(p)} \overline{F}_2(t)dt \le \int_0^{F_1^{-1}(p)} \overline{F}_1(t)dt,\ 0<p<1, \text{since } F_2^{-1}(p) \le F_1^{-1}(p)\\ &\Rightarrow X_2 \le_{ttt} X_1.\end{aligned}$$

$$\begin{aligned}(iii)\ F_1 \text{ is NBUE} \ &\Rightarrow\ e^{-st}\overline{F}_2(t) \le e^{-st}\overline{F}_1(t),\ t \ge 0,\ s \ge 0, \text{using (2)}\\ &\Rightarrow\ \int_0^\infty e^{-st}\overline{F}_2(t)dt \le \int_0^\infty e^{-st}\overline{F}_1(t)dt,\ s \ge 0,\\ &\Rightarrow\ X_2 \le_L X_1.\end{aligned}$$

□

The next property is a consequence of the NBU property.

Theorem 2.10. *If $X \sim F$ is NBU, then $X_n \le_{st} X,\ n = 1,2,3,$ where $X_1 \sim F_1$ is the equilibrium rv derived from X, $X_n \sim F_n$ is the equilibrium rv derived from $X_{n-1}, n = 2,3$.*

Remark 2.11. *The statement appears to be true for other values of $n \ge 4$, but we have not been able to prove it using induction.*

Proof. We denote the mean of F by μ_F and that of F_k by $\mu_{F_k}, k = 1,2,3$.

$$\begin{aligned}\text{We have } X_1 \sim F_1(t) \ &\Leftrightarrow\ \overline{F}_1(t) = \frac{1}{\mu_F}\int_t^\infty \overline{F}(u)du,\ t \ge 0\\ &\Leftrightarrow\ \overline{F}_1(t) = \frac{1}{\mu_F}\int_0^\infty \overline{F}(s+t)ds, t \ge 0,\\ &\Rightarrow\ \overline{F}_1(t) \le \overline{F}(t),\ t \ge 0,\ \text{if } X \text{ is NBU.}\end{aligned}$$

$$\begin{aligned}\text{We have } X_2 \sim F_2(t) \ &\Leftrightarrow\ \overline{F}_2(t) = \frac{1}{\mu_{F_1}}\int_t^\infty \overline{F}_1(u)du,\ t \ge 0,\\ &\Leftrightarrow\ \overline{F}_2(t) = \frac{2\mu_F}{\mu_{2,F}}\int_t^\infty \overline{F}_1(u)du, t \ge 0,\\ &\quad\ \text{since } \mu_{F_1} = \frac{\mu_{2,F}}{2\mu_F},\ \text{where } \mu_{2,F} = \int_{u=0}^\infty u^2 dF(u),\\ &\Leftrightarrow\ \overline{F}_2(t) = \frac{2\mu_F}{\mu_{2,F}}\int_t^\infty \frac{1}{\mu_F}\int_u^\infty \overline{F}(x)dxdu, t \ge 0,\\ &\Leftrightarrow\ \overline{F}_2(t) = \frac{2}{\mu_{2,F}}\int_{x=t}^\infty \int_{u=t}^\infty du\overline{F}(x)dx, t \ge 0,\\ &\Leftrightarrow\ \overline{F}_2(t) = \frac{2}{\mu_{2,F}}\int_{x=t}^\infty (x-t)\overline{F}(x)dx, t \ge 0,\\ &\Leftrightarrow\ \overline{F}_2(t) = \frac{2}{\mu_{2,F}}\int_{u=0}^\infty u\overline{F}(t+u)du, t \ge 0,\\ &\Rightarrow\ \overline{F}_2(t) \le \frac{2}{\mu_{2,F}}\int_{u=0}^\infty u\overline{F}(t)\overline{F}(u)du,\ t \ge 0,\ \text{if } F \text{ is NBU,}\\ &\Rightarrow\ \overline{F}_2(t) \le \frac{2\overline{F}(t)}{\mu_{2,F}}\int_{u=0}^\infty u\overline{F}(u)du,\ t \ge 0,\\ &\Rightarrow\ \overline{F}_2(t) \le \frac{2\overline{F}(t)}{\mu_{2,F}}\frac{\mu_{2,F}}{2} = \overline{F}(t),\ t \ge 0, \Rightarrow X_2 \le_{st} X.\end{aligned}$$

The proof for the case $n = 3$ is similar and is omitted. □

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