Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 304092, 11 pages
http://dx.doi.org/10.1155/2013/304092

Research Article

Adaptive Fault-Tolerant H_∞ Controller Design for Networked Systems with Actuator Saturation

Wei Guan

School of Automation, Shenyang Aerospace University, Shenyang, Liaoning 110136, China

Correspondence should be addressed to Wei Guan; guanweihaha@163.com

Received 10 January 2013; Accepted 5 March 2013

Academic Editor: Engang Tian

In this paper, an indirect adaptive fault-tolerant H_∞ controller design method is proposed for networked systems in the presence of actuator saturation. Based on the on-line estimation of eventual faults, the parameters of controller are being updated automatically to compensate the fault effects on systems. The designs are given in linear matrix inequalities (LMIs) approach, which can guarantee the disturbance tolerance level and adaptive H_∞ performances of networked systems in the cases of actuator saturation and actuator failures. An example is given to illustrate the efficiency of the design method.

1. Introduction

With the rapid developments in network technologies, more and more communication networks are used in control systems; especially, the stability analysis for systems with time delays have become an active research area. However, networked control systems with actuator saturation and *time delays* are often encountered in many practical systems such as electrical heaters and long transmission lines in pneumatic, hydraulic, and rolling mill systems. Since the existence of time delay and actuator saturation in a physical system often induces instability of poor performance, research on time delay systems with actuator saturation is a topic of great practical and theoretical importance. If the saturation and time delay are ignored in system analysis and design, the performance of the overall system can be degenerated. More seriously, saturation and time delay can cause instability of the overall system. Therefore, over the last several decades, many researchers have considered various control problems of disturbance rejection for linear systems subject to actuator saturation [1–11]. Papers [4, 5] carried out the L_2 gain analysis and minimization. Although there are plenty of papers that are devoted to dealing with different problems for systems with actuator saturation, the main difference and difficulty lie in their treatment of saturation nonlinearity. In paper [2], authors gave a method for maximization of an ellipsoid which is invariant under input saturation, but persistent disturbances. The works of [1, 3, 6–8] consider the situation where disturbance are bounded in energy. The works of [1, 6, 7] formulated and solved the problem of stability analysis and design. In [9, 10], authors presented LMI-based methods for regional stability and performance of linear antiwindup compensators for linear control systems. [12] presents a method for the analysis and control design of linear systems in the presence of actuator saturation and L_2 disturbances. During the last few years, problems about actuator saturation have been extended to many other fields of automatic control, such as singular systems [13], systems with parameters uncertainty [14], Markovian jump systems [15], decentralized control systems [16], and Hamiltonian systems [17].

Time delays are frequently encountered in almost all networked systems. Since the existence of a delay in a physical system often induces instability of poor performance, research on time-delay systems is a topic of great practical and theoretical importance. During the last decade, the control problem of systems with time delay has received considerable attention. The main methods can be classified into two types: delay-independent ones and delay-dependent ones.

On the other hand, fault tolerant has become a hot research area because of its importance in practical engineering [18–28]. And the design approach can be broadly classified into two types: Passive approach and Active approach. A passive fault-tolerant controller commonly has a simple structure and is easily implemented [18–22]. The system performances in normal and fault modes can be optimized. Some of these active fault-tolerant control methods may readjust controller parameters or change controller structure to compensate the fault effects on systems. Some of these methods include a strategy involving a fast subsystem for fault detection and isolation (FDI) and a supervisory system that chooses the corresponding controller for a particular type of fault. Most of the results in adaptive fault-tolerant control are based on model reference adaptive control (MRAC) [29–31], but the disturbance attenuation performances of systems have not been addressed yet within the MRAC framework. Paper [32] considered the problem of adaptive reliable controller via state feedback and dynamic output feedback, respectively, for linear time-delay systems against actuator faults. However, when actuator saturation problem is considered, the methods of [32] cannot be used.

As we all know, actuator faults and saturation always happen at the same time for networked systems. However, noting all above results, there is no work that deals with this problem. There are only a few papers that considered the problems about systems with actuator saturation and faults [13, 33, 34]. Motivated by the above observations, this paper studies the problem of designing adaptive fault-tolerant H_∞ controllers for networked systems with actuator saturation. The designs are developed in the framework of LMIs approach, which can guarantee the disturbance tolerance level and adaptive H_∞ performances of networked systems in the cases of actuator saturation and actuator failures. The difference between this paper and some existing results is that in this paper the fault tolerant and saturation are considered at same time for networked systems.

The remainder of this paper is organized as follows. Section 2 introduces notation to be used in the paper, and problem statement is given in it. An adaptive fault-tolerant H_∞ controller design method is described for networked system in Section 3. In Section 4, an example is given to illustrate the efficiency of the design method. The paper will be concluded in Section 5.

2. Problem Statement and Preliminaries

In this paper, the following LTI plant will be considered:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-\tau(t)) + B_1\omega(t) + B_2\sigma(u), \\ z(t) &= Cx(t) + D\,\mathrm{sat}(u), \\ x(t) &= \phi(t) \quad t\in[-h,0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ and x_t is the plant state at time t defined by $x_t(s) = x(t+s)$, $s\in[-h,0]$, $\mathrm{sat}(u)\in R^m$ is the saturated control input, $z(t)\in R^s$ is the regulated output, and $\omega(t)\in R^d$ is an exogenous disturbance in $L_2[0,\infty]$, respectively. A, A_1, B_1, B_2, C, and D, are known constant matrices of appropriate dimensions. For simplicity only, we take single delay $\tau(t)$. The results of this paper can be easily applied to the case of multiple delays.

The following case for *time-varying delay* $\tau(t)$ is considered. That is, $\tau(t)$ is differentiable funcion

$$0\le\tau(t)\le h,\quad \dot{\tau}(t)\le d<1,\quad \text{satisfying } \forall t\ge 0, \tag{2}$$

where d is an upper bound on the derivative of $\tau(t)$.

In this paper, we formulate the fault-tolerant control problem by using the following model form which is considered in [19, 22]:

$$\begin{aligned} u_{jq}^F(t) &= \left(1-\rho_j^q\right)\sigma\left(u_j(t)\right),\quad 0\le\underline{\rho}_j^q\le\rho_j^q\le\overline{\rho}_j^q, \\ &\qquad j\in\mathbf{I}[1,m],\ q\in\mathbf{I}[1,L], \end{aligned} \tag{3}$$

where $u_{jq}^F(t)$ represents the signal from the jth actuator which has failed in the qth fault mode, ρ_j^q is an unknown constant, the L is the total fault modes, and $\underline{\rho}_j^q$ and $\overline{\rho}_j^q$ represent the lower and upper bounds of ρ_j^q, respectively. Denote that

$$\begin{aligned} u_q^F(t) &= \left[u_{1q}^F(t), u_{2q}^F(t),\ldots,u_{mq}^F(t)\right]^T \\ &= \left(I-\rho^q\right)\sigma(u(t)), \end{aligned} \tag{4}$$

where $\rho^q = \mathrm{diag}[\rho_1^q,\rho_2^q,\ldots,\rho_m^q]$, $q\in\mathbf{I}[1,L]$. Considering the lower and upper bounds $\underline{\rho}_j^q$ and $\overline{\rho}_j^q$, the following set can be defined:

$$N_{\rho^q} = \left\{\rho^q\rho^q = \mathrm{diag}\left[\rho_1^q,\rho_2^q,\ldots,\rho_m^q\right], \rho_j^q=\underline{\rho}_j^q \text{ or } \rho_j^q=\overline{\rho}_j^q\right\}. \tag{5}$$

For convenience, the following uniform actuator fault model is exploited:

$$u^F(t) = (I-\rho)\,\mathrm{sat}(u(t)),\quad \rho\in\left\{\rho^1\cdots\rho^L\right\}, \tag{6}$$

where ρ is described by $\rho = \mathrm{diag}[\rho_1,\rho_2,\ldots,\rho_m]$. The following definitions and lemmas will be used in the sequel.

Definition 1 (see [33]). Consider the following system:

$$\begin{aligned} \dot{x}(t) &= A_a\left(\widehat{\rho}(t),\rho\right)x(t) + B_a\left(\widehat{\rho}(t),\rho\right)\omega(t), \\ z(t) &= C_a\left(\widehat{\rho}(t),\rho\right)x(t),\quad x(0)=0, \end{aligned} \tag{7}$$

where ρ is parameter vector and $\widehat{\rho}(t)$ is a time-varying parameter vector to be chosen. Let $r > 0$ be a given constant, then system (7) is said to be with an adaptive H_∞ performance index no larger than r if for any $\varepsilon > 0$, there exists a $\widehat{\rho}(t)$ such that the following inequality holds:

$$\int_0^\infty z^T(t)z(t)\,dt \le r^2\int_0^\infty \omega^T(t)\omega(t)\,dt+\varepsilon. \tag{8}$$

Definition 2. For a matrix $C_{cl}\in R^{m\times n}$, denote the jth row of C_{cl} as C_{clj}, and define

$$\wp\left(C_{cl}\right) = \left\{x\in R^n : \left|C_{clj}x\right|\le 1, j\in\mathbf{I}[1,m]\right\}. \tag{9}$$

Lemma 3 (see [32]). *If there exists a symmetric matrix* Θ *with*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix} \quad (10)$$

and Θ_{11}, $\Theta_{22} \in R^{Nn\times Nn}$ *such that the following inequalities hold:*

$$\begin{gathered} \Theta_{22jj} \leq 0, \quad j \in \mathbf{I}[1, N], \\ \Theta_{11} + \Theta_{12}\Delta(\delta) + (\Theta_{12}\Delta(\delta))^T + \Delta(\delta)\Theta_{22}\Delta(\delta) \geq 0, \\ \delta \in \Delta_v, \\ \begin{bmatrix} Q & E \\ E^T & F \end{bmatrix} + U^T U + G^T \Theta G < 0, \\ \rho \in \{\rho^1 \cdots \rho^L\}, \quad \rho^q \in N_{\rho^q}, \end{gathered} \quad (11)$$

then inequality

$$\begin{aligned} L(\delta) = {} & Q + \sum_{j=1}^{N} \delta_j E_j + \left(\sum_{j=1}^{N} \delta_j E_j\right)^T \\ & + \sum_{j=1}^{N}\sum_{p=1}^{N} \delta_j \delta_p F_{jp} + \left(U_0 + \sum_{j=1}^{N} \delta_j U_j\right)^T \\ & \times \left(U_0 + \sum_{j=1}^{N} \delta_j U_j\right) < 0 \end{aligned} \quad (12)$$

holds for all $\delta_j \in [\underline{\delta}_j \quad \overline{\delta}_j]$, *where* $Q = Q^T \in R^{n\times n}$ *and* $F_{pj} = F_{pj}^T \in R^{n\times n}$, $E_j \in R^{n\times n}$

$$\begin{gathered} \Delta(\delta) = \operatorname{diag}[\delta_1 I_{n\times n} \cdots \delta_N I_{n\times n}], \\ E = [E_1 \quad E_2 \quad \cdots \quad E_N], \qquad U = [U_0 \quad U_1 \quad \cdots \quad U_N], \\ F = \begin{bmatrix} F_{11} & \cdots & F_{1N} \\ \cdots & \cdots & \cdots \\ F_{N1} & \cdots & F_{NN} \end{bmatrix}, \qquad G = \begin{bmatrix} \begin{bmatrix} I_{n\times n} \\ \cdots \\ I_{n\times n} \end{bmatrix} & 0 \\ 0 & I_{Nn\times Nn} \end{bmatrix}. \end{gathered} \quad (13)$$

Let $\mathbf{D}$ *be a set of* $m \times m$ *diagonal matrices whose diagonal elements are either 1 or 0. There are* 2^m *elements in* $\mathbf{D}$, *and one denotes its elements as* D_i, $i \in \mathbf{I}[0, 2^m - 1]$, *where for* $i = z_1 2^{m-1} + Z_2 2^{m-2} + \cdots + z_m$ *with* $z_j \in \{0, 1\}$, *the diagonal elements of* D_i *are* $\{1-z_1, 1-z_2, \ldots, 1-z_m\}$. *Denote* $D_i^- = I - D_i$. *It is easy to see that* $D_i^- \in \mathbf{D}$. *Then, one has the following.*

Lemma 4 (see [35]). *For two vectors* $u, v \in R^m$. *Suppose that* $\|v\|_\infty \leq 1$. *Then,*

$$\operatorname{sat}(u) \in \operatorname{co}\{D_i u + D_i^- v : i \in [0, 2^m - 1]\}, \quad (14)$$

where co *denotes the convex hull.*

For a networked system, the performance of closed-loop system can be measured by the L_2 gain. However, this gain cannot be well defined for closed-loop system, since a sufficiently large disturbance may lead to unstable closed-loop system. For this reason, we need to consider a class of disturbances whose energy is bounded by a given value; that is,

$$\mathfrak{M}_\delta := \left\{\omega : R_+ \longrightarrow R^d : \int_0^\infty \omega^T(t)\,\omega(t)\,dt \leq \delta\right\}. \quad (15)$$

In this paper, we will consider the following problems. The first question is, what is the maximal value of δ such that the state will be bounded for all $\omega \in \mathfrak{M}_\delta$ for systems with time delay? This question can be referred to as disturbance tolerance level. The system performance can be measured by the restricted L_2 gain over $\mathfrak{M}_\delta$. In this paper, L_2 gain and $\mathfrak{M}_\delta$ will be considered at same time for networked system.

3. Main Results

The dynamics with actuator faults (6) and saturation are described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau(t)) + B_1\omega(t) \\ &\quad + B_2(I - \rho)\operatorname{sat}(u(t)), \\ z(t) &= Cx(t) + D(I - \rho)\operatorname{sat}(u(t)), \\ x(t) &= \phi(t) \quad t \in [-h, 0]. \end{aligned} \quad (16)$$

Rewrite (16) as

$$\begin{aligned} \dot{x}(t) &= y(t), \\ y(t) &= (A + A_1)x(t) + B_1\omega(t) \\ &\quad + B_2(I - \rho)\operatorname{sat}(u(t)) - A_1\int_{t-\tau(t)}^{t} y(s)\,ds, \\ z(t) &= Cx(t) + D(I - \rho)\operatorname{sat}(u(t)). \end{aligned} \quad (17)$$

The controller structure is given as

$$u(t) = K(\hat{\rho}(t))x(t) = (K_0 + K_a(\hat{\rho}(t)) + K_b(\hat{\rho}(t)))x(t), \quad (18)$$

where $\hat{\rho}(t)$ is used to estimate ρ, $K_a(\hat{\rho}(t)) = \sum_{j=1}^{m} K_{aj}\hat{\rho}_j(t)$ and $K_b(\hat{\rho}(t)) = \sum_{j=1}^{m} K_{bj}\hat{\rho}_j(t)$.

By Lemma 3, the following equality is given, with $x \in \wp(H(\hat{\rho}))$:

$$\begin{aligned} &\operatorname{sat}(K(\hat{\rho}(t))x(t)) \\ &\quad = \sum_{i=0}^{2^m-1} \eta_i [D_i K(\hat{\rho}(t)) + D_i^- H(\hat{\rho}(t))]x(t), \end{aligned} \quad (19)$$

for some scalars $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m - 1]$, such that $\sum_{i=0}^{2^m-1} \eta_i = 1$, and the following equality holds:

$$\begin{aligned}
&(I-\rho)\operatorname{sat}(u(t)) \\
&= \sum_{i=0}^{2^m-1} \eta_i \left[(I-\rho) D_i K_0 + D_i K_a(\rho) \right. \\
&\quad - \rho D_i K_a(\widehat{\rho}) + (I-\widehat{\rho}(t)) D_i K_b(\widehat{\rho}(t)) \\
&\quad + D_i K_a(\widetilde{\rho}(t)) + \widetilde{\rho} D_i K_b(\widehat{\rho}(t)) + (I-\rho) D_i^- H_0 \\
&\quad + D_i^- H_a(\rho) - \rho D_i^- H_a(\widehat{\rho}) \\
&\quad + (I-\widehat{\rho}(t)) D_i^- H_b(\widehat{\rho}(t)) \\
&\quad \left. + D_i^- H_a(\widetilde{\rho}(t)) + \widetilde{\rho} D_i^- H_b(\widehat{\rho}(t)) \right] x(t),
\end{aligned} \tag{20}$$

where $\widetilde{\rho}(t) = \widehat{\rho}(t) - \rho$. Denote $B^j = [0 \cdots b^j \cdots 0]$ with $B = [b^1 \cdots b^m]$, and

$$\Delta_{\widehat{\rho}} = \left\{ \widehat{\rho} = (\widehat{\rho}_1 \cdots \widehat{\rho}_m) : \widehat{\rho}_j \in \left\{ \min_q \left\{ \underline{\rho}_j^q \right\}, \max_q \left\{ \overline{\rho}_j^q \right\} \right\}, \right.$$
$$\left. q \in \mathbf{I}[1, L] \right\}. \tag{21}$$

Definition 5. Let $P_1 \in R^{n\times n}$ be a positive-definite matrix. Denote

$$\begin{aligned}
\varepsilon(P_1, \delta) &= \left\{ x \in R^n : x^T P_1 x \le \delta \right\}, \\
\varepsilon^*(P_1, \delta) &= \left\{ x \in R^n : x^T P_1 x + \sum_{j=1}^m \frac{\widetilde{\rho}_j^2(t)}{l_j} \le \delta \right\}.
\end{aligned} \tag{22}$$

Assuming that $l_j > 0$ is given, we denote $\delta^* = \delta + \max\{\sum_{j=1}^m (\widetilde{\rho}_j^2(t)/l_j)\}$.

The following lemma provides a method for choosing of η_i's, which are Lipschitzian functions in x and $\widehat{\rho}$ and thus are useful in controller design method.

Lemma 6 (see [11]). *Let $x \in \wp(H(\widehat{\rho}))$. For each $j \in \mathbf{I}[1, m]$,*

$$\lambda_j(x(t), \widehat{\rho}(t)) = \begin{cases} 1, \\ \text{if } K(\widehat{\rho}(t))_j x(t) = H(\widehat{\rho}(t))_j x(t) \\ \dfrac{\sigma\left(K(\widehat{\rho}(t))_j x(t)\right) - H(\widehat{\rho}(t))_j x(t)}{\left(K(\widehat{\rho}(t))_j - H(\widehat{\rho}(t))_j\right) x(t)}, \\ \text{otherwise}, \end{cases} \tag{23}$$

and for each $i \in \mathbf{I}[0, 2^m - 1]$, let $z_j \in \{0, 1\}$ such that $i = z_1 2^{m-1} + z_2 2^{m-2} + \cdots + z_m$, and define

$$\eta_i(x(t), \widehat{\rho}(t)) = \prod_{j=1}^m \left[z_j \left(1 - \lambda_j(x, \widehat{\rho})\right) + \left(1 - z_j\right) \lambda_j(x, \widehat{\rho}) \right]. \tag{24}$$

Then, η_i's are functions Lipschitz in x and $\widehat{\rho}$, such that, $\sum_{i=0}^{2^m-1} \eta_i = 1$, $0 \le \eta_i \le 1$, $i \in \mathbf{I}[0, 2^m - 1]$. Moreover, they satisfy relation (19).

By using the functions $\eta_i(x(t), \widehat{\rho}(t))$'s and controller (18)*, plant* (16) *can be written in a quasi-LPV form as follows:*

$$\begin{aligned}
y(t) &= (A + A_1) x(t) + B_1 \omega(t) \\
&\quad + B_2 \sum_{i=0}^{2^m-1} \eta_i \left[(I-\rho) D_i \left(K_0 + K_a(\widehat{\rho}(t)) + K_b(\widehat{\rho}(t))\right) \right. \\
&\quad + (I-\rho) D_i^- \left(H_0 + H_a(\widehat{\rho}(t)) x(t) \right. \\
&\quad \left. \left. + H_b(\widehat{\rho}(t))\right)\right] \\
&\quad - A_1 \int_{t-\tau(t)}^t y(s)\, ds.
\end{aligned} \tag{25}$$

Lemma 7 (see [36]). *For any $a \in R^n$, $b \in R^{2n}$, $Z_0 \in R^{2n\times n}$, $R \in R^{n\times n}$, $Y \in R^{n\times 2n}$, and $Z \in R^{2n\times 2n}$, the following holds:*

$$-2b^T F a \le \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & Y - Z_0^T \\ Y^T - Z_0 & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{26}$$

where $\left[\begin{smallmatrix} R & Y \\ Y^T & Z \end{smallmatrix}\right] \ge 0$.

Definition 8. Firstly, for system (25), consider the following *Lyapunov-Krasovskii* functional

$$V = V_1 + V_2 + V_3 + \sum_{j=1}^m \frac{\widetilde{\rho}_j^2(t)}{l_j}, \tag{27}$$

where

$$\begin{aligned}
V_1 &= \overline{x}^T(t) E P \overline{x}(t), \qquad V_2 = \int_h^0 \int_{t+\theta}^t y^T(s) R y(s)\, ds\, d\theta, \\
V_3 &= \int_{t-\tau(t)}^t x^T(s) S x(s)\, ds, \\
E &= \begin{bmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \qquad P_1 = P_1^T > 0.
\end{aligned} \tag{28}$$

Then, the following set can be given:

$$M_V(\delta) = \left\{ x(t) \in R^n : V = V_1 + V_2 + V_3 + \sum_{j=1}^m \frac{\widetilde{\rho}_j^2(t)}{l_j} \le \delta \right\}. \tag{29}$$

Remark 9. By Definitions 5 and 8, we can draw the conclusion that $M_V(\delta) \subset \varepsilon^*(P_1, \delta) \subset \varepsilon(P_1, \delta)$.

By Lemma 6, we analyze the auxiliary LPV system as follows, of which the closed-loop system comprising of (25) and (18) is a special case, for all $x(t) \in M_V(\delta) \subset \wp(H(\widehat{\rho}))$:

$$\dot{x}(t) = A(\eta) x(t) + B_1 \omega, \quad \eta \in \Gamma, \tag{30}$$

where $\eta = [\eta_0, \eta_1, \ldots, \eta_{2^m-1}]$, and

$$\Gamma = \left\{ \eta \in R^{2^m} : \sum_{i=0}^{2^m-1} \eta_i = 1, 0 \le \eta_i \le 1, i \in \mathbf{I}\left[0, 2^m - 1\right] \right\},$$

$$\begin{aligned} A(\eta) = {} & A + A_1 \\ & + B_2 \sum_{i=0}^{2^m-1} \eta_i \left[(I-\rho) D_i K_0 + D_i K_a(\rho) \right. \\ & \qquad - \rho D_i K_a(\widehat{\rho}) + (I - \widehat{\rho}(t)) D_i K_b(\widehat{\rho}(t)) \\ & \qquad + D_i K_a(\widetilde{\rho}(t)) + \widetilde{\rho} D_i K_b(\widehat{\rho}(t)) \\ & \qquad + (I-\rho) D_i^- H_0 + D_i^- H_a(\rho) \\ & \qquad - \rho D_i^- H_a(\widehat{\rho}) + (I - \widehat{\rho}(t)) D_i^- H_b(\widehat{\rho}(t)) \\ & \qquad \left. + D_i^- H_a(\widetilde{\rho}(t)) + \widetilde{\rho} D_i^- H_b(\widehat{\rho}(t)) \right] \\ & - A_1 \int_{t-\tau(t)}^{t} y(s)\, ds. \end{aligned} \tag{31}$$

Theorem 10. *Let $r_f > 0$, $r_n > 0$, $d, h > 0$ and $\delta > 0$ be given constants, then the following two conditions are satisfied.*

(I) *The trajectories of the closed-loop system that start from the origin will remain inside the domain $M_V(\delta)$ for every $\omega \in \mathfrak{M}_\delta$.*

(II) *In normal case, that is, $\rho = 0$, for $x(0) = 0$,*

$$\int_0^\infty z^T(t) z(t)\, dt \le r_n^2 \int_0^\infty \omega^T(t)\omega(t)\, dt + r_n^2 \sum_{j=1}^{m} \frac{\widetilde{\rho}_j^2(0)}{l_j}, \tag{32}$$

and in actuator failures cases, that is, $\rho \in \{\rho^1 \cdots \rho^L\}$, for $x(0) = 0$,

$$\int_0^\infty z^T(t) z(t)\, dt \le r_f^2 \int_0^\infty \omega^T(t)\omega(t)\, dt + r_f^2 \sum_{j=1}^{m} \frac{\widetilde{\rho}_j^2(0)}{l_j}, \tag{33}$$

where $\widetilde{\rho}(t) = \operatorname{diag}\{\widetilde{\rho}_1(t) \cdots \widetilde{\rho}_m(t)\}$, $\widetilde{\rho}_j(t) = \widehat{\rho}_j(t) - \rho_j$ if there exist matrices $Q_1 > 0, Q_2, Q_3, \overline{S}, \overline{R}, \overline{Z}_1, \overline{Z}_2, \overline{Z}_3, \overline{O}_0, \overline{O}_{aj}, \overline{O}_{bj}, \overline{Y}_0, \overline{Y}_{aj}, \overline{Y}_{bj}$, $j \in \mathbf{I}[1, m]$ and symmetric matrixes Θ^i, $i \in \mathbf{I}[0, 2^m - 1]$, with

$$\Theta^i = \begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i \\ \Theta_{12}^{iT} & \Theta_{22}^i \end{bmatrix} \tag{34}$$

and $\Theta_{11}^i, \Theta_{22}^i \in R^{mn \times mn}$ such that the following inequalities hold for all $D_i \in \mathbf{D}$ and $M_V(\delta) \subset \wp(H(\widehat{\rho}))$; that is, $|H(\widehat{\rho})_j x| \le 1$ for all $x \in M_V(\delta)$, $j \in \mathbf{I}[1, m]$:

$$\begin{gathered} \Theta_{22jj}^i \le 0, \quad j \in \mathbf{I}[1, m], \; i \in \mathbf{I}\left[0, 2^m - 1\right] \\ \Theta_{11}^i + \Theta_{12}^i \Delta(\widehat{\rho}) + \left(\Theta_{12}^i \Delta(\widehat{\rho})\right)^T \\ + \Delta(\widehat{\rho}) \Theta_{22}^i \Delta(\widehat{\rho}) \ge 0, \quad \widehat{\rho} \in \Delta_{\widehat{\rho}} \end{gathered} \tag{35}$$

$$\begin{bmatrix} M_i & V_{0i}^T & \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_1 (I-\varepsilon)\overline{S} \\ 0 \end{bmatrix} & \begin{bmatrix} Q_1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} hQ_2^T \\ hQ_3^T \\ 0 \end{bmatrix} \\ * & -r_n^2 I & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -(1-d)\overline{S} & 0 & 0 \\ * & * & * & * & -\overline{S} & 0 \\ * & * & * & * & * & -h\overline{R} \end{bmatrix} \quad \rho = 0, \tag{36}$$

$$\begin{bmatrix} M_i & V_{0i}^T & \begin{bmatrix} 0 \\ B_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_1 (I-\varepsilon)\overline{S} \\ 0 \end{bmatrix} & \begin{bmatrix} Q_1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} hQ_2^T \\ hQ_3^T \\ 0 \end{bmatrix} \\ * & -r_f^2 I & 0 & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -(1-d)\overline{S} & 0 & 0 \\ * & * & * & * & -\overline{S} & 0 \\ * & * & * & * & * & -h\overline{R} \end{bmatrix} < 0, \quad \rho \in \left\{\rho^1 \cdots \rho^L\right\}, \; \rho^q \in N_{\rho^q} \tag{37}$$

$$\begin{bmatrix} \overline{R} & 0 & \overline{R}\varepsilon A_1^T \\ * & \overline{Z}_1 & \overline{Z}_2 \\ * & * & \overline{Z}_3 \end{bmatrix} \ge 0, \tag{38}$$

where

$$\begin{aligned} M_i &= \begin{bmatrix} N_{0i} & U_i \\ U_i^T & \Omega_i \end{bmatrix} + G^T \Theta^i G, \\ N_{0i} &= \begin{bmatrix} Q_2 + Q_2^T + h\overline{Z}_1 & T_{1i} \\ * & -Q_3 - Q_3^T + h\overline{Z}_3 \end{bmatrix}, \\ U_i &= \begin{bmatrix} U_{1i} & U_{2i} & \cdots & U_{mi} \end{bmatrix}, \\ V_{0i} &= \begin{bmatrix} V_{00i} & V_{01i} & \cdots & V_{0mi} \end{bmatrix}, \\ \Omega_i &= \left[\Omega_{ijp}\right], \quad j, p = 1 \cdots m, \\ T_{1i} &= Q_3 - Q_2^T + Q_1\left(A^T + \varepsilon A_1^T\right) + h\overline{Z}_2 \\ &\quad + (I-\rho) D_i \overline{Y}_0^T + (I-\rho) D_i^- \overline{O}_0^T, \\ V_{00i} &= \left[CQ_1 + D(I-\rho) D_i \overline{Y}_0 + D(I-\rho) D_i^- \overline{O}_0 \quad 0 \right], \end{aligned}$$

$$V_{0ji} = \left[D(I-\rho)D_i\left(\overline{Y}_{aj}+\overline{Y}_{bj}\right)+D(I-\rho)D_i^{-}\left(\overline{O}_{aj}+\overline{O}_{bj}\right)\ 0\right],$$

$$U_{ji} = \begin{bmatrix} 0 & -\overline{Y}_{aj}^T D_i\rho B_2^T + \overline{Y}_{bj}^T D_i B_2^T - \overline{O}_{aj}^T D_i\rho B_2^T + \overline{O}_{bj}^T D_i B_2^T \\ 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} & 0 \\ 0 & I \end{bmatrix},$$

$$\Omega_{ijp} = \begin{bmatrix} 0 & -B_2^j D_i \overline{Y}_{bp} - \overline{Y}_{bj}^T D_i B_2^{pT} - B_2^j D_i \overline{O}_{bp} - \overline{O}_{bj}^T D_i B_2^{pT} \\ 0 & 0 \end{bmatrix},$$

$$\overline{Y}_a(\rho) = \sum_{j=1}^{m} \overline{Y}_{aj}\rho_j, \tag{39}$$

and also $\widehat{\rho}_j(t)$ *is determined according to the adaptive law*

$$\dot{\widehat{\rho}}_j = Proj_{[\min_q\{\underline{\rho}_j^q\}\max_q\{\overline{\rho}_j^q\}]}\left\{L_j\right\}$$

$$= \begin{cases} 0, & if\ \widehat{\rho}_j = \min\limits_q\left\{\underline{\rho}_j^q\right\}, L_j \le 0 \\ & or\ \widehat{\rho}_j = \max\limits_q\left\{\overline{\rho}_j^q\right\}, L_j \ge 0 \\ L_j, & otherwise, \end{cases} \tag{40}$$

where

$$L_j = -l_j \overline{x}^T(t) \begin{bmatrix} 0 & 0 \\ \Lambda_{ji} & 0 \end{bmatrix} \overline{x}(t),$$

$$\Lambda_{ji} = B_2^j\left(\sum_{i=0}^{2^m-1}\eta_i D_i\right)K_b(\widehat{\rho}) + B_2\left(\sum_{i=0}^{2^m-1}\eta_i D_i\right)K_{aj} + B_2^j\left(\sum_{i=0}^{2^m-1}\eta_i D_i^-\right)H_b(\widehat{\rho}) + B_2\left(\sum_{i=0}^{2^m-1}\eta_i D_i^-\right)H_{aj},$$

$$Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \tag{41}$$

where $K_{aj} = \overline{Y}_{aj}Q_1^{-1}$, $K_{bj} = \overline{Y}_{bj}Q_1^{-1}$, $H_{aj} = \overline{O}_{aj}Q_1^{-1}$, *and* $H_{bj} = \overline{O}_{bj}Q_1^{-1} \cdot l_j > 0$ $(j \in \mathbf{I}[1,m])$ *are the adaptive law gains which can be given according to practical applications. Then, the following controller can be given:*

$$K(\widehat{\rho}) = \overline{Y}_0 Q_1^{-1} + \sum_{j=1}^{m}\widehat{\rho}_j\overline{Y}_{aj}Q_1^{-1} + \sum_{j=1}^{m}\widehat{\rho}_j\overline{Y}_{bj}Q_1^{-1}. \tag{42}$$

Proof. See appendix. □

From Theorem 10, we have the following algorithm to optimize the adaptive H_∞ performance in normal and fault cases and the disturbance tolerance level δ with considering time delay.

Algorithm 11. Suppose that r_n and r_f denote the adaptive H_∞ performance indexes for the normal case and fault cases of the closed-loop system (30), respectively. Let δ denote the disturbance tolerance level. Then, r_n, r_f are minimized, and δ is maximized if the following optimization problem is solvable:

$$\begin{aligned} &\min\ \eta = \alpha\eta_n + \beta\eta_f + \gamma\eta_\delta \\ &\text{s.t.}\quad (a)\quad (35),(36),(37),(38) \\ &\qquad\quad (b)\quad M_V(\delta) \subset \wp\left(H(\widehat{\rho})\right), \end{aligned} \tag{43}$$

where $\eta_n = r_n^2$, $\eta_f = r_f^2$, $\eta_\delta = 1/\delta^* = 1/(\delta + \max\{\sum_{j=1}^m(\overline{\rho}_j^2(t)/l_j)\})$, and α, β, and γ are weighting coefficients.

By Definition 2, condition (b) cannot be shown as LMIs directly. However, obviously, $M_V(\delta) \subset \varepsilon^*(P_1,\delta^*) \subset \varepsilon(P_1,\delta^*)$, which implies that (b) can be replaced with (b1) as follows:

$$(b1)\quad \varepsilon(P_1,\delta^*) \subset \wp\left(H(\widehat{\rho})\right). \tag{44}$$

Equation (44) is equivalent to

$$\delta^* h(\widehat{\rho})_j P_1^{-1} h(\widehat{\rho})_j^T \le 1 \Longleftrightarrow \begin{bmatrix} \dfrac{1}{\delta^*} & h(\widehat{\rho})_j P_1^{-1} \\ * & P_1^{-1} \end{bmatrix} \ge 0, \tag{45}$$

for all $j \in \mathbf{I}[1,m]$, where $h(\widehat{\rho})_j$ is the jth row of $H(\widehat{\rho})$. Then, it can be drawn that (45) is equivalent to the following inequalities:

$$(b2)\quad \begin{bmatrix} -\eta_\delta & -\overline{O}_{0s} \\ * & -Q_1 \end{bmatrix} + \sum_{j=1}^{m}\widehat{\rho}_j \begin{bmatrix} 0 & -\overline{O}_{ajs} - \overline{O}_{bjs} \\ * & 0 \end{bmatrix} \le 0, \tag{46}$$

where $\widehat{\rho} \in \Delta_{\widehat{\rho}}$ and $\overline{O}_{ajs}$ is the sth row of $\overline{O}_{aj}$, $s \in \mathbf{I}[1,m]$.

Remark 12. Theorem 10 prevents a condition for the existence of an adaptive fault tolerant H_∞ controller. In Theorem 10, if set $Y_{aj} = 0$, $Y_{bj} = 0$, $O_{aj} = 0$, and $O_{bj} = 0$, $j \in \mathbf{I}[1,m]$, the condition of Theorem 10 is reduced to fixed gains condition. By the following example, we can get that the adaptive controller can guarantee better effect.

4. Examples

Example 13. Consider the system of the form (1) with

$$\begin{aligned} A &= \begin{bmatrix} 3 & 2 \\ 3 & 40 \end{bmatrix}, & A_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}, \\ C &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T, & D &= \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T, \\ \tau(t) &= \frac{1+\sin(t)}{4}, \end{aligned} \tag{47}$$

and the following two possible fault modes.

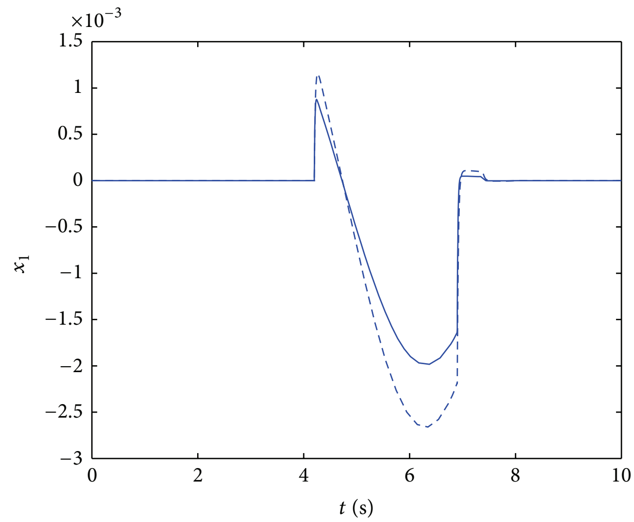

FIGURE 1: Response curve of the first state in normal case with the adaptive controller (solid) and the fixed gain controller (dashed).

Fault mode 1: both of the two actuators are normal; that is $\rho_1^1 = \rho_2^1 = 0$. Fault mode 2: the first actuator is outage, and the second actuator may be normal or loss of effectiveness, described by $\rho_1^2 = 1$, $0 \leq \rho_2^2 \leq a$, where $a = 0.5$ denotes the maximal loss of effectiveness for the second actuator.

Let $\alpha = 10$, $\beta = 1$, and $\gamma = 10$, and let the optimal indexes with fixed controller gains are $\eta_n = 0.1963$, $\eta_f = 9.8933$, $\eta_\delta = 20.5385$, and $\eta = 217.2408$. By solving (43), the optimal indexes can be given as $\eta_n = 0.5881$, $\eta_f = 9.1236$, $\eta_\delta = 9.6701$, and $\eta = 111.7048$. For getting smaller number for every optimal index, we may revise $\alpha = 110$, $\beta = 0.2$, and $\gamma = 0.5$. Then, the indexes can be drawn that $\eta_n = 0.1676$, $\eta_f = 7.1242$, and $\eta_\delta = 18.6399$. For illustrating the efficiency of the design method, the following simulations is given.

During the following simulation, fault case is considered as follows. At 0 second, the first actuator is outage. Here, we choose $l_1 = l_2 = 100$.

Firstly, we consider the H_∞ performance. The disturbance is given as

$$\omega_1(t) = \omega_2(t) = \begin{cases} \cos(t), & 4.2 \leq t \leq 6.9 \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Figures 1 and 2 show the responses curves of the first state in normal and fault case, respectively.

Then, we consider the disturb tolerance problem. The disturbance is given as

$$\omega_1(t) = \omega_2(t) = \begin{cases} 27.2, & 4 \leq t \leq 5 \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

Figures 3 and 4 show the responses curves of the states in normal case.

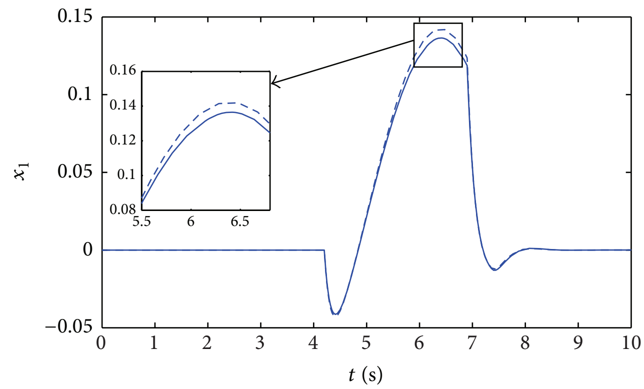

FIGURE 2: Response curve of the first state in fault case with the adaptive controller (solid) and the fixed gain controller (dashed).

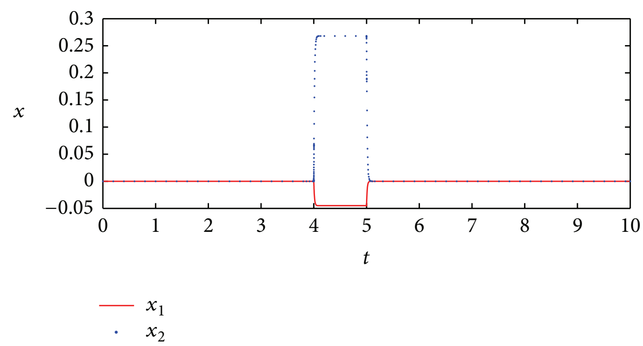

FIGURE 3: Responses curves of the states with adaptive controller in normal case.

5. Conclusions

In this paper, an adaptive fault-tolerant H_∞ controllers design method was given for networked systems with actuator saturation. The designs were proposed in LMIs approach, which could guarantee the disturbance tolerance ability and adaptive H_∞ performances of networked systems in the cases of actuator saturation and actuator failures. An example, has been given to illustrate the efficiency of the design method.

Appendix

Proof of Theorem 10. Item (II) will be proved firstly. By Definition 8, since $\overline{x}^T(t)EP\overline{x}(t) = x^T(t)P_1x(t)$, then

$$\begin{aligned} \frac{d}{dt}\left\{\overline{x}^T(t)EP\overline{x}(t)\right\} &= 2x^T(t)P_1\dot{x}(t) \\ &= 2\overline{x}^T(t)P^T\begin{bmatrix}\dot{x}(t)\\0\end{bmatrix}. \end{aligned} \tag{A.1}$$

From the derivative of $V(t)$ along the closed-loop system (30), it follows that

$$\begin{aligned} \dot{V} &= \overline{x}^T(t)\sum_{i=0}^{2^m-1}(\eta_i\Phi_{1i})\overline{x}(t) + \chi(t) \\ &\quad - (1-d)x^T(t-\tau(t))Sx(t-\tau(t)) \\ &\quad - \int_{t-h}^{t} y^T(s)Ry(s)\,ds + 2\sum_{i=1}^{m}\frac{\tilde{\rho}_i(t)\dot{\tilde{\rho}}_i(t)}{l_i} \\ &\quad + 2\overline{x}^T(t)P^T\begin{bmatrix}0\\B_1\end{bmatrix}\omega(t), \end{aligned} \tag{A.2}$$

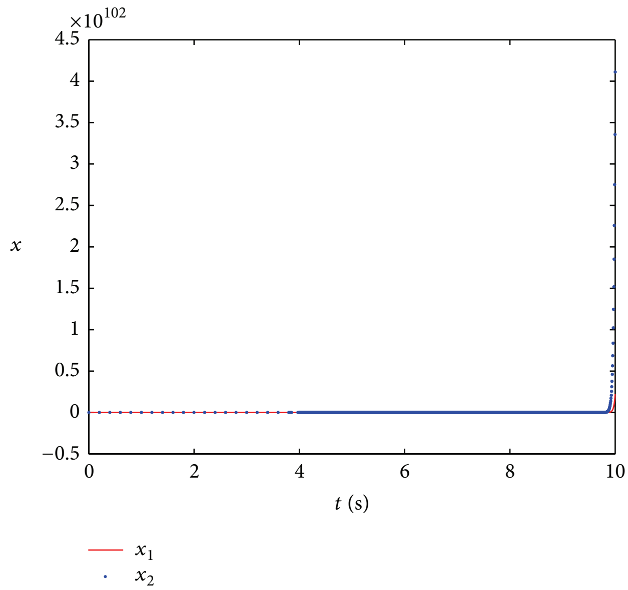

FIGURE 4: Responses curves of the states with fixed gain controller in normal case.

where

$$\Phi_{1i} = P^T\Delta_{0i} + \Delta_{0i}^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix},$$
$$\chi(t) = -2\int_{t-\tau(t)}^{t} \bar{x}^T(t) P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} y(s)\, ds,$$
$$\Delta_{0i} = \begin{bmatrix} 0 & I \\ A + A_1 + B_2(I-\rho)(D_i K(\hat{\rho}) + D_i^- H(\hat{\rho})) & -I \end{bmatrix}. \tag{A.3}$$

By Lemma 7, taking $Z_0 = P^T \left[\begin{smallmatrix} 0 \\ A_1 \end{smallmatrix} \right] a = y(s)$, $b = \bar{x}(t)$, it follows that

$$\begin{aligned}\chi(t) &\le \int_{t-\tau(t)}^{t} \begin{bmatrix} y^T(s) & \bar{x}^T(s) \end{bmatrix} W_1 \begin{bmatrix} y(s) \\ \bar{x}(s) \end{bmatrix} ds \\ &= \int_{t-\tau(t)}^{t} y^T(s) R y(s)\, ds + \int_{t-\tau(t)}^{t} \bar{x}^T(t) Z \bar{x}(t)\, ds \\ &\quad + 2\int_{t-\tau(t)}^{t} y^T(s)\left(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P\right)\bar{x}(t)\, ds \\ &= \int_{t-\tau(t)}^{t} y^T(s) R y(s)\, ds + \tau(t)\bar{x}^T(t) Z \bar{x}(t) \\ &\quad + 2\int_{t-\tau(t)}^{t} \dot{x}^T(s)\left(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P\right)\bar{x}(t)\, ds \\ &\le \int_{t-h}^{t} y^T(s) R y(s)\, ds + 2x^T(t)\left(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P\right)\bar{x}(t) \\ &\quad - 2x^T(t-\tau(t))\left(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P\right)\bar{x}(t) \\ &\quad + h\bar{x}^T(t) Z \bar{x}(t),\end{aligned} \tag{A.4}$$

where $W_1 = \left[\begin{smallmatrix} R & Y-[0\ \ A_1^T]P \\ * & Z \end{smallmatrix} \right]$ and R, Y, Z satisfying $\left[\begin{smallmatrix} R & Y \\ * & Z \end{smallmatrix} \right] \ge 0$.

Furthermore, by (20) it follows that

$$\begin{aligned}&\dot{V} + \frac{1}{\gamma_f^2} z^T(t) z(t) - w^T(t) w(t) \\ &= \bar{x}^T(t) \sum_{i=0}^{2^m-1} (\eta_i \Phi_{2i}) \bar{x}(t) - 2x^T(t-\tau(t))(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P)\bar{x}(t) \\ &\quad + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}i}{li} + \frac{1}{\gamma_f^2} x^T(t) \\ &\quad \times \left[C + D(I-\rho) \sum_{i=0}^{2^m-1} \eta_i (D_i K(\hat{\rho}) + D_i^- H(\hat{\rho})) \right]^T \\ &\quad \times \left[C + D(I-\rho) \sum_{i=0}^{2^m-1} \eta_i (D_i K(\hat{\rho}) + D_i^- H(\hat{\rho})) \right] x(t) \\ &\quad + \bar{x}^T(t) P^T \begin{bmatrix} 0 & B_1^T \end{bmatrix}^T \begin{bmatrix} 0 & B_1^T \end{bmatrix} P\bar{x}(t) \\ &\quad - (1-d) x^T(t-\tau(t)) S x(t-\tau(t)) \\ &\quad - \left(\omega^T - \bar{x}^T(t) P^T \begin{bmatrix} 0 & B_1^T \end{bmatrix}^T\right)\left(\omega - \begin{bmatrix} 0 & B_1^T \end{bmatrix} P\bar{x}\right) \\ &\quad + 2\bar{x}^T(t) P^T \begin{bmatrix} 0 & 0 \\ B_2 \sum_{i=0}^{2^m-1} \eta_i [D_i (K_a(\hat{\rho}) + \tilde{\rho} K_b(\hat{\rho})) + D_i^- (H_a(\hat{\rho}) + \tilde{\rho} H_b(\hat{\rho}))] & 0 \end{bmatrix} \bar{x}(t),\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}\Phi_{2i} &= P^T\Delta_{1i} + \Delta_{1i}^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + hZ \\ &\quad + \begin{bmatrix} Y^T & 0 \end{bmatrix}^T + \begin{bmatrix} Y^T & 0 \end{bmatrix}\end{aligned} \tag{A.6}$$

with $\Delta_{1i} = \left[\begin{smallmatrix} 0 & I \\ W_{2i} & -I \end{smallmatrix} \right]$, $W_{2i} = A + B_2 \sum_{i=0}^{2^m-1} \eta_i [(I-\rho) D_i K_0 + D_i K_a(\rho) - \rho D_i K_a(\hat{\rho}) + (I-\rho) D_i K_b(\hat{\rho}) + (I-\rho) D_i^- H_0 + D_i^- H_a(\rho) - \rho D_i^- H_a(\hat{\rho}) + (I-\rho) D_i^- H_b(\hat{\rho})]$.

Then,

$$\begin{aligned}&\dot{V} + \frac{1}{\gamma_f^2} z^T(t) z(t) - w^T(t) w(t) \\ &\le -2x^T(t-\tau(t))\left(Y - \begin{bmatrix} 0 & A_1^T \end{bmatrix} P\right)\bar{x}(t) \\ &\quad + 2\sum_{i=1}^{m} \frac{\tilde{\rho}_i(t)\dot{\hat{\rho}}_i(t)}{l_i} + \bar{x}^T \sum_{i=0}^{2^m-1} (\eta_i \Phi_{3i}) \bar{x} \\ &\quad + 2\bar{x}^T(t) P^T \begin{bmatrix} 0 & 0 \\ \Phi_4 & 0 \end{bmatrix} \bar{x}(t),\end{aligned} \tag{A.7}$$

where

$$\Phi_{3i} = P^T\Delta_{1i} + \Delta_{1i}^T P + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + P^T\begin{bmatrix} 0 & B_1^T \end{bmatrix}^T \begin{bmatrix} 0 & B_1^T \end{bmatrix} P + hZ + \begin{bmatrix} Y^T & 0 \end{bmatrix} + \begin{bmatrix} Y^T & 0 \end{bmatrix}^T + \frac{1}{\gamma_f^2}\begin{bmatrix} \Phi_5^T \\ 0 \end{bmatrix}\begin{bmatrix} \Phi_5 & 0 \end{bmatrix}, \tag{A.8}$$

$$\Phi_4 = B_2 \sum_{i=0}^{2^m-1} \eta_i \left[D_i\left(K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})\right) + D_i^-\left(H_a(\tilde{\rho}) + \tilde{\rho}H_b(\hat{\rho})\right)\right],$$

$$\Phi_5 = C + D(I-\rho)\left(D_iK(\hat{\rho}) + D_i^- H(\hat{\rho})\right).$$

Let $B_2 = [b^1 \cdots b^m] B_2^j = [0 \cdots b^j \cdots 0]$, then we have

$$\begin{aligned} B_2\tilde{\rho}D_iK_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j B_2^j D_i K_b(\hat{\rho}), \\ B_2\tilde{\rho}D_i^- H_b(\hat{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j B_2^j D_i^- H_b(\hat{\rho}), \\ B_2 D_i K_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j B_2 D_i K_{aj}, \\ B_2 D_i^- H_a(\tilde{\rho}) &= \sum_{j=1}^{m} \tilde{\rho}_j B_2 D_i^- H_{aj}. \end{aligned} \tag{A.9}$$

In fact, ρ_i is an unknow constant which denotes the loss of effectiveness of the ith actuator. So, from $\tilde{\rho}_j(t) = \hat{\rho}_j(t) - \rho$, it follows that $\dot{\tilde{\rho}}_j(t) = \dot{\hat{\rho}}$. Now, if the adaptive laws are chosen as (40), then

$$2\overline{x}^T P^T \begin{bmatrix} 0 & 0 \\ B_2 \sum_{i=0}^{2^m-1} \eta_i \left[D_i\left(K_a(\tilde{\rho}) + \tilde{\rho}K_b(\hat{\rho})\right) + D_i^-\left(H_a(\tilde{\rho}) + \tilde{\rho}H_b(\hat{\rho})\right)\right] & 0 \end{bmatrix} \overline{x} + 2\sum_{j=1}^{m} \frac{\tilde{\rho}_i\dot{\hat{\rho}}_i}{l_j} \leq 0. \tag{A.10}$$

Let $\xi(t) = \operatorname{col}\begin{bmatrix} x(t) & y(t) & x(t-\tau(t)) \end{bmatrix}$, then

$$\dot{V}(t) + \frac{1}{\gamma_f^2} z^T(t) z(t) - \omega^T(t)\omega(t) \leq \xi^T(t) \sum_{i=1}^{2^m-1} (\eta_i \Psi_i)\, \xi(t), \tag{A.11}$$

where

$$\Psi_i = \begin{bmatrix} \Phi_{3i} & P^T\begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y^T \\ * & -S(1-d) \end{bmatrix}. \tag{A.12}$$

Furthermore, the problem $\dot{V}(t) + (1/\gamma_f^2) z^T(t)z(t) - \omega^T(t)\omega(t) \leq 0$ reduces to

$$\Psi_i < 0, \qquad \begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \geq 0, \quad i \in \mathbf{I}\left[0, 2^m - 1\right]. \tag{A.13}$$

It is obvious from the requirement of $0 < P_1$ and the fact that in (A.13) $-(P_3 + P_3^T)$ must be negative and P is nonsingular.

Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \qquad \Pi = \operatorname{diag}\{Q, I\}. \tag{A.14}$$

We multiply Ψ_i by Π^T and Π, on the left and the right, respectively. Applying Fisher's lemma to the emerging quadratic term in Q, denoting $\overline{S} = S^{-1}\overline{Z} = \begin{bmatrix} \overline{Z}_1 & \overline{Z}_2 \\ \overline{Z}_2^T & \overline{Z}_3 \end{bmatrix} = Q^T Z Q$, $\overline{R} = R^{-1}$, and choosing $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = \varepsilon A_1^T \begin{bmatrix} P_2 & P_3 \end{bmatrix}$, where $\varepsilon \in R^{n\times n}$ is a diagonal matrix, we obtain the following: $\Psi_i < 0$ is equivalent to

$$\begin{bmatrix} \Xi_{0i} + Q_1 S Q_1 + hQ_2^T R Q_2 & \Xi_{1i} + hQ_2^T R Q_3 \\ * & \Xi_2 \end{bmatrix} < 0 \tag{A.15}$$

with

$$\begin{aligned} \Xi_{0i} &= \frac{1}{\gamma_f^2}\Xi_{3i}^T \Xi_{3i} + Q_2 + Q_2^T + h\overline{Z}_1, \\ \Xi_{1i} &= Q_3 - Q_2^T + Q_1\left(A^T + \varepsilon A_1^T\right) + h\overline{Z}_2 \\ &\quad + \left[(I-\rho)(D_i\overline{Y}_0 + D_i^-\overline{O}_0)\right]^T B_2^T \\ &\quad + \left[D_i\overline{Y}_a(\rho) + D_i^-\overline{O}_a(\rho)\right]^T B_2^T \\ &\quad - \left[\rho(D_i\overline{Y}_a(\hat{\rho}) + D_i^-\overline{O}_a(\hat{\rho}))\right]^T B_2^T \\ &\quad + \left[(I-\hat{\rho})\left(D_i\overline{Y}_b(\hat{\rho}) + D_i^-\overline{O}_b(\hat{\rho})\right)\right]^T B_2^T, \\ \Xi_2 &= -Q_3 - Q_3^T + h\overline{Z}_3 + \Upsilon_0 + hQ_3^T R Q_3, \\ \Xi_{3i} &= CQ_1 + D(I-\rho)\left(D_i\overline{Y}(\hat{\rho}) + D_i^-\overline{O}(\hat{\rho})\right), \\ \Upsilon_0 &= A_1(I_n - \varepsilon)(1-d)^{-1}(I_n - \varepsilon)A_1^T + B_1B_1^T, \\ \overline{Y}_0 &= K_0Q_1, \qquad \overline{Y}_{aj} = K_{aj}Q_1, \qquad \overline{Y}_{bj} = K_{bj}Q_1, \\ \overline{Y}_a(\rho) &= \sum_{j=1}^{m}\overline{Y}_{aj}\rho_j, \qquad \overline{Y}_a(\hat{\rho}) = \sum_{j=1}^{m}\overline{Y}_{aj}\hat{\rho}_j, \\ \overline{Y}_b(\hat{\rho}) &= \sum_{j=1}^{m}\overline{Y}_{bj}\hat{\rho}_j, \end{aligned}$$

$$\begin{aligned}
&\overline{Y}(\widehat{\rho}) = \overline{Y}_0 + \overline{Y}_a(\widehat{\rho}) + \overline{Y}_b(\widehat{\rho}), \\
&\overline{O}_0 = H_0 Q_1, \qquad \overline{O}_{aj} = H_{aj} Q_1, \qquad \overline{O}_{bj} = H_{bj} Q_1, \\
&\overline{O}_a(\rho) = \sum_{j=1}^{m} \overline{O}_{aj} \rho_j, \\
&\overline{O}_a(\widehat{\rho}) = \sum_{j=1}^{m} \overline{O}_{aj} \widehat{\rho}_j, \qquad \overline{O}_b(\widehat{\rho}) = \sum_{j=1}^{m} \overline{O}_{bj} \widehat{\rho}_j, \\
&\overline{O}(\widehat{\rho}) = \overline{O}_0 + \overline{O}_a(\widehat{\rho}) + \overline{O}_b(\widehat{\rho}).
\end{aligned} \tag{A.16}$$

Furthermore, (A.15) can be described by

$$\begin{aligned}
M_i(\widehat{\rho}) = {} & N_{1i} + \sum_{j=1}^{m} \widehat{\rho}_j U_{ji} + \left(\sum_{j=1}^{m} \widehat{\rho}_j U_{ji} \right)^T + \sum_{j=1}^{m} \sum_{p=1}^{m} \widehat{\rho}_j \widehat{\rho}_p \Omega_{ijp} \\
& + \left(V_{00i} + \sum_{j=1}^{m} \widehat{\rho}_j V_{0ji} \right)^T \left(V_{00i} + \sum_{p=1}^{m} \widehat{\rho}_j V_{0ji} \right) < 0,
\end{aligned} \tag{A.17}$$

where

$$N_{1i} = N_{0i} + \begin{bmatrix} Q_1 S Q_1 + h Q_2^T R Q_2 & h Q_2^T R Q_3 \\ * & \Upsilon_0 + h Q_3^T R Q_3 \end{bmatrix} \tag{A.18}$$

and $U_{ji}, \Omega_{ijp}, V_{00i}, V_{0ji}$, $j = 1 \cdots m$ are defined in (39).

If we multiply $\left[\begin{smallmatrix} R & Y \\ * & Z \end{smallmatrix}\right] \geq 0$, on the left and on the right, by diag $\{R^{-1}, Q^T\}$ and diag $\{R^{-1}, Q\}$, then it follows that $\left[\begin{smallmatrix} \overline{R} & 0 & \overline{R}\varepsilon A_1^T \\ * & \overline{Z}_1 & \overline{Z}_2 \\ * & * & \overline{Z}_3 \end{smallmatrix}\right] \geq 0$. By Lemmas 3 and 7, it is easy to see if conditions (34), (37), and (38) hold, then (A.17) and $\left[\begin{smallmatrix} R & Y \\ * & Z \end{smallmatrix}\right] \geq 0$ are satisfied, which implies that $\dot{V}(t) \leq 0$. Furthermore, by Lemma 3 and (37), it follows that $\dot{V}(t) + (1/\gamma_f^2) z^T(t) z(t) - \omega^T(t)\omega(t) \leq 0$ holds for any $x \in \wp(H(\widehat{\rho}))$, $\rho \in \{\rho^1 \cdots \rho^L\}$, $\rho^q \in N_{\rho^q}$, and $\widehat{\rho}$ satisfying (40). The proofs for the normal case of closed-loop system (16) are similar and omitted here.

To prove item (I): by the proof of item (II), we have $\dot{V} \leq \omega^T \omega$, which implies that

$$V(x(t)) \leq \int_0^{\infty} \omega^T(t)\,\omega(t)\,dt + \sum_{j=1}^{m} \frac{\widetilde{\rho}_j^2(0)}{l_j} \leq \delta^*, \qquad \text{for } x(0) = 0. \tag{A.19}$$

Then, the conclusion can be drawn that trajectories of the closed-loop system that start from the origin will remain inside $\varepsilon^*(P, \delta^*)$ for every $\omega \in \mathfrak{M}_\delta$. □

Acknowledgments

This work was supported in part by Program for the Funds of National Science of China (Grant nos. 61203087 and 61074090), the Liaoning Education Department Plan Project of China (no. L2010426), and the Liaoning Students Innovation Training Plan Project of China (nos. 201210143012 and 2012106).

References

[1] H. Hindi and S. Boyd, "Analysis of linear systems with saturation using convex optimization," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 903–908, December 1998.

[2] T. Hu, Z. Lin, and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance," *Automatica*, vol. 38, no. 2, pp. 351–359, 2002.

[3] A. Megretski, "L_2 BIBO output feedback stabilization with saturated control," in *Proceedings of the 13th IFAC World Congress*, pp. 435–440, 1996.

[4] T. Nguyen and F. Jabbari, "Disturbance attenuation for systems with input saturation: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 852–857, 1999.

[5] T. Nguyen and F. Jabbari, "Output feedback controllers for disturbance attenuation with bounded control," in *Proceedings of the 36th IEEE Conference on Decision and Control*, pp. 177–182, December 1997.

[6] C. Paim, S. Tarbouriech, J. M. Gomes da Silva Jr., and E. B. Castelan, "Control design for linear systems with saturating actuators and L_2-bounded disturbances," in *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 4148–4153, December 2002.

[7] C. W. Scherer, H. Chen, and F. AllgHower, "Disturbance attenuation withactuator constraints by hybrid state-feedback control," in *Proceedings of the 41st IEEE Conference on Decision and Controls*, pp. 4134–4138, 2002.

[8] R. Suárez, J. Alvarez-Ramírez, M. Sznaier, and C. Ibarra-Valdez, "L_2-disturbance attenuation for linear systems with bounded controls: an ARE-based approach," in *Control of Uncertain Systems with Bounded Inputs*, S. Tarbouriech and G. Garcia, Eds., vol. 227 of *Lecture Notes in Control and Information Sciences*, pp. 25–38, Springer, Berlin, Germany, 1997.

[9] T. Hu, A. R. Teel, and L. Zaccarian, "Stability and performance for saturated systems via quadratic and nonquadratic Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1770–1786, 2006.

[10] T. Hu, A. R. Teel, and L. Zaccarian, "Anti-windup synthesis for linear control systems with input saturation: achieving regional, nonlinear performance," *Automatica*, vol. 44, no. 2, pp. 512–519, 2008.

[11] F. Wu, Z. Lin, and Q. Zheng, "Output feedback stabilization of linear systems with actuator saturation," *IEEE Transactions on Automatic Control*, vol. 52, no. 1, pp. 122–128, 2007.

[12] H. Fang, Z. Lin, and T. Hu, "Analysis of linear systems in the presence of actuator saturation and L_2-disturbances," *Automatica*, vol. 40, no. 7, pp. 1229–1238, 2004.

[13] Z. Zuo, D. W. C. Ho, and Y. Wang, "Fault tolerant control for singular systems with actuator saturation and nonlinear perturbation," *Automatica*, vol. 46, no. 3, pp. 569–576, 2010.

[14] H. Huang, D. Li, Z. Lin, and Y. Xi, "An improved robust model predictive control design in the presence of actuator saturation," *Automatica*, vol. 47, no. 4, pp. 861–864, 2011.

[15] Y. Wang, C. Wang, and Z. Zuo, "Controller synthesis for Markovian jump systems with incomplete knowledge of transition probabilities and actuator saturation," *Journal of the Franklin Institute*, vol. 348, no. 9, pp. 2417–2429, 2011.

[16] L. Lu, Z. Lin, and A. Bateman, "Decentralised state feedback design for large-scale linear systems subject to input saturation," *IET Control Theory & Applications*, vol. 4, no. 2, pp. 206–227, 2010.

[17] A. Wei and Y. Wang, "Stabilization and H_∞ control of nonlinear port-controlled Hamiltonian systems subject to actuator saturation," *Automatica*, vol. 46, no. 12, pp. 2008–2013, 2010.

[18] F. Liao, J. L. Wang, and G. H. Yang, "Reliable robust flight tracking control: an LMI approach," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 1, pp. 76–89, 2002.

[19] R. J. Veillette, "Reliable linear-quadratic state-feedback control," *Automatica*, vol. 31, no. 1, pp. 137–143, 1995.

[20] G.-H. Yang, J. L. Wang, and Y. C. Soh, "Reliable H_∞ controller design for linear systems," *Automatica*, vol. 37, no. 5, pp. 717–725, 2001.

[21] G. H. Yang and K. Y. Lum, "Fault-tolerant flight tracking control with stuck faults," in *Proceedings of the American Control Conference*, pp. 521–526, June 2003.

[22] Q. Zhao and J. Jiang, "Reliable state feedback control system design against actuator failures," *Automatica*, vol. 34, no. 10, pp. 1267–1272, 1998.

[23] J. D. Boskovic and R. K. Mehra, "Stable multiple model adaptive flight control for accommodation of a large class of control effector failures," in *Proceedings of the American Control Conference (ACC '99)*, pp. 1920–1924, June 1999.

[24] M. A. Demetriou, "Adaptive reorganization of switched systems with faulty actuators," in *Proceedings of the 40th IEEE Conference on Decision and Control (CDC '01)*, pp. 1879–1884, December 2001.

[25] K. S. Kim, K. J. Lee, and Y. Kim, "Reconfigurable flight control system design using direct adaptive method," *Journal of Guidance, Control, and Dynamics*, vol. 26, no. 4, pp. 543–550, 2003.

[26] M. Maki and K. Hagino, "Robust control with adaptation mechanism for improving transient behaviour," *International Journal of Control*, vol. 72, no. 13, pp. 1218–1226, 1999.

[27] G. Tao, S. M. Joshi, and X. Ma, "Adaptive state feedback and tracking control of systems with actuator failures," *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 78–95, 2001.

[28] Y. Zhang and J. I. N. Jiang, "Integrated active fault-tolerant control using IMM approach," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 37, no. 4, pp. 1221–1235, 2001.

[29] J. D. Boskovic and R. K. Mehra, "Stable multiple model adaptive flight control for accommodation of a large class of control effector failures," in *Proceedings of the American Control Conference (ACC '99)*, pp. 1920–1924, San Diego, Calif, USA, June 1999.

[30] J. D. Boskovic and R. K. Mehra, "An adaptive Retrofit Reconfigurable Flight Controller," in *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 1257–1262, Las Vegas, Nev, USA, 2002.

[31] G. Tao, S. M. Joshi, and X. Ma, "Adaptive state feedback and tracking control of systems with actuator failures," *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 78–95, 2001.

[32] G. H. Yang and D. Ye, *Reliable Control and Filtering of Linear Systems with Adaptive Mechanisms*, Taylor and Francis Group, 2011.

[33] W. Guan and G. Yang, "Adaptive fault-tolerant control of linear systems with actuator saturation and L_2-disturbances," *Journal of Control Theory and Applications*, vol. 7, no. 2, pp. 119–126, 2009.

[34] W. Guan and G. Yang, "Adaptive fault-tolerant control of linear time-invariant systems in the presence of actuator saturation," *Journal of Control Theory and Applications*, vol. 7, no. 3, pp. 321–327, 2009.

[35] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, BirkhTauser, Boston, Mass, USA, 2001.

[36] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *International Journal of Control*, vol. 74, no. 14, pp. 1447–1455, 2001.