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Plane-Strain Surface Waves in a Laminated Composite

Elastic wave propagation in plane strain in a periodically layered half space is considered, with the layering parallel to the bounding traction-free plane. Attention is focused on surface waves of the Rayleigh-type propagating parallel to the bounding plane. It is found that such waves are highly dispersive and that higher branches may be discontinuous.

Introduction

We are concerned here with a periodically layered half space which is formed by an infinite number of cells and bounded by a traction-free plane which is parallel to the layering, whose location with respect to the cell structure, however, is arbitrary. Each cell consists of two layers of different materials and different thicknesses, perfectly bonded together, which are, however, assumed to be homogeneous, isotropic, and linearly elastic.

In considering propagation of horizontally polarized shear waves (i.e., *SH* waves involving a state of antiplane strain) it has been recently shown [1–3] that there exist a new class of surface waves which, like Love waves, propagate along the free plane of the layered composite described in the foregoing. A method was presented for deriving these horizontally polarized shear-wave solutions from plane-wave solutions by applying Floquet's theorem to an infinite periodically layered medium [4].

It is the purpose of the present work to consider the propagation of compressional and vertically polarized shear waves (i.e., *P* and *SV* waves involving a state of plane strain). The work is based on results of recent analyses of wave solutions in plane strain obtained for an infinite layered medium, again by applying Floquet's theorem [5–7] and summarized in the next section.

Emphasis is placed on determining properties of Rayleigh-type surface waves propagating parallel to the bounding plane. For this purpose, an infinite periodically layered medium is considered first and the associated dispersion equation is briefly discussed. Next, the condition is imposed that some plane parallel to the layering be traction-free, which leads to the problem of reflection at a free boundary. Different types of waves in a half-space are discussed in the folowing section, and then narrowed down to a consideration of surface waves. The presentation concludes with an analysis of the properties of the lowest, and thus most interesting branch of the surface wave spectrum.

Periodically Layered Medium

The system considered consists of an infinite sequence of two alternating layers, each of which is linearly elastic, homogeneous, isotropic, and perfectly bonded to the adjoining layers. A unit cell is defined as the union of any two adjacent layers. The properties of two lamellae of a typical unit cell will be distinguished by using simple (unprimed) and primed notation: Lamé's elastic constants λ, μ; λ', μ', thicknesses $2h$, $2h'$ and densities ρ; ρ' (Fig. 1).

In the sequel, the numerical values assigned to these constants will be such that the phase velocities in the primed layer will be greater than in the unprimed layer. For this reason the primed layer will be called the fast layer, while the unprimed layer will be called the slow layer.

The most general two-dimensional motion in such a medium uncouples in motions in plane strain and in antiplane strain. Each of these has been discussed recently in [4–7], and the dispersion equations have been derived. In the following, steps leading to the derivation of the dispersion equation in plane strain are summarized first.

Let us consider a typical unit (*N*th) cell and write the equations of motion for each layer in its own local coor-

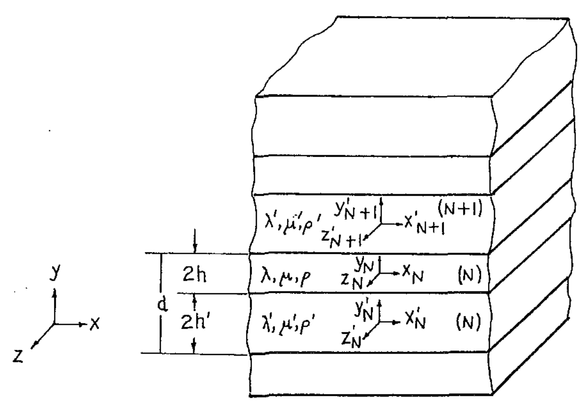

Fig. 1 ***N*th unit cell**

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dinates. For the unprimed Nth layer we have the displacements

$$u = u(x, y_N, t) \quad -h \le y_N \le h$$
$$v = v(x, y_N, t) \tag{1}$$

In terms of the displacement potential φ and ψ

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y_N}$$
$$v = \frac{\partial\varphi}{\partial y_N} - \frac{\partial\psi}{\partial x} \tag{2}$$

The equations of motion are

$$(\lambda+2\mu)\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y_N^2}\right) = \rho\ddot{\varphi}$$
$$\mu\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y_N^2}\right) = \rho\ddot{\psi} \tag{3}$$

and their solution may be written as

$$\varphi = \left(C_1\, e^{i\pi\beta y_N/2h} + C_2\, e^{-i\pi\beta y_N/2h}\right) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right] \quad P \text{ waves}$$
$$\psi = \left(C_3\, e^{i\pi\alpha y_N/2h} + C_4\, e^{-i\pi\alpha y_N/2h}\right) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right] \quad SV \text{ waves} \tag{4}$$

where

$$c_1 = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$$
$$c_2 = \left(\frac{\mu}{\rho}\right)^{1/2}$$
$$\Omega = \frac{2h}{\pi}\frac{\omega}{c_2}$$
$$\zeta = \frac{2h}{\pi}k_x$$
$$\alpha = \sqrt{\Omega^2 - \zeta^2} = \frac{2h}{\pi}\sqrt{(\omega/c_2)^2 - k_x^2}$$
$$\beta = \sqrt{l\Omega^2 - \zeta^2} = \frac{2h}{\pi}\sqrt{(\omega/c_1)^2 - k_x^2}$$
$$l = \frac{1-2\nu}{2(1-\nu)} \tag{5}$$

Here ω is the frequency; k_x is the wave number in the x direction; $C_1, \ldots, C_4$ are arbitrary constants; ζ is the nondimensional wave number in the x direction; and α and β are nondimensional wave numbers in the y direction for SV and P waves, respectively; ν is Poisson's ratio.

The expressions for displacements and the two needed components of stress are

$$u = \frac{i\pi}{2h}\left(\zeta C_1\, e^{i\pi\beta y_N/2h} + \zeta C_2\, e^{-i\pi\beta y_N/2h} + \alpha C_3\, e^{i\pi\alpha y_N/2h} - \alpha C_4\, e^{-i\pi\alpha y_N/2h}\right)\exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right]$$
$$v = \frac{i\pi}{2h}\left(\beta C_1\, e^{i\pi\beta y_N/2h} + \beta C_2\, e^{-i\pi\beta y_N/2h} - \zeta C_3\, e^{i\pi\alpha y_N/2h} - \zeta C_4\, e^{-i\pi\alpha y_N/2h}\right)\exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right]$$
$$\tau_{xy} = \frac{-\pi^2\mu}{4h^2}\left(2C_1\beta\zeta\, e^{i\pi\beta y_N/2h} - 2C_2\beta\zeta e^{-i\pi\beta y_N/2h} + C_3\varphi_1\, e^{i\pi\alpha y_N/2h} + C_4\varphi_1\, e^{-i\pi\alpha y_N/2h}\right)\exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right]$$
$$\tau_{yy} = \frac{-\pi^2\mu}{4h^2}\left(C_1\varphi_1\, e^{i\pi\beta y_N/2h} + C_2\varphi_1\, e^{-i\pi\beta y_N/2h} - 2\alpha\zeta C_3\, e^{i\pi\alpha y_N/2h} + 2\alpha\zeta C_4\, e^{-i\pi\alpha y_N/2h}\right)\exp\left[\frac{i\pi}{2h}(\zeta x - c_2\Omega t)\right] \tag{6}$$

where $\varphi_1 = \alpha^2 - \zeta^2$.

A similar procedure for the fast layer produces the corresponding solution for that layer. The expressions for the displacements and tractions are similar to those derived in the foregoing with the corresponding primed parameters replacing the unprimed ones. The continuity of the displacements and tractions across the interface of the two layers within a cell and across the interface of the two adjacent cells supply eight equations. Periodicity of the coefficients of the wave equations enable us to make use of the Floquet theorem and relate the solution in two adjacent cells. In this way the eight equations can be written only in terms of parameters of the Nth unit cell. These are eight equations in eight unknowns, which, in this case, are the arbitrary constants appearing in the general solution to the wave equations in each layer. They may be written in matrix notation as

$$\begin{bmatrix}
\zeta B_- & \zeta B_+ & \alpha A_- & -\alpha A_+ & -\zeta B'_+ & -\zeta B'_- & -\alpha' A'_+ & \alpha' A'_- \\
\beta B_- & -\beta B_+ & -\zeta A_- & -\zeta A_+ & -\beta' B'_+ & \beta' B'_- & \zeta A'_+ & \zeta A'_- \\
2\gamma\beta\zeta B_- & -2\gamma\beta\zeta B_+ & \gamma\varphi A_- & \gamma\varphi A_+ & -2\beta'\zeta B'_+ & 2\beta'\zeta B'_- & -\varphi' A'_+ & -\varphi' A'_- \\
\gamma\varphi B_- & \gamma\varphi B_+ & -2\gamma\alpha\zeta A_- & 2\gamma\alpha\zeta A_+ & -\varphi' B'_+ & -\varphi' B'_- & 2\alpha'\zeta A'_+ & -2\alpha'\zeta A'_- \\
\zeta B_+ & \zeta B_- & \alpha A_+ & -\alpha A_- & -\zeta B'_-\tau & -\zeta B'_+\tau & -\alpha' A'_-\tau & \alpha' A'_+\tau \\
\beta B_+ & -\beta B_- & -\zeta A_+ & -\zeta A_- & -\beta' B'_-\tau & \beta' B'_+\tau & \zeta A'_-\tau & \zeta A'_+\tau \\
2\gamma\beta\zeta B_+ & -2\gamma\beta\zeta B_- & \gamma\varphi A_+ & \gamma\varphi A_- & -2\beta'\zeta B'_-\tau & 2\beta'\zeta B'_+\tau & -\varphi' A'_-\tau & -\varphi' A'_+\tau \\
\gamma\varphi B_+ & \gamma\varphi B_- & -2\gamma\alpha\zeta A_+ & 2\gamma\alpha\zeta A_- & -\varphi' B'_-\tau & -\varphi' B'_+\tau & 2\alpha'\zeta A'_-\tau & -2\alpha'\zeta A'_+\tau
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C'_1 \\ C'_2 \\ C'_3 \\ C'_4 \end{bmatrix} = 0 \tag{7}$$

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where

$$A_{\pm}=\exp\,[\pm i\pi\alpha/2]$$
$$B_{\pm}=\exp\,[\pm i\beta/2]$$
$$\varphi=\alpha^2-\zeta^2$$
$$\tau=\exp\,[ip\eta]$$
$$A_{\pm}=\exp\,[\pm i\pi\alpha/2]$$
$$B_{\pm}=\exp\,[\pm i\pi\beta/2]$$
$$\varphi'=\alpha'^2-\zeta^2$$
$$\epsilon=\frac{h'}{h}$$
$$\gamma=\frac{\mu}{\mu'}$$
$$\sigma^2=\frac{\mu\rho'}{\mu'\rho}=\frac{c_2^2}{c_2'^2}$$
$$l'=\frac{1-2\nu'}{2(1-\nu')}$$
$$\eta=\frac{2h}{\pi}k_y$$
$$p=\pi(1+\epsilon)$$
$$\alpha'=\sqrt{\sigma^2\Omega^2-\zeta^2}=\frac{2h}{\pi}\sqrt{(\omega/c_2')^2-k_x^2}$$
$$\beta'=\sqrt{\sigma^2\Omega^2 l'-\zeta^2}=\frac{2h}{\pi}\sqrt{(\omega/c_1')^2-k_x^2} \tag{8}$$

where k_y is the Floquet wave number.

For a nontrivial solution to exist, the determinant of the system must be zero, thus leading to the dispersion equation. This determinant has been expanded and is given in closed form in reference [7] as

$$\begin{aligned}&2\tau^2[-L_1(c_\alpha c_\beta+c_{\alpha'}c_{\beta'})+L_2 s_\alpha s_\beta(1-c_{\alpha'}c_{\beta'})\\&+L_3 s_{\alpha'}s_{\beta'}(1-c_\alpha c_\beta)-L_5 c_\alpha s_\beta c_{\alpha'}s_{\beta'}-L_6 s_\alpha c_\beta s_{\alpha'}c_{\beta'}\\&-L_8 c_\alpha s_\beta s_{\alpha'}c_{\beta'}-L_9 s_\alpha c_\beta c_{\alpha'}s_{\beta'}+(L_{11}+L_{12})c_\alpha c_\beta c_{\alpha'}c_{\beta'}\\&+\frac{1}{2}L_{10}s_\alpha s_\beta s_{\alpha'}s_{\beta'}]+\tau(\tau^2+1)\times[-L_4(c_\alpha c_{\alpha'}+c_\beta c_{\beta'})\\&+L_5 s_\beta s_{\beta'}+L_6 s_\alpha s_{\alpha'}+L_7(c_\alpha c_{\beta'}+c_\beta c_{\alpha'})+L_8 s_\beta s_{\alpha'}\\&+L_9 s_\alpha s_{\beta'}]+2\tau^2 L_{11}+(\tau^4+1)L_{12}=0\end{aligned} \tag{9}$$

where

$$c_\alpha=\cos\pi\alpha$$
$$c_\beta=\cos\pi\beta$$
$$c_{\alpha'}=\cos\pi\alpha'\epsilon$$
$$c_{\beta'}=\cos\pi\beta'\epsilon$$
$$s_\alpha=\sin\pi\alpha$$
$$s_\beta=\sin\pi\beta$$
$$s_{\alpha'}=\sin\pi\alpha'\epsilon$$
$$s_{\beta'}=\sin\pi\beta'\epsilon \tag{10}$$

$$L_1=2\alpha\beta\alpha'\beta' P_1P_2P_4P_5$$
$$L_2=\alpha'\beta'[(\alpha\beta P_1P_2)^2+(P_4P_5)^2]$$
$$L_3=\alpha\beta[(P_1P_4)^2+(\alpha'\beta' P_2P_5)^2]$$
$$L_4=2\alpha\beta\alpha'\beta' P_1P_5P_3P_6$$
$$L_5=\alpha\alpha' P_3P_6[(P_1\beta)^2+(P_5\beta')^2]$$
$$L_6=\beta\beta' P_3P_6[(P_1\alpha)^2+(P_5\alpha')^2]$$
$$L_7=2\alpha\beta\alpha'\beta' P_3P_6P_2P_4$$
$$L_8=\alpha\beta' P_3P_6[(P_2\beta\alpha')^2+P_4^2]$$
$$L_9=\beta\alpha' P_3P_6[(P_2\alpha\beta')^2+P_4^2]$$
$$L_{10}=(\alpha\beta)^2P_1^4+(\alpha\beta\alpha'\beta')^2P_2^4+P_4^4+(\alpha'\beta')^2P_5^4+(P_3P_6)^2[(\alpha\beta')^2+(\alpha'\beta)^2]$$
$$L_{11}=\alpha\beta\alpha'\beta'[(P_1P_5)^2+(P_2P_4)^2]$$
$$L_{12}=\alpha\beta\alpha'\beta'(P_3P_6)^2 \tag{11}$$

and

$$P_1=\sigma^2\Omega^2+2\zeta^2(\gamma-1)$$
$$P_2=-2\zeta(\gamma-1)$$
$$P_3=\gamma\Omega^2$$
$$P_4=\zeta\,[2\zeta^2(\gamma-1)+\Omega^2(\sigma^2-\gamma)]$$
$$P_5=\gamma\Omega^2-2\zeta^2(\gamma-1)$$
$$P_6=\sigma^2\Omega^2$$
$$\tau^2=\cos 2p\eta+i\sin 2p\eta$$
$$\tau^3+\tau=2\cos p\eta(\cos 2p\eta+i\sin 2p\eta)$$
$$\tau^4+1=2\cos 2p\eta(\cos 2p\eta+i\sin 2p\eta)$$
$$\tau^2+1=2\cos p\eta(\cos p\eta+i\sin p\eta)$$
$$(\tau^2+1)^2=4\cos^2 p\eta(\cos 2p\eta+i\sin 2p\eta) \tag{12}$$

Discussion of the Dispersion Equation

Equation (9) is the dispersion equation for the propagation of harmonic waves in a periodically layered infinite medium, which relates the frequency Ω to the two wave numbers ζ and η. It is a fourth-order equation in $\tau = e^{ip\eta}$ and therefore has four roots in τ. As can be seen from this relation the roots for η are periodic, such that if η is a root, $\eta + 2n/(1+\epsilon)$ $(n = 1, 2, 3, \ldots)$ is also a root. The basic period is, therefore, $2/(1+\epsilon)$. The four nondimensional Floquet wave numbers obtained for a pair of Ω, ζ can be categorized as

(*i*) $\eta_1, -\eta_1, \eta_2, -\eta_2$
η_1, η_2 real numbers
$0 < \eta_1 \leq 1/(1+\epsilon), 0 < \eta_2 \leq 1/(1+\epsilon)$

(*ii*) $\eta_1, -\eta_1, \eta_2, \bar{\eta}_2$
η_1 real number, η_2 complex number
$0 \leq \eta_1 \leq 1/(1+\epsilon)$
real part of $\eta_2 = 0$ or $1/(1+\epsilon)$

(*iii*) $\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2$
η_1, η_2 complex numbers
real part of $\eta_1 = 0$ or $1/(1+\epsilon)$
real part of $\eta_2 = 0$ or $1/(1+\epsilon)$

(*iv*) $\eta, -\eta, \bar{\eta}, -\bar{\eta}$
η complex
$0 <$ real part of $\eta \leq 1/(1+\epsilon)$ (13)

A bar designates the complex conjugate quantity.

Consider a harmonic wave traveling in the layered medium in the state of plane strain with frequency Ω and let the x component of the wave number be ζ. Equation (9) then gives the component of the wave number in the direction normal to the layers (the Floquet wave number η). There are four Floquet wave numbers for a pair of Ω, ζ numbers. In the case of wave propagation in a homogeneous medium it is known

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that for a pair of Ω, ζ numbers there are four wave number components in the y direction, i.e., α, $-\alpha$, β, $-\beta$. This shows the similarity between the two media with regard to harmonic wave propagation.

Reflection of Waves at a Free Boundary

To study the propagation of harmonic waves in a periodically layered half space we must derive the general expressions for τ_{xy}, τ_{yy} in infinite space and set them equal to zero at some plane parallel to the layering. Imagine that for a pair of numbers Ω, ζ, (equation (9)) is solved and the four wave numbers in the direction normal to the layers are obtained. The system (7) can be solved for each triplet $\Omega - \zeta - \eta$ and the expressions for τ_{xy}, τ_{yy} can derived.

The general expressions for τ_{xy} and τ_{yy} will be the superposition of the four solutions, each corresponding to one Floquet wave number. In the soft layer we obtain

$$\tau_{xy} = (Cf_1 + Df_3 + Ef_5 + Ff_7) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right]$$

$$\tau_{yy} = (Cf_2 + Df_4 + Ef_6 + Ff_8) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right] \quad (14)$$

where C, D, E, and F are arbitrary constants and $f_j = f_j(\Omega, \zeta, y_N)$, $j = 1, 8$. In the stiff layer f_j must be replaced by f_j'.

The expressions for tractions at a cell a distance nd ($n = 1, 2, 3, \ldots$) away from the cell in which the solution was obtained are, considering the Floquet theorem,

$$\tau_{xy} = \left(Cf_1\, e^{ip\eta_1 n} + Df_3\, e^{-ip\eta_1 n} + Ef_5\, e^{ip\eta_2 n} + Ff_7\, e^{-ip\eta_2 n}\right) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right]$$

$$\tau_{yy} = \left(Cf_2\, e^{ip\eta_1 n} + Df_4\, e^{-ip\eta_1 n} + Ef_6\, e^{ip\eta_2 n} + Ff_8\, e^{-ip\eta_2 n}\right) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right] \quad (15)$$

The expressions for τ_{xy}' and τ_{yy}' can be obtained for replacing f_j by f_j'. Thus having the solution in one cell is tantamount to having the solution in all cells.

Imagine now that at a certain location $y_N = y_N^*$ in the slow layer the tractions vanish

$$Cf_1^* + Df_3^* + Ef_5^* + Ff_7^* = 0$$
$$Cf_2^* + Df_4^* + Ef_6^* + Ff_8^* = 0 \quad (16)$$

where $f_j^* = f_j(\Omega, \zeta, y_N^*)$, $j = 1, 8$. By setting $E = 0$ the following wave system can be envisaged. An incident wave with the y component of the wave number η_1 and amplitude C is reflected as two waves having η_1, η_2 as wave number components in the y direction and D and F as their amplitudes, respectively.

$$Cf_1^* + Df_3^* + Ff_7^* = 0$$
$$Cf_2^* + Df_4^* + Ff_8^* = 0$$
$$D = -\frac{f_1^* f_8^* - f_7^* f_2^*}{f_3^* f_8^* - f_7^* f_4^*} C$$
$$F = -\frac{f_3^* f_2^* - f_1^* f_4^*}{f_3^* f_8^* - f_7^* f_4^*} C \quad (17)$$

By setting $C = 0$, another system of waves can be envisaged which can be similarly described. An incident wave with y component of the wave number η_2 and amplitude E is reflected as two waves having η_1, u_2 as y components of the wave numbers and D and F as amplitudes, respectively.

$$Df_3^* + Ff_7^* = -Ef_5^*$$
$$Df_4^* + Ff_8^* = -Ef_6^*$$
$$D = -\frac{f_5^* f_8^* - f_6^* f_7^*}{f_3^* f_8^* - f_7^* f_4^*} E$$
$$F = -\frac{f_3^* f_6^* - f_5^* f_4^*}{f_3^* f_8^* - f_7^* f_4^*} E \quad (18)$$

Consider the equation

$$f_3^* f_8^* - f_7^* f_4^* = 0 \quad (19)$$

The expression on the left-hand side of this equation is the determinant of the system of equation (17) or (18). For a solution to exist it must be nonzero. If it is zero, however, by setting C and E equal to zero in the systems (17) and (18), respectively, we obtain the following homogeneous system of equations

$$Df_3^* + Ff_7^* = 0$$
$$Df_4^* + Ff_8^* = 0 \quad (20)$$

$$F = -\frac{f_3^*}{f_7^*} D$$
$$\tau_{xy} = (Df_3 + Ff_7) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right] \quad (21)$$
$$\tau_{yy} = (Df_4 + Ff_8) \exp\left[\frac{i\pi}{2h}(\zeta x - c_2 \Omega t)\right]$$

For the wave represented by the preceding system to decay away from the surface, both η_1 and η_2 must have imaginary parts. It has been found that in the domains of the $\Omega - \zeta$ plane where equation (19) is valid, the foregoing condition is satisfied, which means system (20) represents a surface wave traveling in the layered medium. Equation (19) is the dispersion relations for that wave.

Types of Waves in a Half Space

In the case of harmonic wave propagation in the homogeneous half space, it is noted that the $\Omega - \zeta$ plane can be divided into three regions, in each of which certain types of waves travel. The regions are separated by the P line ($\beta = 0$) and by the SV line ($\alpha = 0$). A similar division can be introduced in the $\Omega - \zeta$ plane for the layered half space. Here the lines of division are given by $\eta_1 = 0$, $\eta_1 = 1/(1+\epsilon)$, $\eta_2 = 0$, $\eta_2 = 1/(1+\epsilon)$. These curves represent wave motions in the infinite layered medium parallel to the layers. The four types of such curves correspond to the symmetry or antisymmetry with respect to the centerline of the displacements u and u' in the soft or fast layers, respectively. With respect to the roots of equation (7) region *I* corresponds to the category (*i*) of roots, region *II* corresponds to the category (*ii*), and region *III* corresponds to categories (*iii*) and (*iv*). Region *IIIa* corresponds to category (*iii*) of the roots of equation (7) and region *IIIb* corresponds to category (*iv*) of these roots. As can be seen from Fig. 2 between the boundaries of region *IIIb* and *II* there may be regions that correspond to category (*i*) of roots. On the boundaries of region *IIIb* the two Floquet wave numbers are equal.

The computation was carried out for the flowing values of the parameters

$$\gamma = 0.02$$
$$\sigma^2 = 0.06$$
$$\epsilon = 4$$
$$\nu = 0.3$$
$$\nu' = 0.35$$

In the layered half space for points in region *I* (Fig. 2), both Floquet wave numbers η_1 and η_2 are real which means

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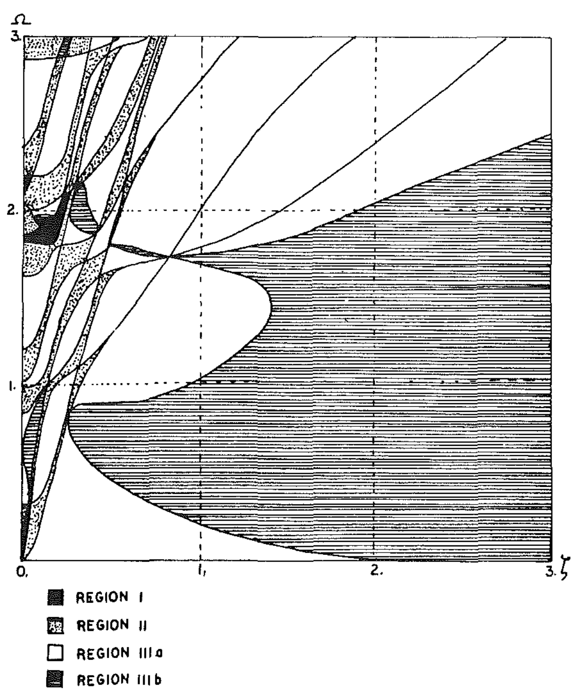

Fig. 2 Regions of $\Omega - \zeta$ plane

propagation of bulk waves in the half space. For points in region *II* either η_1 or η_2 has an imaginary part, which means an incident bulk wave reflects as a bulk wave and as a surface wave. For points in region *III* both η_1 and η_2 have imaginary parts. Therefore, an incident wave grows away from the surface while the two reflected waves decay. The reflection patterns described in the foregoing are governed by systems of equations (17) or (18).

It is only in region *III*, where the two reflected waves decay that we expect the determinant of the systems (17) or (18) to vanish along some curves so that a system of surface waves may result, similar to Rayleigh waves in a homogeneous medium. As seen from Fig. 2, the regions II and *III* have multiple boundaries and since surface waves can exist only in region *III* we expect the dispersion curve for such surface waves to start and end on the boundaries of the two regions.

Surface Waves in Layered Medium

The system of equations (17) and (18) is solvable if the determinant of the coefficients does not vanish. However, if it vanishes the situation is similar to the vanishing of the determinant of systems in a homogeneous medium which resulted in Rayleigh waves. The Rayleigh line represents points along which the determinant vanishes.

In the present investigation curves were sought along which a similar phenomenon occurs in the layered half space. Equation (19) was solved numerically and the dispersion curve thus obtained is shown in Fig. 3, and on a magnified scale in Fig. 4. The dispersion curve shown in this figure is for a traction-free plane at $y_N = -h$.

As can be seen from Fig. 3, there is an infinite number of branches for the dispersion curve. Whenever the dispersion branches reach region *II* they come to an end. The determinant is then no longer zero and the system of waves governed by equations (17) or (18) prevails. As one moves along a dispersion branch, the closer one gets to the end points the weaker is the decay rate and the deeper does the wave penetrate into the half space. At the end point itself a transition from surface waves to bulk waves occurs as region *II* is reached.

For $\zeta = 0$ the system of plane strain waves traveling in the layered half space and described by equation (7) decouples

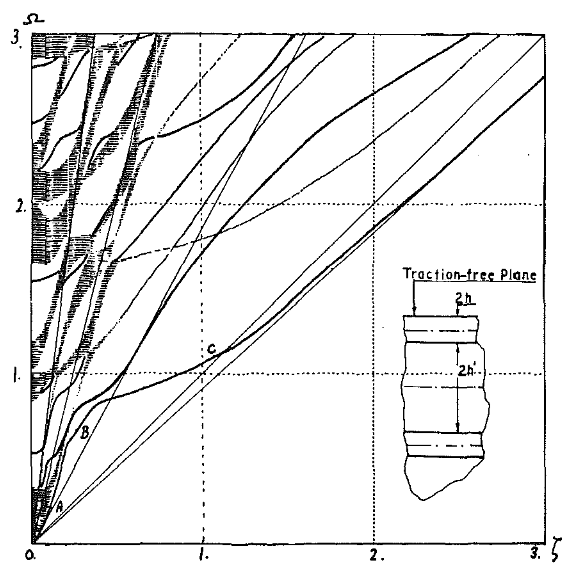

Fig. 3 Dispersion curve for surface waves. Slow layer at the surface.

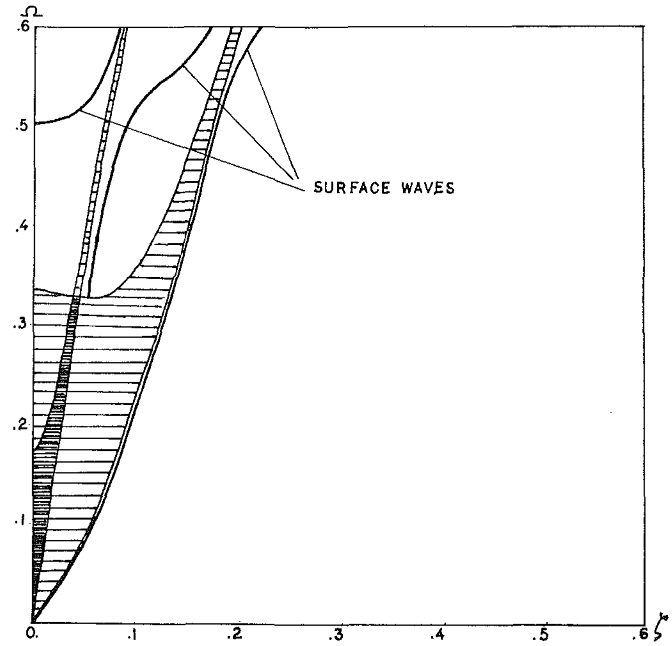

Fig. 4 Dispersion curve for surface waves. Magnified view of part of Fig. 3.

into its constituent parts, i.e., a P and as SV wave traveling normal to the layers. At such frequencies the traction-free surface repeats itself periodically. Therefore there is an infinite number of traction-free surfaces in contrast to the general case for nonzero ζ where there is only one traction-free surface.

As the frequency Ω and the wave number ζ tend to zero the system of equations (7) must be solved in the vicinity of the origin. The phase velocity (which is the slope of the tangent to the dispersion curve at the origin) computed in this manner agrees with the one predicted by the effective modulus theory for a homogeneous, transversely isotropic medium [8–9]. For the parameters chosen in this paper, the slope at the origin is $\Omega/\zeta = 1.32$. As ζ becomes large, depending on the location of the free surface, the lowest branch will tend to the Rayleigh line for the layer in which the free surface resides. As mentioned, Fig. 3 is for the case when the slow layer is the bounding top layer. Results for calculations when the fast layer is the bounding top layer are not given here, but may be found in reference [10].

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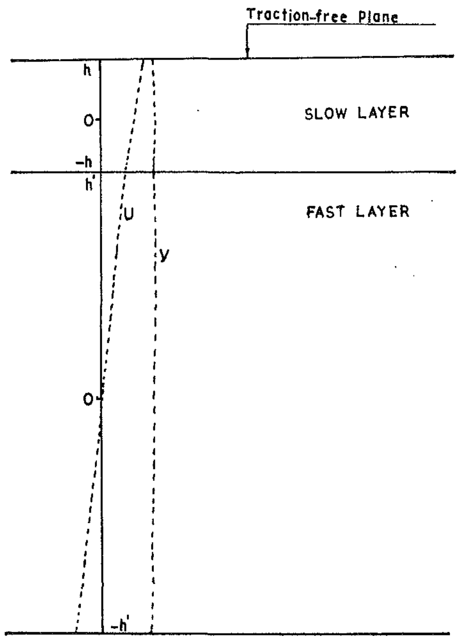

Fig. 5 Horizontal and vertical displacements of the first cell for Ω = 0.2 (point *A* in Fig. 3)

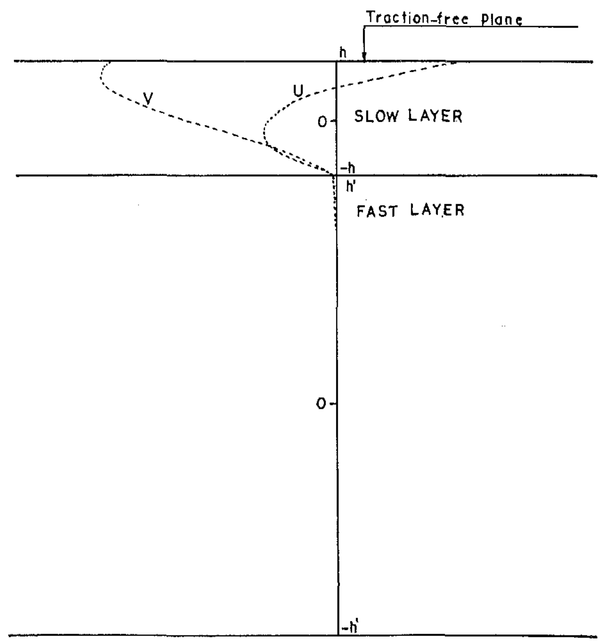

Fig. 7 Horizontal and vertical displacements of the first cell for Ω = 1.2 (point *C* in Fig. 3)

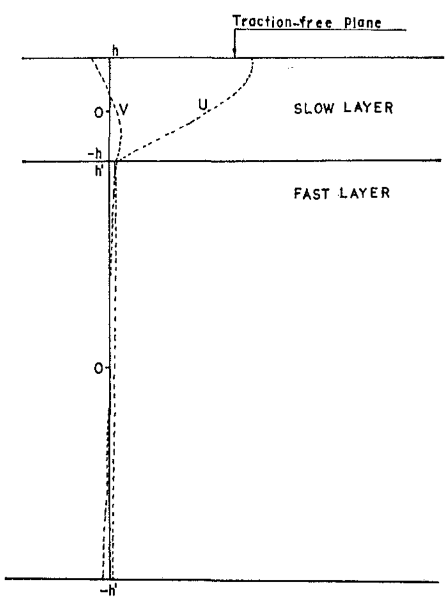

Fig. 6 Horizontal and vertical displacements of the first cell for Ω = 0.65 (point *B* in Fig. 3)

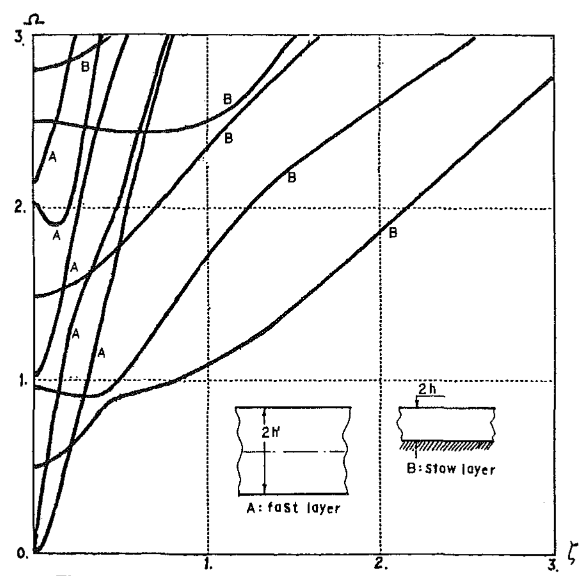

Fig. 8 The superposition of dispersion curve of clamped-free, slow layer and free-free, fast layer

Analysis of the Lowest Surface Wave Branch

To gain some insight into the features of the lowest branch of the surface wave, whose dispersion is given in Fig. 3, it is helpful to calculate the mode shapes and attempt to obtain the reasons for the dispersive behavior. To begin with, it is observed from Fig. 3 that this lowest branch is continuous in the complete range $0 \leq \xi \leq \infty$ and is composed essentially of three segments. Typical points on each of the three segments are denoted by *A*, *B*, and *C*, respectively. Mode shapes for the top two layers, i.e., the top unit cell, are given in Figs. 5–7, for points *A*, *B*, and *C*, respectively. It is noted that at point *A* the fast layer appears to undergo simple bending, while at points *B* and *C* all motion is essentially confined to the top slow layer and the fast layer remains essentially undisplaced.

These properties of modes of the lowest branch of the surface wave suggest an analysis of dispersive properties of two much simpler systems in the expectation that these would be similar to those of the layered half space under discussion in the present paper. The two simple systems consists of: (*A*) a plate with traction-free faces, thickness $2h'$, and properties of the fast layer; (*B*) a plate of thickness $2h$, one face traction-free, the other face clamped, and the properties of the slow layer. Dispersion curves for these two plates are sketched in Fig. 8. It is noted that the lowest continuous curve, consisting, as ξ increases, first of the lowest curve *A* and then of the lowest curve *B*, already greatly resembles the lowest dispersion curve of the layered half space, Fig. 3. The behavior near the origin is represented in magnified form in Fig. 9. Here the second mode of system *A*, namely, the lowest symmetric mode, as well as the second and third branches of the surface wave in the semi-infinite layered medium have also been included.

This figure illustrates not only the closeness of the lowest

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surface wave branch and the lowest asymmetric (bending) mode of the free-free stiff plates for low frequencies up to the first cut-off frequency of a surface wave branch (occurring for the third-lowest branch), but also the feature that the second lowest and the third lowest branches of the surface wave are uncoupled by the lowest symmetric mode of the free-free fast plate and the lowest mode of the clamped-free slow plate.

Returning to Fig. 8 it is observed that the lowest curve B itself appears to consist of two parts separated by the approximate value of $\zeta = 0.5$. It is thus tempting to investigate more closely the properties of those two regimes. With this aim in mind it is noted that system B, i.e., clamped-free layer of thickness $2h$, inherently lacks symmetry, and thus its modes cannot be grouped into symmetric and asymmetric ones. To alleviate this undesirable feature we consider a substitute system consisting of a slow layer of thickness $4h$ clamped at both faces. This permits one to consider the lowest symmmetric and the lowest asymmetric modes of this plate, whose dispersion is plotted as curves 2 and 3 in Fig. 10. The lowest asymmetric mode of the free-free stiff plate is replotted in this figure, as well as the Rayleigh line for the slow material. It is observed that curves 2 and 3 represent, to a good accuracy, the uncoupled dispersion curves of the two lowest modes of system B of Fig. 8, as given by the two lowest curves in that figure.

The lowest dispersion curve for surface waves in the layered medium has also been reproduced in Fig. 10 and it is seen that the desired result has been achieved to a larger extent: the behavior of the fundamental surface wave can now be physically understood over the whole range of wave number. For very long waves and while the wave length is larger than approximately twice the cell thickness $d = 2h + 2h'$, the system behaves essentially as a free-free plate of thickness $2h'$. If the wave length is reduced further, to approximately the cell thickness d, the system is in the intermediate regime denoted by point B in Fig. 3 as discussed in the foregoing. For small wave lengths the system behaves first as one of the lowest modes of system B and then asymptotically approaches the Rayleigh line of the slow, top layer.

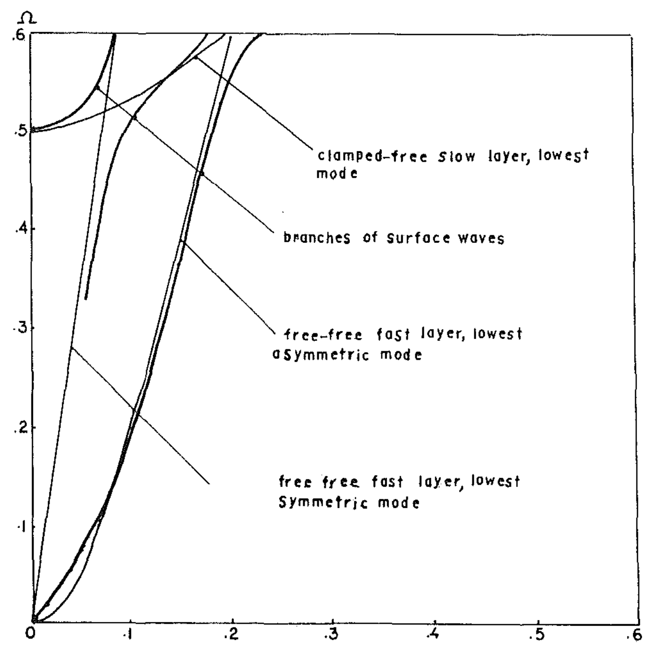

Fig. 9 Surface waves dispersion branches and uncoupling branches of a clamped-free, slow layer and free-free, fast layer

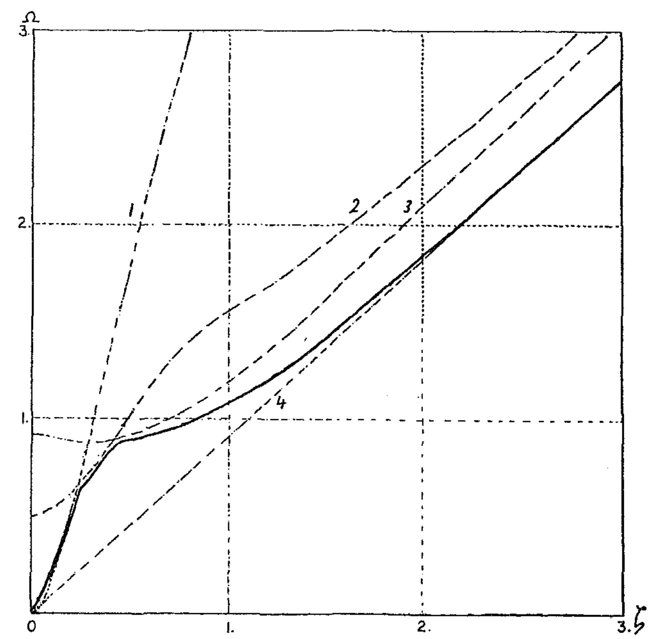

Fig. 10 The lowest branch of the dispersion curve with the fast layer at the surface and the uncoupling curves. 1—Lowest asymmetric branch of the free-free stiff plate. 2,3—Lowest branches of a clamped-clamped soft plate two times the thickness of the slow layer. 4—Rayleigh line for the slow medium.

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