Bianchi Type III Cosmological Model with Binary Mixture of Perfect Fluid and Dark Energy

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Abstract. The paper deals with Bianchi type III cosmological models filled with perfect fluid and dark energy components. The two sources are assumed to interact minimally so that their energy momentum tensors are conserved separately. The exact solution of Einstein's field equations is obtained by assuming the expansion θ in the model is proportional to the shear (σ). This condition leads to $A = B^m$, where A and B are metric coefficients and m is a constant. The physical and geometrical behaviors of the models are also discussed.

1 Introduction

In the 1990's two teams of astronomers, the Supernova Cosmology Project (Lawrence Berkeley National Laboratory) and the High-Z Supernova Search (international), were looking for distant Type Ia supernovae in order to measure the expansion rate of the universe with time. They expected that the expansion would be slowing, and it would be indicated by the supernovae being brighter than their red shifts indicate. Instead, they found the supernovae to be fainter than expected. Hence, the expansion of the universe was accelerating. In addition, measurements of the cosmic microwave background indicate that the universe has a flat geometry on large scales. Because there is not enough matter in the universe either ordinary or dark matter to produce this flatness, the difference must be attributed to a "dark energy" (DE). This same dark energy causes the acceleration of the expansion of the universe. In addition, the effect of dark energy seems to vary with the expansion of the universe slowing down and speeding up over different times. The accelerating expansion of the universe is driven by mysterious energy with negative pressure known as Dark Energy (DE). This was observed with SNe [1], WMAP [2], SDSS [3, 4] and X-ray [5]. This acceleration is triggered by more than 70% of dark energy. There are many proposals

to explain the dark energy (DE). The nature of dark energy as well as of dark matter is unknown, and many radically different models have been proposed, such as: a tiny positive cosmological constant, quintessence [6-8], DGP branes [9,10], the non-linear E(R) models [11,12], and dark energy in brane worlds, among many authors [13-29]; the review articles [30,31]. Since the observation of small anisotropy in the microwave background radiation (CMB) [32] and the large scale structures [3] it becomes clear that a pure Friedmann-Lemaitire-Robertson-Walker (FLRW) cosmology could not explain all the properties of the universe. It is, therefore, natural to consider anisotropic cosmological models that allow FLRW universes as special cases. Many cosmologists believe that the simplest candidate for the DE is the cosmological constant (Λ) or vacuum energy since it fits the observational data well. During the cosmological evolution, the Λ-term has the constant energy density and pressure $p^{(de)} = -\rho^{(de)}$, where the superscript *(de)* stands for DE. However, one has the reason to dislike the cosmological constant since it always suffers from the theoretical problems, such as the "fine-tuning" and "cosmic coincidence" puzzles [30]. That is why, the different forms of dynamically changing DE with an effective equation of state (EoS), $\omega^{(de)} = p^{(de)}/\rho^{(de)} < -1/3$ have been proposed in the literature.

Other possible forms of DE include quintessence, $\omega^{(de)} > -1$ [8], phantom, [$\omega^{(de)} < -1$ [33], *etc*. While the possibility $\omega^{(de)} \ll -1$ is ruled out by current cosmological data from SN Ila (Supernovae Legacy Survey, Gold sample of Hubble Space Telescope) [34,35], CMBR (WMAP, BOOMERANG) [36,37] and large scale structure (Sloan Digital Sky Survey) data [38], the dynamically evolving DE crossing the phantom divide line (PDL), $\omega^{(de)} = -1$, is mildly favored. SN Ia data collaborated with CMBR anisotropy and galaxy clustering statistics suggest that $-1.33 < \omega^{(de)} < -0.79$ (see [4]). For instance, quintessence models involving scalar fields give rise to time dependent EoS parameter $\omega^{(de)}$ [39]. Recently dark energy model with variable EoS parameter has been intensively studied [40-43].

In this paper, we have investigated the Bianchi type III models with perfect fluid and DE components. This paper is organized as follows: In Section 2 the metric and field equations. Section 3 deals with the exact solutions of the field equations, in Section 4 geometrical and physical behavior of the model are discussed. Finally, concluding remarks have been given in Section 5.

2   The Metric and Field Equations

We consider the Bianchi type III space-time in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{-2\alpha x} dy^2 - C^2 dz^2, \tag{1}$$

where A, B and C are the scale factors (metric tensors) and functions of the cosmic time t, $\alpha \neq 0$ is a constant. The Einstein field equations in natural units ($8\pi G = c = 1$) are

$$R_{ij} - \frac{1}{2}g_{ij}R = -T_{ij}, \tag{2}$$

where $T_{ij} = T_{ij}^{(m)} + T_{ij}^{(de)}$ is the overall energy momentum tensor with $T_{ij}^{(m)}$ and $T_{ij}^{(de)}$ as a energy momentum tensor of ordinary matter and DE, respectively. These are given by

$$
\begin{aligned}
T_j^{(m)\,i} &= \mathrm{diag}[\rho^{(m)}, -p^{(m)}, -p^{(m)}, -p^{(m)}] \\
&= \mathrm{diag}[1, -\omega^{(m)}, -\omega^{(m)}, -\omega^{(m)}]\rho^{(m)} \tag{3}
\end{aligned}
$$

$$
\begin{aligned}
T_j^{(m)\,i} &= \mathrm{diag}[\rho^{(de)}, -p^{(de)}, -p^{(de)}, -p^{(de)}] \\
&= \mathrm{diag}[1, -\omega^{(de)}, -\omega^{(de)}, -\omega^{(de)}]\rho^{(de)}, \tag{4}
\end{aligned}
$$

where $\rho^{(m)}$ and $p^{(m)}$ are respectively the energy density and pressure of the perfect fluid component or ordinary baryonic matter while $\omega^{(m)} = p^{(m)}/\rho^{(m)}$ is its EoS parameters. Similarly, $\rho^{(de)}$ and $p^{(de)}$ are respectively the energy density and pressure of the DE component while $\omega^{(de)} = p^{(de)}/\rho^{(de)}$ is the corresponding EoS parameters.

In a co-moving co-ordinate system, the Einstein field equations (2), with (3) and (4) for the metric (1) subsequently lead to the following system of equations:

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{A^2} = \rho^{(m)} + \rho^{(de)}, \tag{5}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}, \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}, \tag{8}$$

$$\alpha\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0. \tag{9}$$

Here, and in what follows, a dot indicated ordinary differentiation with respect to t. The energy conservation equation $T_{;j}^{ij} = 0$ yields

$$\dot{\rho}^{(m)} + 3(1+\omega^{(m)})\rho^{(m)}H + \dot{\rho}^{(de)} + 3(1+\omega^{(de)})\rho^{(de)}H = 0, \tag{10}$$

where H is the Hubble parameter.

The average scale factor R of Bianchi type III model is defined as

$$R = (ABCe^{\alpha x})^{1/3}. \tag{11}$$

A volume scale factor is given by

$$R = (ABCe^{\alpha x}). \tag{12}$$

We define the generalized mean Hubble 's parameter H as

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{13}$$

where $H_1 = \dot{A}/A$, $H_2 = \dot{B}/B$ and $H_3 = \dot{C}/C$ are the directional Hubble parameters H_i in the direction of x, y, and z, respectively. From (11)-(13), we obtain

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{\dot{R}}{R} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \tag{14}$$

The physical quantities, *i.e.* the scalar expansion θ, the anisotropy parameter A_m, the shear scalar σ and the deceleration parameter q are defined as

$$\theta = 3H, \tag{15}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2, \tag{16}$$

where $\Delta H_i = H_i - H$ ($i = 1, 2, 3$),

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - 3H^2\right) = \frac{3}{2}A_m H^2, \tag{17}$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1. \tag{18}$$

3   Solution of the Field Equations

From equation (9), we get

$$A = mB, \tag{19}$$

where m is a positive constant of integration. Using (19) in equations (5)–(9), we obtain

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{m^2 B^2} = \rho^{(m)} + \rho^{(de)}, \tag{20}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}, \tag{21}$$

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{\alpha^2}{m^2 B^2} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}. \tag{22}$$

Now we have initially six variables (B, C, $\omega^{(m)}$, $\omega^{(de)}$, $\rho^{(m)}$, $\rho^{(de)}$) and three linearly independent equations (20)–(22). The system is thus initially undetermined and we need additional constraints to close the system.

First, we assume that perfect fluid and DE components interact minimally. Therefore, the energy conservation equation of the two sources may be conserved separately.

The energy conservation equation $T^{(m)\;ij}_{\;\;\;;j} = 0$ of the perfect fluid yields

$$\dot{\rho}^{(m)} + 3(1 + \omega^{(m)})\rho^{(m)}H = 0, \tag{23}$$

whereas the energy conservation equation $T^{(de)\;ij}_{\;\;\;;j} = 0$ of the DE component yields

$$\dot{\rho}^{(de)} + 3(1 + \omega^{(de)})\rho^{(de)}H = 0. \tag{24}$$

The EoS parameter of the perfect fluid has been assumed to be constant.

$$\omega^{(m)} = \frac{p^{(m)}}{\rho^{(m)}} = \text{const.} \tag{25}$$

Here $\omega^{(m)}$ is a constant and lies in the interval $\omega^{(m)} \in [0, 1]$. Depending on its numerical value, the $\omega^{(m)}$ describes the following types of universe:

$$\omega^{(m)} = 0 \quad \text{(Dust)}, \tag{25a}$$

$$\omega^{(m)} = \frac{1}{3} \quad \text{(Radiation universe)}, \tag{25b}$$

$$\omega^{(m)} \in (1/3, 1) \quad \text{(Hard universe)}, \tag{25c}$$

$$\omega^{(m)} = 1 \quad \text{(Zeldovich universe or Stiff matter)}. \tag{25d}$$

Whereas $\omega^{(de)}$ is allowed to be a function of the cosmic time, since the current cosmological data from SN Ia, CMB and the large scale structures mildly favor dynamically evolving DE crossing the PDL as mentioned in Section 1. Hence, since $\omega^{(de)}(t)$ has not been constrained, we still need one more constraint to close the system. We assume that the expansion (θ) in the model is proportional to the shear (σ). This condition leads to

$$B = C^n, \tag{26}$$

where $n\ (> 0)$ is a constant. According to Thorne [44], the observations of the velocity red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic within about 30% range approximately [45,46] and red-shift studies place the limit $\frac{\sigma}{H} \leq 0.3$ of the ratio of shear σ to Hubble H in the neighborhood of our galaxy today. Collins [47] has discussed the physical significance of this condition for perfect fluid and barotropic EoS in a more general case. In many papers [42,48-50] this condition has been proposed to find the exact solutions of cosmological models.

Subtracting Eq. (21) from (22), we get

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{B}^2}{B^2} - \frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{m^2 B^2} = 0. \tag{27}$$

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Using (26), we get

$$(n-1)\frac{\ddot{C}}{C} + 2n(n-1)\frac{\dot{C}^2}{C^2} - \frac{\alpha^2}{m^2 C^{2n}} = 0. \tag{28}$$

Let $\dot{C} = f(C)$, which implies that

$$\ddot{C} = ff', \tag{29}$$

where $f' = df/dC$.

Using (29), Eq. (28) becomes

$$\frac{d}{dC}(f^2) + 4n\frac{f^2}{C} = \frac{2\alpha^2}{m^2(n-1)}C^{1-2n}. \tag{30}$$

After integration, (30) is reduced to

$$f = \left(\frac{dC}{dt}\right) = \left(\frac{\alpha^2}{m^2(n^2-1)}C^{2(1-n)} + l_0 C^{-4n}\right)^{1/2}, \tag{31}$$

where l_0 is an integrating constant. From (31), we obtain

$$t = \int \frac{dC}{\sqrt{\frac{\alpha^2}{m^2(n^2-1)}C^{2(1-n)} + l_0 C^{-4n}}}. \tag{32}$$

Thus the metric (1) is reduced to

$$ds^2 = \frac{dC^2}{\frac{\alpha^2}{m^2(n^2-1)}C^{2(1-n)} + l_0 C^{-4n}} - m^2 C^{2n} dx^2 - C^{2n} e^{-2\alpha x} dy^2 - C^2 dz^2. \tag{33}$$

After using the suitable transformation of co-ordinates the above metric is written as

$$ds^2 = \frac{dT^2}{\frac{\alpha^2}{m^2(n^2-1)}T^{2(1-n)} + l_0 T^{-4n}} - m^2 T^{2n} dx^2 - T^{2n} e^{-2\alpha x} dy^2 - T^2 dz^2, \tag{34}$$

where $C = T$ and

$$dt = \frac{dT}{\frac{\alpha^2}{m^2(n^2-1)}T^{2(1-n)} + l_0 T^{-4n}}.$$

The solution of (34) is not tenable for $n = 1$.

4 Geometrical and Physical Significance of the Model

The rate of expansion in the direction of x, y and z is given by

$$H_x = H_y = n\Big(\frac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big)^{1/2}, \tag{35}$$

$$H_z = \Big(\frac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big)^{1/2}. \tag{36}$$

Hence the average generalized Hubble's parameter (H) is given by

$$H = \frac{2m+1}{3}\Big(\frac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big)^{1/2}. \tag{37}$$

We see that the dynamical H, H_x, H_z are infinite for earlier time and converges to zero as $T \to \infty$.

The energy density $(\rho^{(m)})$ of perfect fluid, DE density $(p^{(de)})$ and Eos parameter $(\omega^{(de)})$ of DE, for model (34) are found to be

$$\rho^{(m)} = l_1 T^{-(2n+1)(1+\omega^{(m)})}, \tag{38}$$

$$\rho^{(de)} = \frac{(2n+1)\alpha^2}{m^2(n^2-1)}T^{-2n} + n(n+2)l_0 T^{-2(2n+1)} - l_1 T^{-(2n+1)(1+\omega^{(m)})}, \tag{39}$$

$$\omega^{(de)} = \frac{\Big(\frac{-\alpha^2}{m^2(n^2-1)}\Big)T^{-2n} + n(n+2)l_0 T^{-2(2n+1)} - l_1\omega^{(m)}T^{-(2n+1)(1+\omega^{(m)})}}{\frac{(2n+1)\alpha^2}{m^2(n^2-1)}T^{-2n} + n(n+2)l_0 T^{-2(2n+1)} - l_1 T^{-(2n+1)(1+\omega^{(m)})}}. \tag{40}$$

It is noted that the parameters $\rho^{(m)}$ and $\rho^{(de)}$ start off with extremely large values and continue to decrease with expansion of the universe. From equation (40) with $\omega^{(m)} = 1$, we observe that at cosmic time

$$T = \Big[\frac{-n\alpha^2}{m^2(n^2-1)(n(n+2)l_0 - l_1)}\Big]^{2(n+1)}, \quad \omega^{(de)} = -1, \tag{41}$$

(*i.e.*, cosmological constant dominated universe), when

$$T < \Big[\frac{-n\alpha^2}{m^2(n^2-1)(n(n+2)l_0 - l_1)}\Big]^{2(n+1)}, \quad \omega^{(de)} > -1, \tag{42}$$

(*i.e.* quintessence) and when

$$T > \Big[\frac{-n\alpha^2}{m^2(n^2-1)(n(n+2)l_0 - l_1)}\Big]^{2(n+1)}, \quad \omega^{(de)} < -1, \tag{43}$$

(*i.e.* super quintessence or phantom fluid dominated universe), representing the different phases of universe throughout the evolving process.

By using Eqs. (15)–(18), we can express the physical quantities

$$V = me^{-\alpha x}T^{2n+1},\tag{44}$$

$$\theta = (2n+1)\Big(\frac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big)^{1/2},\tag{45}$$

$$A_m = \frac{2(n-1)^2}{(2n+1)^2},\tag{46}$$

$$\sigma^2 = \frac{(n-1)^2}{3}\Big(\frac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big).\tag{47}$$

From Eqs. (46) and (47), we obtain

$$\frac{\sigma}{\theta} = \frac{n-1}{\sqrt{3}(2n+1)}.\tag{48}$$

From the above results, it can be seen that the spatial volume V is zero at initial epoch and increases as $T \to \infty$. The expansion and shear scalar are infinite at $T = 0$ and decreases with the increase in time. Thus the universe starts evolving with zero volume at initial epoch with infinite rate of expansion which slows down for the late time of the universe. Since $\sigma/\theta =$ const. (from early to late time), the model does not approach isotropy through the whole evolution of the universe.

The value of deceleration parameter (q) is found to be

$$q = -1 - \frac{3\Big[\dfrac{n\alpha^2}{m^2(n^2-1)}T^{-2n} + (2n+1)l_0 T^{-2(2n+1)}\Big]}{(2n+1)\Big[\dfrac{\alpha^2}{m^2(n^2-1)}T^{-2n} + l_0 T^{-2(2n+1)}\Big]}.\tag{49}$$

From (49), we conclude the following three cases:

$$(i)\ \text{For}\ T = \Big[\frac{l_0 m^2(1-n^2)(2n+1)}{n\alpha^2}\Big]^{\frac{1}{2(n+1)}},\qquad q = -1,\tag{50}$$

i.e., we get the largest rate of acceleration of the universe;

$$(ii)\ \text{For}\ T < \Big[\frac{l_0 m^2(1-n^2)(2n+1)}{n\alpha^2}\Big]^{\frac{1}{2(n+1)}},\qquad q < 0,\tag{51}$$

i.e., we get the accelerating model of the universe.

$$(iii)\ \text{For}\ T > \Big[\frac{l_0 m^2(1-n^2)(2n+1)}{n\alpha^2}\Big]^{\frac{1}{2(n+1)}},\qquad q > 0,\tag{52}$$

i.e., we get the a model is in decelerating phase.

5 Conclusion

We have investigated the role of DE with variable EoS parameter in the evolution of Bianchi type III space-time by taking into account that the expansion scalar (θ) is proportional to shear (σ). We have discussed some physical and geometrical aspects of the model in detail. The universe does not approach to isotropy. Also our model does not stand for $n = 1$. We obtain the accelerating model of the universe.

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