PRIME IDEALS IN GROUP ALGEBRAS OF POLYCYCLIC-BY-FINITE GROUPS

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Introduction

Group algebras $K[G]$ of polycyclic-by-finite groups are easily-defined, interesting examples of right and left Noetherian rings. Since prime rings and prime ideals are the basic building blocks in the Goldie theory of Noetherian rings, the determination of the structure of the prime ideals of $K[G]$ is certainly of importance. In a recent fundamental paper [**8**], Roseblade proved that G has a characteristic subgroup G_0 of finite index such that the prime ideals of $K[G_0]$ can be described in a particularly nice manner. Furthermore, he showed that this type of description does not, in general, apply to $K[G]$. In this paper, we offer a slightly different, somewhat more complicated formulation which does indeed describe the primes of $K[G]$.

Observe that the group algebra $K[G]$ can be written as $K[G] = K[G_0] * (G/G_0)$, a crossed product of the finite group $H = G/G_0$ over the ring $R = K[G_0]$. Furthermore, a recent paper [**5**] of the authors studies the general crossed product situation $R * H$ with H finite and offers a rather complicated scheme for describing the prime ideals of $R * H$ in terms of those of R. Since the situation here is certainly much more special, the general scheme, as expected, simplifies enormously and allows us to lift information from $R = K[G_0]$ to $K[G]$. Thus the proofs of the main results of this paper use crossed-product techniques, frequently well disguised, and ultimately rest on the structure of the primes in $R = K[G_0]$, that is, on the basic work of Roseblade. To describe these new results, we first require a number of definitions, most of which come from [**8**].

In the following definitions and theorems of the introduction, $K[G]$ will always denote the group algebra of a polycyclic-by-finite group G over a field K.

DEFINITIONS. Let N be a subgroup of G and let I be an ideal of $K[G]$.

(1) N is said to be *orbital* in G if $[G : \mathbf{N}_G(N)] < \infty$, that is, if N has only finitely many conjugates under the action of G. N is an *isolated* orbital subgroup if N is orbital and if there are no orbital subgroups M of G with

$M \supset N$ (where $\supset$ denotes strict inclusion) and $[M:N] < \infty$.

(2) $\Delta(G) = \{x \in G \mid [G:\mathbf{C}_G(x)] < \infty\}$. This is the f.c. centre of G and $G = \Delta(G)$ if and only if G is finite-by-abelian. Furthermore, we let $\nabla_G(N)$ denote the complete inverse image in $\mathbf{N}_G(N)$ of $\Delta(\mathbf{N}_G(N)/N)$.

(3) $I^\dagger = \{x \in G \mid x-1 \in I\}$. Thus $I^\dagger$ is the kernel of the homomorphism $G \to K[G]/I$ so that $I^\dagger$ is a normal subgroup of G and I is the complete inverse image in G of an ideal of $K[G/I^\dagger]$. If $N \lhd G$ we say that I is *almost faithful sub* N if and only if $I^\dagger \subseteq N$ and $[N:I^\dagger] < \infty$.

(4) Let H be a subgroup of G and let L be an ideal of $K[H]$. Then

$$L^G = \mathrm{Id}_{K[G]}(L \cdot K[G]) = \bigcap_{x \in G} L^x \cdot K[G].$$

Here Id stands for 'the largest ideal contained in' and it is a simple matter to see that this definition makes sense. Indeed, suppose I is an ideal contained in $L \cdot K[G]$. Since $I = I^x$ for all $x \in G$, we have $I \subseteq (L \cdot K[G])^x = L^x \cdot K[G]$ and hence $I \subseteq \bigcap_{x \in G} L^x \cdot K[G] = L^G$. On the other hand, the right-hand term L^G is surely a G-invariant right ideal of $K[G]$ and hence $xL^G = L^G x \subseteq L^G$ for all $x \in G$. Thus L^G is a two-sided ideal and in fact the unique largest such contained in $L \cdot K[G]$.

We can now state

THEOREM I (Existence). *If P is a prime ideal of $K[G]$, then there exists an isolated orbital subgroup N of G and an almost faithful sub N prime ideal L of $K[\nabla_G(N)]$ with $P = L^G$.*

DEFINITION. In the above context, if $P = L^G$ we call any such N a *vertex* of P and write $N = \mathrm{vx}(P)$. Moreover, for this N, any such L is a *source* of P.

THEOREM II (Uniqueness). *If P is a prime ideal of $K[G]$, then the vertices of P are unique up to conjugation in G. Furthermore if N is any such vertex, the sources of P for this N are unique up to conjugation by $\mathbf{N}_G(N)$.*

THEOREM III (Converse). *If N is an isolated orbital subgroup of G and if L is an almost faithful sub N prime ideal of $K[\nabla_G(N)]$, then L^G is a prime ideal of $K[G]$.*

At this point, it is necessary for us to assume that the reader is familiar with the basic results and remaining definitions of [**8**].

Furthermore, we note that the main results of the latter paper hold for all polycyclic-by-finite groups, rather than just for polycyclic groups as claimed. We now consider briefly the relationship between the theorems stated above and their predecessors in [**8**].

Let us observe first that if $H \lhd G$ and if L is an ideal of $K[H]$, then it follows from the freeness of $K[G]$ over $K[H]$ that

$$L^G = \bigcap_{x \in G} L^x \cdot K[G] = (\bigcap_{x \in G} L^x) \cdot K[G],$$

and we see that L^G is controlled by H. Thus, since G is an orbitally sound group if and only if each isolated orbital subgroup of G is normal, we see that Theorem I reduces to [**8**, Theorem C1] in the orbitally sound case. Furthermore, if one merely assumes the weaker condition that $\nabla_G(N) = \nabla_G(\mathrm{core}_G N)$ for all isolated orbital subgroups N, then again $\nabla_G(N)$ is normal in G and $\nabla_G(N)/\mathrm{core}_G N$ is an f.c. group so Theorem I reduces to the 'if' part of [**8**, Theorem B]. On the other hand, the 'only if' part of the latter result is surely related to Theorem III.

The following result is not really new. Rather it translates certain invariants computed in [**4**] and [**8**] into the language of vertices and sources. Recall that the *central rank* of P, c.r.P, is the transcendence degree over K of the centre of the classical ring of quotients $\mathcal{Q}(K[G]/P)$ of the prime Noetherian ring $K[G]/P$. Furthermore, the *plinth length* $p_\Gamma(G)$ is defined in [**8**, § 2.3].

THEOREM IV (Invariants). *If P is a prime ideal of $K[G]$ with vertex N and source L, we have that*

(i) c.r.P = c.r.L,

(ii) $\mathrm{hgt}\, P = p_A(\nabla_G(N)) - \mathrm{c.r.}\, L$, *where* $A = \mathbf{N}_G(N)$,

(iii) *if K is a non-absolute field, then P is primitive if and only if* $\dim_K K[\nabla_G(N)]/L < \infty$.

Finally, we have opted to make this paper independent of the crossed-product work in [**5**]. This certainly lengthens the present manuscript, but considerably eases the prerequisites for reading it. On the other hand, for the reader who is somewhat familiar with [**5**], we offer in the next two paragraphs a brief description of the relationship between the latter paper and the proof given here, stressing in particular the way in which this proof was motivated.

Reference [**5**] studies prime ideals in crossed products $R * G$ of finite groups and, without loss of generality, one can assume that R is a G-prime ring. There are then two cases to consider. If R is prime, then [**5**,

§2] introduces a suitable ring of quotients of R and sets up an appropriate one-to-one correspondence describing the prime ideals. In some sense, there are really just two key ingredients determined by this quotient ring, namely the extended centroid of R and the X-inner automorphisms. Several years ago, Formanek [**2**] computed the extended centroid of an arbitrary group ring using Δ-methods and it is clear that one can also determine the X-inner automorphisms in the same way. Thus if R is a group ring, then these Δ-methods should handle the extension from R to $R*G$ quite nicely. Furthermore, it was shown in [**4**] that Δ-methods also work modulo a standard prime, that is modulo the type of prime ideal which occurs in the group algebra of an orbitally sound group. Because of this, it is clear that using these methods we can describe the extensions necessary for this paper. However, as it turns out, there is really very little that has to be done. Indeed [**4**, Proposition 1.4] is all that is required in the course of the proof.

The second case is the general case with R a G-prime ring. Here [**5**, §3] introduces the maps $^{\vee}$ and $^{\delta}$ which afford a one-to-one correspondence between certain prime ideals of $R*G$ and of $R*H$, where H is the stabilizer in G of a minimal prime of R. Admittedly the original formulation of these maps in the above-mentioned paper is useful there from a theoretical point of view, but they are certainly less convenient computationally. Thus, in this case, our concern is with making the map $^{\vee}$ more understandable and, as we will show in §3, for the sake of completeness, we have $L^{\vee} = L^{G}$ in general. With this observation, the results of this paper follow rather quickly from Roseblade's work. Furthermore, the isolated orbital subgroups show up here because, as is pointed out in [**8**, §3.1], among 'nearby' orbital subgroups, their normalizers are largest.

1. Crossed products

In this section we obtain the crossed-product results necessary for the main theorems and, as was indicated in the introduction, this work will not depend upon reference [**5**]. In fact, we are able to offer a completely different argument here, which is a good deal shorter than the original, by assuming throughout that the rings involved are Noetherian. Many of the preliminary lemmas are well known in a slightly different context. Furthermore, the proof of the key result here is conceptually quite simple. However, since we are not computing invariants but rather determining ideals set-theoretically, it is necessary carefully and precisely to describe certain embeddings. In other words, it is not sufficient just to know that certain rings are isomorphic; we must really understand these isomorphisms.

LEMMA 1.1. *Let S be a ring.*

(i) *Suppose $1 = e_1 + e_2 + \dots + e_n$ is a decomposition of 1 in S into a sum of n orthogonal idempotents. Let U be a group of units of S and suppose that U permutes the set $\{e_1, e_2, \dots, e_n\}$ transitively by conjugation. Then $S = M_n(T)$, the ring of $n \times n$ matrices over a suitable subring $T \simeq e_1 S e_1$, and e_1 plays the role of the matrix unit e_{11}.*

(ii) *Suppose $S = M_n(T)$ and let $\{e_{ij}\}$ denote the set of matrix units of S. Then the maps $I \to e_{11} I e_{11}$ and $L \to SLS$ yield a one-to-one correspondence between the two-sided ideals I of S and the two-sided ideals L of the ring $e_{11} S e_{11} \simeq T$. Furthermore, I is prime if and only if $e_{11} I e_{11}$ is prime.*

Proof. (i) This is [7, Lemma 6.1.6].

(ii) This follows from the well-known fact that every ideal I of $M_n(T)$ is of the form $M_n(J)$ for some ideal J of T.

DEFINITION. If T is an additive subgroup of a ring S, we let $\mathrm{Id}_S(T)$ denote the sum of all ideals of S contained in T. This is surely the largest ideal of S contained in T.

Now let $R * G$ denote a crossed product of the group G over the ring R, so that $R * G$ is a free R-module with basis $\bar{G} = \{\bar{x} \mid x \in G\}$. If H is a subgroup of G and if L is an ideal of $R * H$, we define $L^G = \mathrm{Id}_{R*G}(L\bar{G})$. Thus ^{G} determines a map from $\mathscr{I}(R * H)$, the set of ideals of $R * H$, to $\mathscr{I}(R * G)$.

LEMMA 1.2. *Let H be a subgroup of G and let L be an ideal of $R * H$.*

(i) *We have*

$$L^G = \mathrm{Id}_{R*G}(L\bar{G}) = \bigcap_{x \in G} (L\bar{G})^{\bar{x}} = \bigcap_{x \in G} L^{\bar{x}}\bar{G}.$$

(ii) *If $H \lhd G$, then $L^G = (\bigcap_{x \in G} L^{\bar{x}})\bar{G}$.*

(iii) *If $H \subseteq A \subseteq G$, then $L^G = (L^A)^G$.*

Proof. (i) If I is an ideal of $R * G$ contained in $L\bar{G}$, then, since I is G-invariant, we have $I \subseteq \bigcap_{x \in G} (L\bar{G})^{\bar{x}}$. On the other hand, since $\bigcap_{x \in G} (L\bar{G})^{\bar{x}} = \bigcap_{x \in G} L^{\bar{x}}\bar{G}$, this intersection is clearly a left R-module, a right $R * G$-module, and it is G-invariant. Thus it is a two-sided ideal and hence equal to $\mathrm{Id}_{R*G}(L\bar{G}) = L^G$.

(ii) If $H \lhd G$, then each $L^{\bar{x}}$ is an ideal of $R * H$. Hence it follows from the freeness of $R * G$ over $R * H$, that

$$L^G = \bigcap_{x \in G} L^x \bar{G} = (\bigcap_{x \in G} L^x)\bar{G}.$$

(iii) Set

$$M = L^A = \bigcap_{a \in A} (L\bar{A})^{\bar{a}} = \bigcap_{a \in A} L^{\bar{a}}\bar{A}.$$

Since $R * G$ is free over $R * A$, it then follows that

$$M\bar{G} = (\bigcap_{a \in A} L^{\bar{a}}\bar{A})\bar{G} = \bigcap_{a \in A} L^{\bar{a}}\bar{G},$$

and hence clearly

$$(L^A)^G = M^G = \bigcap_{x \in G} (M\bar{G})^{\bar{x}} = \bigcap_{x \in G} (L\bar{G})^{\bar{x}} = L^G.$$

At this point, and for the remainder of this section, we assume that $R * G$ is a crossed product with G a finite group and with R a right Noetherian ring. Thus $R * G$, being a finitely generated R-module, is also right Noetherian. We note that G permutes the ideals of R and if L is a G-invariant ideal of R, then $L * G = L\bar{G}$ is an ideal of $R * G$ and we have the natural isomorphism $(R * G)/(L * G) \simeq (R/L) * G$. Furthermore, the G-invariant ideal L is said to be *G-prime* if and only if, for all G-invariant ideals J_1 and J_2 of R, $J_1 J_2 \subseteq L$ implies $J_1 \subseteq L$ or $J_2 \subseteq L$. In the following lemma, part (ii) is Incomparability and part (iv) is Going Down.

LEMMA 1.3. *Let $R * G$ be given.*

(i) *If I is an ideal of $R * G$, then $I \cap R$ is a G-invariant ideal of R. Furthermore, if I is prime, then $I \cap R$ is G-prime.*

(ii) *If $I \supset P$ are ideals of $R * G$ with P prime, then $I \cap R \supset P \cap R$.*

(iii) *Let $J \supseteq L$ be ideals of R with L being G-invariant and let*

$$\tilde{\ }: R * G \longrightarrow (R * G)/(L * G) = \tilde{R} * G$$

be the natural map. If I is an ideal of $R * G$ with $I \supseteq J$, then $I \cap R = J$ if and only if $\tilde{I} \cap \tilde{R} = \tilde{J}$.

(iv) *If L is a G-prime ideal of R, then L is a semiprime ideal. Moreover, if P is a prime ideal of $R * G$ with $P \supseteq L$, then P is a minimal covering prime of $L * G$ if and only if $P \cap R = L$.*

Proof. (i) Since I and R are G-invariant, so is $I \cap R$. Furthermore, suppose I is prime and that J_1 and J_2 are G-invariant ideals of R with

$J_1 J_2 \subseteq I$. Then $(J_1 * G)(J_2 * G) \subseteq I$ and, since I is prime, we have $J_i * G \subseteq I$ for some i and therefore $J_i \subseteq I \cap R$.

(ii) Since $R * G$ is right Noetherian, the proof of [**3**, Theorem 1.37] asserts that there exists an element $\alpha \in I$ which is regular modulo P. Moreover, since $R * G$ is a Noetherian R-module, there exists an integer $m \geqslant 1$ with

$$\alpha^m \in R + \alpha R + \alpha^2 R + \ldots + \alpha^{m-1} R.$$

This yields a monic equation of the form $\alpha^m + \alpha^{m-1} r_{m-1} + \ldots + r_0 = 0$ with $r_i \in R$. In particular, this expression is also contained in P and we can now choose $n \geqslant 1$ minimal such that $\alpha^n + \alpha^{n-1} s_{n-1} + \ldots + s_0 \in P$ for some $s_i \in R$. If $s_0 \in P$, then, since α is regular modulo P, we obtain either $1 \in P$ if $n = 1$ or $\alpha^{n-1} + \alpha^{n-2} s_{n-1} + \ldots + s_1 \in P$ if $n > 1$ and both of these are contradictions. Thus $s_0 \notin P$. On the other hand, $P \subseteq I$ and $\alpha \in I$ so $s_0 \in I$. Therefore, $s_0 \in I \cap R$ and $s_0 \notin P \cap R$.

(iii) Assume $\tilde{I} \cap \tilde{R} = \tilde{J}$. Then $(I \cap R)^{\sim} \subseteq \tilde{I} \cap \tilde{R} = \tilde{J}$ so

$$I \cap R \subseteq (J + L * G) \cap R = J \quad \text{and} \quad I \cap R = J.$$

Conversely, let $I \cap R = J$ and let $\tilde{\alpha} \in \tilde{I} \cap \tilde{R}$. We can of course assume that $\alpha \in R$. But $I \supseteq L * G$, so I is the complete inverse image of $\tilde{I}$ and hence $\alpha \in I$ also. Thus $\alpha \in I \cap R = J$, $\tilde{\alpha} \in \tilde{J}$, and we conclude that $\tilde{I} \cap \tilde{R} = \tilde{J}$.

(iv) Suppose first that $L = 0$ and hence that R is a G-prime ring. Then R is semiprime since the nilpotent radical of R is a G-invariant nilpotent ideal and hence must be zero. If $P \cap R = 0$, then it follows immediately from (ii) above that P is a minimal prime of $R * G$. Conversely, suppose $P \cap R \neq 0$. Since $l_R(P \cap R) \cdot (P \cap R) = 0$ and since both these ideals are G-invariant, we conclude that $l_R(P \cap R) = 0$ and hence that $P \cap R$ is an essential right ideal of R. It now follows from [**3**, Theorem 1.37, proof] that $P \cap R$ contains a regular element $r \in R$. Then r is clearly also regular in $R * G$ and since r, being an element of P, is not regular modulo P, we conclude from [**3**, Theorem 2.5 and Lemma 2.17] that P is not a minimal prime of $R * G$.

Finally, let L be arbitrary and use the notation of (iii) above with $J = L$. In particular, if $P \supseteq L$, then $P \cap R = L$ if and only if $\tilde{P} \cap \tilde{R} = 0$. Since P is a minimal covering prime of $L * G$ if and only if $\tilde{P}$ is a minimal prime of $\tilde{R} * G$, the work of the preceding paragraph yields the result.

If R is a right Ore ring, then, by definition, R has a classical right ring of quotients $\mathcal{Q}(R)$. In particular, by [**3**, Theorem 1.37], this always occurs when R is semiprime and furthermore in this case $\mathcal{Q}(R)$ is necessarily a semisimple Artinian ring.

In the next three lemmas we make the further assumption that R is a G-prime ring. We fix notation so that Q is a minimal prime ideal of R and so that H is the stabilizer of Q in G. Additional notation is introduced in the following lemma.

LEMMA 1.4. *Let R, G, Q and H be as above.*

(i) *Then R is a semiprime ring and in fact*

$$\bigcap_{x \in G} Q^{\bar{x}} = 0.$$

Thus the minimal primes of R are precisely the G-conjugates of Q. Furthermore, if $N = \bigcap_{x \notin H} Q^{\bar{x}}$, then $Q = \operatorname{ann} N$ and $N = \operatorname{ann} Q$.

(ii) *The natural map $R \to R/Q$ extends to an epimorphism $\mathscr{Q}(R) \to \mathscr{Q}(R/Q)$ with kernel $Q\mathscr{Q}(R)$.*

(iii) *If $Q\mathscr{Q}(R) = (1-e)\mathscr{Q}(R)$ for some central idempotent $e \in \mathscr{Q}(R)$, then e is in fact centrally primitive and $e\mathscr{Q}(R) \simeq \mathscr{Q}(R/Q)$.*

Proof. (i) Since R is a G-prime ring, it is semiprime by Lemma 1.3 (iv). Thus if $Q = Q_1, Q_2, \ldots, Q_m$ are the finitely many minimal primes of R, then these are permuted by G and we have the irredundant intersection $0 = \bigcap Q_i$. Let $I = \bigcap_{x \in G} Q^{\bar{x}}$ and let J be the intersection of the remaining minimal primes. Then, since I and J are both G-invariant and $IJ = 0$, it follows that $I = 0$ and that the $Q^{\bar{x}}$ are all the minimal primes of R.

Now observe that $NQ = 0$. Thus if $A = \operatorname{ann} Q$, then $A \supseteq N$ and for all $x \notin H$ we have $Q^{\bar{x}} \supseteq 0 = AQ$. Hence we see that $Q^{\bar{x}} \supseteq A$ so $N \supseteq A$ and $N = \operatorname{ann} Q$. Similarly we can deduce that $Q = \operatorname{ann} N$.

(ii) Since R is semiprime, $\mathscr{Q}(R) = RT^{-1}$ exists, where T denotes the set of regular elements of R. Let $\tilde{\ }: R \to R/Q$ denote the natural epimorphism. Then it follows from [**3**, Lemma 2.17] that $\tilde{T}$ consists of regular elements of $\tilde{R}$. Furthermore, since T is a right divisor set in R and $\tilde{\ }$ is an epimorphism, it is clear that $\tilde{T}$ is a right divisor set in $\tilde{R}$. Hence the ring $\tilde{R}\tilde{T}^{-1}$ exists. It is now trivial to verify that the map $RT^{-1} \to \tilde{R}\tilde{T}^{-1}$ given by $rt^{-1} \to \tilde{r}\tilde{t}^{-1}$ is a well-defined ring epimorphism extending the original map $R \to \tilde{R}$. But $RT^{-1} = \mathscr{Q}(R)$ is a semisimple Artinian ring, so we see that $\tilde{R}$ is an order in the Artinian ring $\tilde{R}\tilde{T}^{-1}$ and hence clearly $\mathscr{Q}(\tilde{R}) = \tilde{R}\tilde{T}^{-1}$. Since the kernel of the map $\mathscr{Q}(R) \to \mathscr{Q}(\tilde{R})$ is $QT^{-1} = Q\mathscr{Q}(R)$, this part is proved.

(iii) Since $\mathscr{Q}(R)$ is a semisimple Artinian ring and $Q\mathscr{Q}(R)$ is a two-sided ideal, by (ii) above, we know that $Q\mathscr{Q}(R) = (1-e)\mathscr{Q}(R)$ for some central idempotent e. Thus by (ii) again, $e\mathscr{Q}(R) \simeq \mathscr{Q}(R/Q)$ and, by [**3**, Corollary 1.38] the latter ring is simple since R/Q is prime. Thus we see that e is a centrally primitive idempotent of $\mathscr{Q}(R)$.

We know that $R*G$ is right Noetherian, but of course it need not be a semiprime ring even if R is G-prime. Thus there is no a priori reason to believe that $\mathcal{Q}(R*G)$ exists. However, we show that it does in the next lemma. Note that parts (i) and (ii) below only require that R be a semiprime ring.

LEMMA 1.5. *We use the above notation.*

(i) *$\mathcal{Q}(R*G) = \mathcal{Q}(R)*G$ exists and is a right Artinian ring.*

(ii) *The maps $P \to P\mathcal{Q}(R)$ and $P' \to P' \cap (R*G)$ yield a one-to-one correspondence between the minimal primes P of $R*G$ and the primes P' of $\mathcal{Q}(R*G)$.*

(iii) *$\mathcal{Q}(R*G) = M_n(S)$ where $n = [G:H]$. Furthermore, e plays the role of the matrix unit e_{11}, e is central in $\mathcal{Q}(R)*H$, and*

$$S \simeq e\mathcal{Q}(R*G)e = e\mathcal{Q}(R)*H.$$

Proof. (i) If T denotes the set of regular elements of R, then T is easily seen to be a right divisor set of regular elements of $R*G$. Indeed the elements of T are surely regular in $R*G$ and if $t \in T$ and $\alpha = \sum r_x\bar{x} \in R*G$, then the fractions $(t^{-1}r_x)^{\bar{x}} = (t^{\bar{x}})^{-1}r_x{}^{\bar{x}} \in \mathcal{Q}(R)$ can all be written with a common denominator. Hence, for all x in the support of α, there exist $s_x \in R$ and $\tilde{t} \in T$ with $(t^{-1}r_x)^{\bar{x}} = s_x\tilde{t}^{-1}$. Setting $\beta = \sum \bar{x}s_x \in R*G$, we see immediately that $t\beta = \alpha\tilde{t}$ and T is indeed a right divisor set. We conclude that $(R*G)T^{-1}$ exists and then we see easily that this ring is just $(RT^{-1})*G = \mathcal{Q}(R)*G$. In other words, $R*G$ is an order in the Artinian ring $\mathcal{Q}(R)*G$ and from this it follows that $\mathcal{Q}(R*G) = \mathcal{Q}(R)*G$.

(ii) Observe that $P\mathcal{Q}(R*G) = P(R*G)T^{-1} = P\mathcal{Q}(R)$. This one-to-one correspondence now follows immediately from [**3**, Lemma 2.18] since $\mathcal{Q}(R*G)$ is Artinian.

(iii) Let U denote the group of units of R and set $\mathfrak{G} = \{u\bar{x} \mid u \in U, x \in G\}$ so that $\mathfrak{G}$ is a group of units of $R*G$ and hence of $\mathcal{Q}(R*G)$. Clearly $\mathfrak{G}$ acts by conjugation on $\mathcal{Q}(R)$. Now R is a G-prime ring, so $\mathcal{Q}(R)$ is also G-prime and hence G-simple. It follows that $\mathfrak{G}$ permutes transitively all the centrally primitive idempotents of $\mathcal{Q}(R)$. If the latter are $e = e_1, e_2, \ldots, e_n$, then since $1 = e_1 + e_2 + \ldots + e_n$ is an orthogonal decomposition of 1 in $\mathcal{Q}(R*G)$, we conclude from Lemma 1.1(i) that $\mathcal{Q}(R*G) = M_n(S)$, where $S \simeq e\mathcal{Q}(R*G)e$. It remains to identify n and the ring $e\mathcal{Q}(R*G)e$.

Since $Q\mathcal{Q}(R) \cap R = Q$, by Lemma 1.4(ii), it is clear that the stabilizer of $Q\mathcal{Q}(R)$ in $\mathfrak{G}$ is $\mathfrak{H} = \{u\bar{x} \mid u \in U, x \in H\}$. But $1-e$ is the unique identity element of $Q\mathcal{Q}(R)$ so we see that $\mathfrak{H}$ is also the stabilizer of e in $\mathfrak{G}$ and hence $n = [\mathfrak{G} : \mathfrak{H}] = [G:H]$. Finally, if $\alpha = \sum_{x \in G} a_x \bar{x} \in \mathcal{Q}(R) * G$ with $a_x \in \mathcal{Q}(R)$, then

$$e\alpha e = \sum_{x \in G} a_x e\bar{x}e = \sum_{x \in G} a_x e e^{\bar{x}^{-1}}.\bar{x} = \sum_{x \in H} a_x e\bar{x},$$

since e is orthogonal to its distinct G-conjugates. Hence it follows that $e\mathcal{Q}(R * G)e = e\mathcal{Q}(R) * H$ and of course e is central in $\mathcal{Q}(R) * H$.

We now consider the ring $e\mathcal{Q}(R) * H$ in more detail. In the course of the following proof, we will apply earlier lemmas to the crossed products $R * H$ and $(R/Q) * H = (R * H)/(Q * H)$. Observe that in the latter case, R/Q is indeed an H-prime ring.

LEMMA 1.6. *Again, we use the above notation.*

(i) $e\mathcal{Q}(R) * H \simeq \mathcal{Q}((R/Q) * H)$.

(ii) *Let $L_1, L_2, \ldots, L_m$ be the distinct prime ideals of $R * H$ with $L_i \cap R = Q$. Then the expressions $eL_i\mathcal{Q}(R)$ are all distinct and yield all the prime ideals of $e\mathcal{Q}(R) * H$. Furthermore, $L_i\mathcal{Q}(R) \cap (R * H) = L_i$.*

Proof. Observe that, by Lemma 1.4, the map

$$\tilde{}: R * H \to (R * H)/(Q * H) = (R/Q) * H$$

extends to a map $\mathcal{Q}(R) * H \to \mathcal{Q}(R/Q) * H$ with kernel

$$Q\mathcal{Q}(R) * H = (1-e)\mathcal{Q}(R) * H.$$

Thus we see that $e\mathcal{Q}(R) * H \simeq \mathcal{Q}(R/Q) * H$. Moreover, the latter ring is isomorphic to $\mathcal{Q}((R/Q) * H)$, by Lemma 1.5(i), so (i) follows.

Now, by Lemma 1.3(iv), $\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_m$ are the distinct minimal primes of $\tilde{R} * H$ and hence they are indeed finite in number. Furthermore, by Lemma 1.5(ii), the expressions $\tilde{L}_i\mathcal{Q}(\tilde{R})$ yield all the distinct primes of $\mathcal{Q}(\tilde{R}) * H = \mathcal{Q}(\tilde{R} * H)$. Thus the complete inverse images of these ideals in $\mathcal{Q}(R) * H$ are also distinct and prime. Since the complete inverse image of $\tilde{L}_i\mathcal{Q}(\tilde{R})$ is

$$L_i\mathcal{Q}(R) + (1-e)\mathcal{Q}(R) * H = eL_i\mathcal{Q}(R) + (1-e)\mathcal{Q}(R) * H,$$

we see that the expressions $eL_i\mathcal{Q}(R)$ are all distinct prime ideals of $e\mathcal{Q}(R) * H$. Moreover, in view of the isomorphism of (i), these account for all the prime ideals.

Finally, let T denote the set of regular elements of R. Then by [**3**, Lemma 2.17] these elements are regular modulo Q and hence, as elements of $R * H$, they are regular modulo $Q * H$. But L_i is a minimal covering prime of $Q * H$ so, by [**3**, Theorem 2.5 and Lemma 2.17], the elements of T are also regular modulo L_i. Now any finite number of elements of $\mathcal{Q}(R)$ can be written with a common denominator and hence it follows that any element of $L_i\mathcal{Q}(R)$ is of the form αt^{-1} with $\alpha \in L_i$ and $t \in T$. Thus if $\beta \in L_i\mathcal{Q}(R) \cap (R * H)$, then $\beta = \alpha t^{-1}$ so $\beta t = \alpha \in L_i$ and, since t is regular modulo L_i, we conclude that $\beta \in L_i$. Hence surely $L_i\mathcal{Q}(R) \cap (R * H) = L_i$.

We are now ready to prove the main result of this section.

THEOREM 1.7. *Let $R * G$ be a crossed product of the finite group G over the right Noetherian ring R. Let Q be a prime ideal of R, set $H = \{x \in G \mid Q^{\bar{x}} = Q\}$, and let A be a fixed subgroup of G with $A \supseteq H$.*

(i) *The map ${}^G\colon \mathcal{I}(R * A) \to \mathcal{I}(R * G)$ yields a one-to-one correspondence between the prime ideals L of $R * A$ with $L \cap R = \bigcap_{a \in A} Q^{\bar{a}}$ and the prime ideals P of $R * G$ with $P \cap R = \bigcap_{x \in G} Q^{\bar{x}}$.*

(ii) *Set $J = \bigcap_{x \notin A} Q^{\bar{x}}$. If $P = L^G$ as above, then $JL \subseteq P \cap (R * A) \subseteq L$ and L is the unique minimal covering prime of $P \cap (R * A)$ with $J \nsubseteq L \cap R$.*

Proof. We proceed in a series of steps.

Step 1. Suppose that $H = A$ and that $\bigcap_{x \in G} Q^{\bar{x}} = 0$. It follows from the latter assumption that R is a G-prime ring and that Q is a minimal prime of R. Hence all the notation of the preceding three lemmas can apply. In particular, if $L_1, L_2, \ldots, L_m$ denote the distinct prime ideals of $R * H$ with $L_i \cap R = Q$, then, by Lemma 1.6(ii), the expressions $eL_i\mathcal{Q}(R)$ yield all the distinct prime ideals of $e\mathcal{Q}(R) * H$. Thus since $\mathcal{Q}(R * G) = M_n(S)$, by Lemma 1.5(iii), with e playing the role of the matrix unit e_{11} and with $e\mathcal{Q}(R * G)e = e\mathcal{Q}(R) * H$, it follows from Lemma 1.1(ii) that the expressions

$$P'_i = \mathcal{Q}(R * G)eL_i\mathcal{Q}(R)\mathcal{Q}(R * G)$$

are distinct and yield all the primes of $\mathcal{Q}(R * G)$. Hence, by Lemmas 1.3(iv) and 1.5(ii), the expressions $P_i = P'_i \cap (R * G)$ are distinct and yield all the prime ideals of $R * G$ with $P_i \cap R = 0$. It remains to obtain a simpler formulation for P'_i and P_i.

Observe that $\mathcal{Q}(R) \cdot \mathcal{Q}(R * G) = \mathcal{Q}(R) * G$ and that $eL_i\mathcal{Q}(R)$ is an ideal of $e\mathcal{Q}(R * G)e = e\mathcal{Q}(R) * H$. Thus

$$\begin{aligned} eP'_i &= e\mathcal{Q}(R * G)e \cdot eL_i\mathcal{Q}(R) \cdot \mathcal{Q}(R * G) \\ &= eL_i\mathcal{Q}(R) \cdot \mathcal{Q}(R * G) = eL_i\mathcal{Q}(R) * G. \end{aligned}$$

Now $(1-e)\mathcal{Q}(R) = Q\mathcal{Q}(R)$, so

$$\mathcal{Q}(R)e = e\mathcal{Q}(R) = \operatorname{ann}_{\mathcal{Q}(R)}Q\mathcal{Q}(R) = \operatorname{ann}_{\mathcal{Q}(R)}Q,$$

and hence, by Lemma 1.4(i),

$$\mathcal{Q}(R)e \cap R = \operatorname{ann}_R Q = N.$$

We now multiply the above expression for eP'_i by any element $s \in \mathcal{Q}(R)$ with $se \in N$. Since $(se)L_i \subseteq L_i$ we deduce immediately, by considering all such s, that $NP'_i \subseteq L_i\mathcal{Q}(R) * G$ and hence that

$$NP_i \subseteq L_i\mathcal{Q}(R) * G \cap (R * G).$$

Let $\alpha = \sum \alpha_j \bar{x}_j \in P_i$ with $\alpha_j \in R * H$ and with $\{x_j\}$ a right transversal for H in G. Then it follows from the above and from Lemma 1.6(ii) that, for each j,

$$N\alpha_j \subseteq L_i\mathcal{Q}(R) \cap (R * H) = L_i.$$

But L_i is a prime ideal of $R * H$ and $L_i \not\supseteq N * H$, since $L_i \cap R = Q$ is disjoint from N. Thus since $(N * H)\alpha_j \subseteq L_i$ we conclude that $\alpha_j \in L_i$ and hence that $\alpha \in L_i\bar{G}$. In other words, $P_i \subseteq L_i\bar{G}$ and, since P_i is an ideal, we have $P_i \subseteq L_i{}^G$, by definition. Finally, $L_i{}^G$ is an ideal of $R * G$ and

$$L_i{}^G \cap R \subseteq L_i\bar{G} \cap R = L_i \cap R = Q.$$

Thus since $L_i{}^G \cap R$ is G-invariant and $\bigcap_{x \in G} Q^{\bar{x}} = 0$, we deduce that $L_i{}^G \cap R = 0$. Since P_i is prime and $P_i \subseteq L_i{}^G$, it now follows immediately from Incomparability, Lemma 1.3(ii), that $P_i = L_i{}^G$ and this special case is proved.

Step 2. Suppose that $H = A$ but that $I = \bigcap_{x \in G} Q^{\bar{x}}$ is arbitrary. Let $\tilde{}: R * G \to (R * G)/(I * G)$ denote the natural map. Observe that, by Lemma 1.3(iii), the prime ideals L of $R * H$ with $L \cap R = Q$ correspond in a one-to-one manner with the prime ideals $\tilde{L}$ of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = \tilde{Q}$. Similarly, the prime ideals P of $R * G$ with $P \cap R = I$ correspond in a one-to-one manner with the prime ideals $\tilde{P}$ of $\tilde{R} * G$ with $\tilde{P} \cap \tilde{R} = 0$. Note that $(L\bar{G})^{\sim} = \tilde{L}\bar{G}$, so since $L^G \supseteq I * G$ we have clearly

$$(L^G)^{\sim} = (\operatorname{Id}_{R*G}(L\bar{G}))^{\sim} = \operatorname{Id}_{\tilde{R}*G}(\tilde{L}\bar{G}) = \tilde{L}^G.$$

Hence since H is also the stabilizer of $\tilde{Q}$ in G, the correspondence in $\tilde{R} * G$, proved in Step 1, pulls back to yield the appropriate correspondence in this case.

Step 3. Suppose that $A \supseteq H$ is arbitrary. Now, by Step 2, the map ^{G} yields a one-to-one correspondence between the prime ideals M of $R * H$

with $M \cap R = Q$ and the primes P of $R * G$ with $P \cap R = \bigcap_{x \in G} Q^{\bar{x}}$. Furthermore, by considering the relationship $R * A \supseteq R * H$, we see that the map ^{A} yields a one-to-one correspondence between the prime ideals M of $R * H$ with $M \cap R = Q$ and the primes L of $R * A$ with $L \cap R = \bigcap_{a \in A} Q^{\bar{a}}$. Since

$$P = M^G = (M^A)^G = L^G,$$

by Lemma 1.2(iii), part (i) is clearly proved.

Step 4. The back map. If $\bar{x} \in A$, then $JL \subseteq L \subseteq L^{\bar{x}}\bar{G}$. If $x \notin A$, then $J \subseteq (\bigcap_{a \in A} Q^{\bar{a}})^{\bar{x}} \subseteq L^{\bar{x}}$ so since $L^{\bar{x}}\bar{G}$ is a right ideal of $R * G$ we have $JL \subseteq L^{\bar{x}}\bar{G}$. Thus we see that $JL \subseteq \bigcap_{x \in G} L^{\bar{x}}\bar{G} = L^G = P$ and hence $JL \subseteq P \cap (R * A)$. On the other hand, $P \subseteq L\bar{G}$ so

$$P \cap (R * A) \subseteq L\bar{G} \cap (R * A) = L.$$

Now let I be a minimal covering prime of $P \cap (R * A)$ with $I \cap R \not\supseteq J$. Then

$$I \supseteq P \cap (R * A) \supseteq JL = (J * A)L,$$

and since $I \not\supseteq J * A$ we have $I \supseteq L$. On the other hand, since $L \supseteq P \cap (R * A)$ we can choose L' a minimal covering prime of $P \cap (R * A)$ with $L \supseteq L'$. Since $\bigcap_{a \in A} Q^{\bar{a}} = L \cap R \supseteq L' \cap R$, and since $A \supseteq H$, we see that $L' \cap R \not\supseteq J$. Hence, by the above, $L' \supseteq L$, so $L' = L$ and the theorem is proved.

We remark that Theorem 1.7 holds for arbitrary rings and does not require the Noetherian assumption. This is a consequence of [**5**, Theorem 3.6] and Proposition 3.6 of this paper.

2. Group algebras

This section is devoted to proving the main theorems of the paper. Indeed, we will actually obtain a number of more detailed results which contain Theorems I–IV and which give a good deal of additional information. Throughout this section, G will be assumed to be a polycyclic-by-finite group and K will denote a commutative field.

DEFINITIONS. (1) Recall from [**8**] that an ideal I of the group algebra $K[G]$ is called *faithful* if $I^\dagger = \{x \in G \mid x - 1 \in I\} = \langle 1 \rangle$ and *almost faithful* if $I^\dagger$ is finite. More generally, if $N \lhd G$, then I is said to be *almost faithful mod N* if $I^\dagger \supseteq N$ and $[I^\dagger : N] < \infty$. This of course occurs if and only if I is the complete inverse image of an almost faithful ideal of $K[G/N]$.

(2) An ideal I of $K[G]$ is said to be *standard* [4] if

(i) $I = (I \cap K[\Delta(G)])K[G]$,

(ii) $I \cap K[\Delta(G)]$ is an intersection of almost faithful prime ideals of $K[\Delta(G)]$.

Note that, by [4, Lemma 1.1], the intersection in (ii) above can always be taken to be finite. A *standard prime* of $K[G]$ is of course a standard ideal which is prime.

Recall, from Lemma 1.2, that for any subgroup H of G and any ideal L of $K[H]$, we have

$$L^G = \mathrm{Id}_{K[G]}(LK[G]) = \bigcap_{x \in G} (LK[G])^x = \bigcap_{x \in G} L^x K[G].$$

Furthermore, if $H \lhd G$ then the last expression becomes $L^G = (\bigcap_{x \in G} L^x)K[G]$. The following lemma is a reformulation of [4, Proposition 1.4]. Its content is in the second part which asserts that L^G is prime.

LEMMA 2.1. *Any standard prime ideal P of $K[G]$ can be written as $P = L^G$ with L an almost faithful prime ideal of $K[\Delta(G)]$. Conversely if L is an almost faithful prime of $K[\Delta(G)]$, then L^G is a standard prime ideal of $K[G]$.*

The next lemma describes the behaviour of standard primes of $K[G]$ under restriction to $K[H]$ where H is a normal subgroup of G of finite index. In this situation, $K[G]$ can be written as $K[G] = K[H] * (G/H)$, a crossed product of the finite group G/H over the ring $K[H]$. Thus results on crossed products of finite groups apply when dealing with the extensions $K[H] \subseteq K[G]$. In particular, we know from Lemmas 1.3(i) and 1.4(i) that for any prime P of $K[G]$ we have $P \cap K[H] = \bigcap_{x \in G} Q^x$, where Q is a prime ideal of $K[H]$ which is unique up to G-conjugacy.

LEMMA 2.2. *Let H be a normal subgroup of G of finite index and let P be a prime ideal of $K[G]$. Write $P \cap K[H] = \bigcap_{x \in G} Q^x$ with Q a prime ideal of $K[H]$.*

(i) *P is standard if and only if Q is standard.*

(ii) *Assume that P is standard and write $P = L^G$ with L an almost faithful prime of $K[\Delta(G)]$. If J is a minimal covering prime of $L \cap K[\Delta(H)]$, then J^H is a minimal covering prime of $P \cap K[H]$ and is standard.*

Proof. We first prove (ii). If $U \subseteq G$, let π_U: $K[G] \to K[U]$ denote the natural projection. Set $D = \Delta(H)$ and $\Delta = \Delta(G)$, so that $D = H \cap \Delta$ since $[G:H] < \infty$. Now, by assumption, P is standard so

$P = (P \cap K[\Delta])K[G]$ implies that $\pi_\Delta(P) \subseteq P$. Hence, since $D = H \cap \Delta$, we have

$$\pi_D(P \cap K[H]) = \pi_\Delta(P \cap K[H]) \subseteq P \cap K[H].$$

It follows from [**7**, Lemma 1.1.5(ii)] that $I = P \cap K[H]$ is controlled by D, that is, $I = (I \cap K[D])K[H]$. Moreover, since

$$P \cap K[\Delta] = L^G \cap K[\Delta] = \textstyle\bigcap_{x \in G} L^x,$$

we see that

$$I \cap K[D] = P \cap K[D] = \bigcap_{x \in G}(L^x \cap K[D]) = \bigcap_{x \in G}(L \cap K[D])^x.$$

We claim now that any minimal covering prime J of $L \cap K[D]$ is almost faithful. Indeed, since $L \cap K[D]$ is a Δ-prime ideal of $K[D]$, we have $L \cap K[D] = \bigcap_{y \in \Delta} J^y$ and hence $L^\dagger \cap D = \bigcap_{y \in \Delta}(J^\dagger)^y$. In particular, if $Z = \mathbf{Z}(\Delta) \cap D$, then $J^\dagger \cap Z = L^\dagger \cap Z \subseteq L^\dagger$. But L is almost faithful and $[D:Z] < \infty$, so we conclude that $J^\dagger$ is finite.

It now follows from Lemma 2.1 that J^H is a standard prime of $K[H]$. Moreover, since $D \lhd G$,

$$\begin{aligned} P \cap K[H] = I = (I \cap K[D])K[H] &= \bigcap_{x \in G}(L \cap K[D])^x \cdot K[H] \\ &= \bigcap_{x \in G} J^x K[H] = \bigcap_{x \in G}\Big(\bigcap_{y \in H} J^y K[H]\Big)^x \\ &= \bigcap_{x \in G}(J^H)^x, \end{aligned}$$

so that J^H is clearly a minimal covering prime of $P \cap K[H]$. This proves (ii) and one implication in (i) since the uniqueness of Q implies that Q is G-conjugate to J^H and hence is clearly standard.

Conversely, assume that Q is standard and, by [**8**, Lemma 5], write $P \cap K[\Delta] = \bigcap_{x \in G} N^x$ with N a prime ideal of $K[\Delta]$. Since $Q = M^H$ for some almost faithful prime ideal M of $K[D]$, we have $Q \cap K[D] = \bigcap_{y \in H} M^y$ and hence

$$\begin{aligned} \bigcap_{x \in G}(N \cap K[D])^x &= (P \cap K[\Delta]) \cap K[D] = (P \cap K[H]) \cap K[D] \\ &= \bigcap_{x \in G}(Q^x \cap K[D]) = \bigcap_{x \in G} M^x. \end{aligned}$$

Thus since these intersections are finite and M is prime, we have $N \cap K[D] \subseteq M^x$ for some $x \in G$ and it follows that $N^\dagger \cap D \subseteq (M^x)^\dagger$. Since the latter group is finite and $[\Delta : D] < \infty$, we conclude that N is an almost faithful prime of $K[\Delta]$. Thus it follows from Lemma 2.1 that

$$P' = N^G = \Big(\bigcap_{x \in G} N^x\Big)K[G] = (P \cap K[\Delta])K[G]$$

is a standard prime ideal of $K[G]$ which is clearly contained in P. Finally observe that since $Q = (\bigcap_{y \in H} M^y)K[H]$, we have

$$\begin{aligned}(P \cap K[H])K[G] &= (\bigcap_{x \in G} Q^x)K[G] = (\bigcap_{x \in G} M^x)K[G] \\ &\subseteq (P \cap K[\Delta])K[G] = P',\end{aligned}$$

so that $(P \cap K[H])K[G] \subseteq P' \subseteq P$. It now follows immediately from Incomparability, Lemma 1.3(ii), that $P = P'$. Thus P is standard and the lemma is proved.

Let us review and slightly extend some definitions from the introduction.

DEFINITION. [8] A subgroup N of G is called *orbital* if $[G : \mathbf{N}_G(N)] < \infty$. In addition, N is an *isolated orbital* subgroup if and only if N is orbital and for all orbital subgroups $M \supset N$ we have $[M : N] = \infty$. If N is orbital, then its *isolator* $i_G(N)$ is defined by

$$i_G(N) = \langle M \mid M \text{ is orbital in } G,\ M \supseteq N, \text{ and } [M : N] < \infty \rangle.$$

By [**8**, § 3.1] we know that $[i_G(N) : N] < \infty$, $i_G(N)$ is an isolated orbital subgroup, and $\mathbf{N}_G(i_G(N)) \supseteq \mathbf{N}_G(N)$.

The group G is called *orbitally sound* if and only if all its isolated orbital subgroups are normal. By [**8**, Theorem C2], the intersection of the normalizers of the isolated orbital subgroups of G, for any polycyclic-by-finite group G, has finite index in G. Following [**8**] we denote this latter subgroup by $\operatorname{nio}(G)$ so that $\operatorname{nio}(G)$ is a characteristic orbitally sound subgroup of G of finite index.

We are now ready to prove the following result which contains within it Theorem I.

THEOREM 2.3. *Let G be a polycyclic-by-finite group and let P be a prime ideal of $K[G]$. Suppose H is a normal orbitally sound subgroup of G of finite index and write $P \cap K[H] = \bigcap_{x \in G} Q^x$ with Q a prime of $K[H]$. Define $N = i_G(Q^\dagger)$, $A = \mathbf{N}_G(N)$, and $B = \{x \in G \mid Q^x = Q\}$.*

(i) $A \supseteq B \supseteq H$.

(ii) *There exists a unique prime ideal T of $K[A]$ with $P = T^G$ and $T \cap K[H] = \bigcap_{a \in A} Q^a$. Indeed T is the unique minimal covering prime of $P \cap K[A]$ not containing $\bigcap_{x \notin A} Q^x$.*

(iii) *There exists an almost faithful sub N prime ideal L of $K[\nabla_G(N)]$ with $T = L^A$ and hence with $P = L^G$. Indeed, L is a minimal covering prime of $T \cap K[\nabla_G(N)]$.*

Proof. We use the above notation. Recall that such a normal subgroup H of finite index in G does indeed exist, for example we could take $H = \mathrm{nio}(G)$, and that $P \cap K[H]$ does indeed have the above structure with Q unique up to G-conjugation. Furthermore, $Q^\dagger \lhd H$ so $Q^\dagger$ is orbital in G and $N = i_G(Q^\dagger)$ is an isolated orbital subgroup of G with $[N : Q^\dagger] < \infty$ and $\mathbf{N}_G(N) \supseteq \mathbf{N}_G(Q^\dagger) \supseteq H$. For convenience, let M be a characteristic subgroup of N of finite index contained in $Q^\dagger$. Then certainly $M \lhd \mathbf{N}_G(N) = A$. Observe that if $Q^x = Q$, then x certainly normalizes $Q^\dagger$ and hence we have $A \supseteq \mathbf{N}_G(Q^\dagger) \supseteq B \supseteq H$. Thus by considering $K[G]$ as a crossed product $K[G] = K[H] * (G/H)$, we see from Theorem 1.7(i) that there exists a unique prime ideal T of $K[H] * (A/H) = K[A]$ with $T \cap K[H] = \bigcap_{a \in A} Q^a$ and with $P = T^G$. Since Theorem 1.7(ii) characterizes T appropriately in terms of $P \cap K[A]$, parts (i) and (ii) are proved.

Let $\tilde{\ }: K[A] \to K[A/M]$ denote the natural map. Since $M \lhd H$ and $[Q^\dagger : M] < \infty$, it follows that Q is almost faithful mod M. Furthermore, H is orbitally sound so $\tilde{H}$ is orbitally sound and hence, by [**8**, Theorem C_1], $\tilde{Q}$ is a standard prime of $K[\tilde{H}]$. Moreover, since $T \cap K[H] = \bigcap_{a \in A} Q^a$, we see by Lemma 1.3(iii) that $\tilde{T}$ is a prime ideal of $K[\tilde{A}]$ with $\tilde{T} \cap K[\tilde{H}] = \bigcap_{a \in A} \tilde{Q}^{\tilde{a}}$. We can now conclude from Lemma 2.2(i) that $\tilde{T}$ is a standard prime of $K[\tilde{A}]$. In other words, there exists an almost faithful prime $\tilde{L}$ of $K[\Delta(\tilde{A})]$ with

$$\tilde{T} = (\bigcap_{\tilde{a} \in \tilde{A}} \tilde{L}^{\tilde{a}})K[\tilde{A}] = \tilde{L}^{\tilde{A}}.$$

By lifting the above expression back to $K[A]$, we see that there exists an almost faithful mod M prime ideal L of $K[\nabla_A(M)]$ with

$$T = (\bigcap_{a \in A} L^a)K[A] = L^A.$$

Observe that $N, M \lhd A$ and $[N : M] < \infty$. Hence it follows easily that $\nabla_A(M) = \nabla_A(N) = \nabla_G(N)$. Note that $i_G(M) \supseteq N \supseteq M$ so $i_G(M) = N$. Furthermore, since $L^\dagger$ is orbital in A, it is orbital in G, and from $L^\dagger \supseteq M$ and $[L^\dagger : M] < \infty$, we conclude that $L^\dagger \subseteq i_G(M) = N$ and that $[N : L^\dagger] < \infty$. Thus we see that L is almost faithful sub N. Finally we have $T = L^A$ and $P = T^G$ so Lemma 1.2(iii) yields $P = (L^A)^G = L^G$. Since $T \cap K[\nabla_G(N)] = \bigcap_{a \in A} L^a$, a finite intersection of primes, L is clearly a minimal covering prime of $T \cap K[\nabla_G(N)]$. The result follows.

DEFINITION. Let P be a prime ideal of $K[G]$, N an isolated orbital subgroup of G, and L an almost faithful sub N prime ideal of $K[\nabla_G(N)]$. If $P = L^G$, then N is said to be a *vertex* of P and we write $N = \mathrm{vx}_G(P)$. Furthermore, for this N, L is said to be a *source* of P.

The previous theorem gives a means of finding at least one vertex and source for P. This will become all the more useful once we prove a uniqueness result. This is in fact done in the next theorem which contains within it Theorems II and III.

THEOREM 2.4. *Let N be an isolated orbital subgroup of G and let L be an almost faithful sub N prime ideal of $K[\nabla_G(N)]$. Then $P = L^G$ is a prime ideal of $K[G]$. Furthermore, if $H = \mathrm{nio}(G)$ and if*

$$P \cap K[H] = Q_1 \cap Q_2 \cap \ldots \cap Q_n,$$

an intersection of G-conjugate primes, then we have that

(i) *for some $Q = Q_i$, $N = i_G(Q^\dagger)$ so that N is uniquely determined by P up to conjugation in G,*

(ii) *for this N, the ideal L is uniquely determined by P up to conjugation by $A = \mathbf{N}_G(N)$.*

(iii) *if J is any minimal covering prime of $L \cap K[\nabla_G(N) \cap H]$, then $J^H = Q^a$ for some $a \in A$. Furthermore, $\nabla_G(N) \cap H = \nabla_H(Q^\dagger)$.*

Proof. Set $A = \mathbf{N}_G(N)$ and let $H = \mathrm{nio}(G)$ so that, by definition, $H \subseteq A$. Since $[N : N \cap H] < \infty$ and $[N : L^\dagger] < \infty$, we can choose M to be a characteristic subgroup of N of finite index with $M \subseteq N \cap H$ and $M \subseteq L^\dagger$. Thus $M \lhd A$ and therefore also $M \lhd H$. Furthermore, since $[N : M] < \infty$ we have $i_G(M) = N$, so $\mathbf{N}_G(M) = A$, and then clearly $\nabla_G(M) = \nabla_G(N)$.

Let $\tilde{}: K[A] \to K[A/M]$ denote the natural map. Then $\tilde{L}$ is an almost faithful prime of $K[\Delta(\tilde{A})]$, since $\nabla_G(M) = \nabla_G(N)$, and we conclude from Lemma 2.1 that

$$\tilde{T} = \tilde{L}^{\tilde{A}} = (\bigcap_{\tilde{a} \in \tilde{A}} \tilde{L}^{\tilde{a}})K[\tilde{A}]$$

is a standard prime of $K[\tilde{A}]$. Furthermore, by Lemma 2.2(i), $\tilde{T} \cap K[\tilde{H}] = \bigcap_{\tilde{a} \in \tilde{A}} \tilde{Q}^{\tilde{a}}$ is a finite intersection of $\tilde{A}$-conjugate standard primes of $K[\tilde{H}]$. Lifting this information to $K[A]$, we see immediately that

$$T = L^A = (\bigcap_{a \in A} L^a)K[A]$$

is a prime ideal of $K[A]$ and that $T \cap K[H] = \bigcap_{a \in A} Q^a$ is a finite intersection of A-conjugate primes of $K[H]$ which are almost faithful mod M.

Set $B = \{x \in G \mid Q^x = Q\}$ so that $H \subseteq B \subseteq G$ and surely $B \subseteq \mathbf{N}_G(Q^\dagger)$. Observe that $[Q^\dagger : M] < \infty$ and $Q^\dagger$ is orbital in G so $i_G(Q^\dagger) = i_G(M) = N$

and hence $B \subseteq \mathbf{N}_G(Q^\dagger) \subseteq \mathbf{N}_G(N) = A$. We now view $K[G]$ as the crossed product $K[H] * (G/H)$. Then, since $B \subseteq A$, it follows from Theorem 1.7(i) that $P = T^G$ is a prime ideal of $K[G]$ and, since $T^G = (L^A)^G = L^G$ by Lemma 1.2(iii), the first assertion is proved. Furthermore, by Theorem 1.7(i) again, Q is a minimal covering prime of $P \cap K[H] = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ so $Q = Q_i$ for some i. Thus since $N = i_G(Q^\dagger)$, (i) is proved.

Now suppose both N and $P = L^G$ are known. We consider those Q_i with $i_G(Q_i^\dagger) = N$. Since $Q_i = Q^x$ for some $x \in G$, we have

$$N = i_G(Q_i^\dagger) = i_G(Q^\dagger)^x = N^x,$$

and hence this occurs if and only if $x \in A$. In particular, the ideal $\bigcap_{a \in A} Q^a$ is the intersection of all the minimal covering primes Q_i of $P \cap K[H]$ with $i_G(Q_i^\dagger) = N$ and this is surely determined by N and P. It now follows from Theorem 1.7(ii) that the prime ideal T of $K[A]$ is uniquely determined by the conditions $T^G = P$ and $T \cap K[H] = \bigcap_{a \in A} Q^a$. But $T = L^A$ and L is certainly a minimal covering prime of $T \cap K[\nabla_G(N)] = \bigcap_{a \in A} L^a$, so L is unique up to conjugation in A. This proves (ii).

Finally, recall that $\tilde{T} = \tilde{L}^{\tilde{A}}$ is a standard prime of $K[\tilde{A}]$. Thus Lemma 2.2(ii) asserts that if $\tilde{J}$ is any minimal covering prime of $\tilde{L} \cap K[\Delta(\tilde{H})]$, then $\tilde{J}^{\tilde{H}}$ is a minimal covering prime of $\tilde{T} \cap K[\tilde{H}] = \bigcap_{\tilde{a} \in \tilde{A}} \tilde{Q}^{\tilde{a}}$ and hence $\tilde{J}^{\tilde{H}} = \tilde{Q}^{\tilde{a}}$ for some $\tilde{a} \in \tilde{A}$. Lifting this information back to $K[A]$, we see that for any minimal covering prime J of $L \cap K[\nabla_H(M)]$, we have $J^H = Q^a$ for some $a \in A$. Since clearly $\nabla_H(M) = \nabla_H(Q^\dagger) = \nabla_G(N) \cap H$, the theorem is proved.

It remains to prove Theorem IV. The following lemma contains some technical details that will be needed in the proof. Recall that the *central rank* of a prime ideal P of $K[G]$ is defined in [4] by

$$\text{c.r. } P = \text{tr. deg.}_K \mathbf{Z}(\mathcal{Q}(K[G]/P)),$$

and that $h(G)$ denotes the *Hirsch number* of the polycyclic-by-finite group G.

LEMMA 2.5. *Let G be a polycyclic-by-finite group and let P be a prime ideal of $K[G]$.*

(i) *Suppose H is a normal subgroup of G of finite index and write $P \cap K[H] = \bigcap_{x \in G} Q^x$ with Q a prime ideal of $K[H]$.* Then

$$\text{c.r. } P = \text{c.r. } Q, \quad \text{hgt } P = \text{hgt } Q.$$

(ii) *Suppose G is an f.c. group. Then*

$$\operatorname{hgt} P = h(G) - \text{c.r.}\, P$$

and c.r. $P = 0$ *if and only if* $\dim_K K[G]/P < \infty$.

Proof. (i) The equality c.r. $P =$ c.r. Q is proved in [**4**, Lemma 4.3 (iii)], and the equality $\operatorname{hgt} P = \operatorname{hgt} Q$ follows easily from Lemma 1.4(i) together with the Incomparability and Going Down results proved in Lemma 1.3(ii)(iv).

(ii) Let G be an f.c. group and let Z be a torsion-free central subgroup of G with $[G:Z] < \infty$. Then $P \cap K[Z]$ is prime in $K[Z]$ and it follows from part (i) that $\operatorname{hgt} P = \operatorname{hgt}(P \cap K[Z])$. But $K[Z]$ is a domain which is a finitely generated commutative K-algebra so the classical Dimension Theorem of commutative algebra [**6**, Theorem 23, pp. 84–85] shows that

$$\operatorname{hgt}(P \cap K[Z]) + \text{c.r.}(P \cap K[Z]) = \text{tr.deg.}_K K[Z] = \operatorname{rank} Z.$$

Of course $\operatorname{rank} Z = h(Z) = h(G)$ and c.r.$(P \cap K[Z]) =$ c.r. P, by part (i). Thus we see that $\operatorname{hgt} P +$ c.r. $P = h(G)$. Finally, c.r. $P = 0$ if and only if c.r.$(P \cap K[Z]) = 0$ and the latter is clearly equivalent to $\dim_K K[Z]/(P \cap K[Z]) < \infty$. But $[G:Z] < \infty$ so this in turn is surely equivalent to $\dim_K K[G]/P < \infty$ and the result follows.

The proof of Theorem IV now follows quite simply. As we pointed out in the introduction, this result merely translates known invariants into the language of vertices and sources. Note that the *plinth length* $p_\Gamma(G)$ is defined in [**8**, § 2.3] for G a polycyclic group. However, the definition extends naturally to all polycyclic-by-finite groups since finite factors are immaterial here.

Proof of Theorem IV. We assume that P is a prime ideal of $K[G]$ with vertex N and source L so that $P = L^G$. Let $H = \operatorname{nio}(G)$ and use the notation and results of Theorem 2.4. In particular, if J is a minimal covering prime of $L \cap K[\nabla_G(N) \cap H]$, then $Q = (J^H)^a$ for some $a \in A = \mathbf{N}_G(N)$. Thus by replacing L by L^a and J by J^a if necessary, we can clearly assume that $a = 1$ and $Q = J^H$. Let $\tilde{}: K[H] \to K[H/Q^\dagger]$ denote the natural map.

(i) By Lemma 2.5(i), c.r. $P =$ c.r. Q and c.r. $L =$ c.r. J. Furthermore, since $\tilde{Q} = \tilde{J}^{\tilde{H}}$ is standard by [**8**, Theorem C1] and since $\nabla_G(N) \cap H = \nabla_H(Q^\dagger)$, it follows from [**4**, Corollary 3.3(ii)] that c.r. $\tilde{Q} =$ c.r. $\tilde{J}$ and hence that c.r. $Q =$ c.r. J. Thus we have c.r. $P =$ c.r. $Q =$ c.r. $J =$ c.r. L.

(ii) The height formula given in [**8**, § 2.4] states that

$$\operatorname{hgt} Q = \operatorname{hgt} \tilde{J} + p_H(Q^\dagger),$$

and Lemma 2.5(ii) yields

$$\operatorname{hgt} \tilde{J} = h(\widetilde{\nabla_H}(Q^\dagger)) - \text{c.r.}\, \tilde{J}.$$

Clearly $\text{c.r.}\, \tilde{J} = \text{c.r.}\, J = \text{c.r.}\, L$ where the latter equality was obtained in (i) above. Since $\operatorname{hgt} P = \operatorname{hgt} Q$, by Lemma 2.5(i), we now have

$$\operatorname{hgt} P = h(\widetilde{\nabla_H}(Q^\dagger)) + p_H(Q^\dagger) - \text{c.r.}\, L.$$

Finally, it follows easily from the definition of plinth length that

$$h(\widetilde{\nabla_H}(Q^\dagger)) = p_H(\nabla_H(Q^\dagger)) - p_H(Q^\dagger)$$

and $p_H(\nabla_H(Q^\dagger)) = p_A(\nabla_G(N))$, where the latter holds since $[A : H] < \infty$ and $\nabla_H(Q^\dagger) = \nabla_G(N) \cap H$. Thus

$$\operatorname{hgt} P = p_A(\nabla_G(N)) - \text{c.r.}\, L$$

and (ii) is proved.

(iii) Let K be a non-absolute field. Then we know, by [**4**, Theorem 5.2] which is a consequence of [**8**, Theorem F2], that P is primitive if and only if $\text{c.r.}\, P = 0$. Hence, in view of part (i), this occurs if and only if $\text{c.r.}\, L = 0$. Let $\bar{\ }: K[\nabla_G(N)] \to K[\nabla_G(N)/L^\dagger]$ denote the natural map. Then $K[\nabla_G(N)]/L \simeq K[\overline{\nabla_G(N)}]/\bar{L}$ so it suffices to show that $\text{c.r.}\, \bar{L} = 0$ if and only if $\dim_K K[\overline{\nabla_G(N)}]/\bar{L} < \infty$. But $[N : L^\dagger] < \infty$ so $\overline{\nabla_G(N)}$ is an f.c. group and thus Lemma 2.5(ii) yields this fact. This concludes the proof of Theorem IV.

3. Further comments on induced ideals

In this brief final section, we comment on some elementary but useful properties of the induced ideals L^G. We also show that, for $A = H$, the correspondence given in Theorem 1.7 is identical to the one of [**5**, Theorem 3.6]. The first lemma, which describes briefly a certain monotonic property of the operator ∇_G, is essentially contained in [**8**, § 7.1]. We include it here to avoid confusion over certain formal differences in definition.

LEMMA 3.1. *Let G be a polycyclic-by-finite group and let $N_1 \subseteq N_2$ be orbital subgroups with N_2 isolated. Then $\nabla_G(N_1) \subseteq \nabla_G(N_2)$.*

Proof. Since $N_1 \subseteq N_2$, it clearly suffices to show that $\nabla_G(N_1) \subseteq \mathbf{N}_G(N_2)$. To this end, let $x \in \nabla_G(N_1)$. Then there certainly exists a normal subgroup H of finite index in G with the commutator $[H, x]$ contained in N_1. Moreover, $H \cap N_2$ is an orbital subgroup of G of finite index in N_2 which is normalized by x, since

$$[H \cap N_2, x] \subseteq H \cap N_1 \subseteq H \cap N_2.$$

Finally, N_2 is isolated so $i_G(H \cap N_2) = N_2$ and x therefore also normalizes N_2.

PROPOSITION 3.2. *Let G be a polycyclic-by-finite group and let P be a prime ideal of $K[G]$. If N is an isolated orbital subgroup of G with $N \supseteq \mathrm{vx}_G(P)$, then $P = I^G$ for some prime ideal I of $K[\mathbf{N}_G(N)]$.*

Proof. Set $V = \mathrm{vx}_G(P)$ so that, by definition, $P = L^G$ for some almost faithful sub V prime ideal L of $K[\nabla_G(V)]$. Since $V \subseteq N$, it follows from the preceding lemma that $\nabla_G(V) \subseteq \nabla_G(N) \subseteq \mathbf{N}_G(N) = A$. Hence by Lemma 1.2(iii), if $I = L^A$, then $P = L^G = (L^A)^G = I^G$. Since $[G:A] < \infty$, it is clear that V is an isolated orbital subgroup of A and that $\nabla_A(V) = \nabla_G(V)$. Thus, by Theorem III, $I = L^A$ is indeed a prime ideal.

We now consider crossed products $R * G$ with R and G arbitrary and we identify the ideals L^G as the annihilators of certain induced modules. If H is a subgroup of G and V is a right $R * H$-module, then as usual we write $V^G = V \otimes_{R * H} R * G$ for the induced $R * G$-module.

LEMMA 3.3. *Let $R * G$ be given, let H be a subgroup of G, and let L be an ideal of $R * H$. If $L = \mathrm{ann}_{R*H}(V)$ for some $R * H$-module V, then $L^G = \mathrm{ann}_{R*G}(V^G)$. In particular, we have $L^G = \mathrm{ann}_{R*G}((R * H/L)^G)$.*

Proof. If $x \in G$, then it is trivial to see that $\alpha \in R * G$ annihilates $V \otimes \bar{x}$ if and only if $\alpha \in L^{\bar{x}}\bar{G}$. Thus since $V^G = \sum_{x \in G} V \otimes \bar{x}$, we conclude from Lemma 1.2(i) that

$$\mathrm{ann}_{R*G} V^G = \bigcap_{x \in G} L^{\bar{x}}\bar{G} = L^G.$$

Finally, since $L = \mathrm{ann}_{R*H}(R * H/L)$, the result follows.

In the next lemma, we collect a number of standard facts concerning induced modules. For example, part (iii) is an obvious extension of the Mackey Decomposition Theorem [**1**, p. 324] to crossed products. Let H be a subgroup of G. If V is a right $R * H$-module and if $x \in G$, then we let $V^{\bar{x}} = V \otimes \bar{x}$ denote the conjugate module for the subring $(R * H)^{\bar{x}} = R * (H^x)$. Furthermore, for any right $R * G$-module W, we let $W|_H$ denote its restriction to $R * H$.

LEMMA 3.4. *Let $R*G$ be given, let H be a subgroup of G and let V, V_1, and V_2 be right $R*H$-modules.*

(i) *If $H \subseteq N \subseteq G$, then $V^G \simeq (V^N)^G$.*

(ii) $(V_1 \oplus V_2)^G \simeq V_1{}^G \oplus V_2{}^G$.

(iii) *Let N be any subgroup of G and let D be a full set of (H, N)-double coset representatives in G. Then*

$$V^G|_N \simeq \oplus \sum_{d \in D} (V^d|_{H^d \cap N})^N.$$

In view of the relationship, described in Lemma 3.3, between induced ideals and induced modules, the above facts immediately translate into corresponding assertions concerning induced ideals. Observe that part (i) below already appears in Lemma 1.2(iii).

LEMMA 3.5. *Let $R*G$ be given, let H be a subgroup of G, and let L, L_1, and L_2 be ideals of $R*H$.*

(i) *If $H \subseteq N \subseteq G$, then $L^G = (L^N)^G$.*

(ii) $(L_1 \cap L_2)^G = L_1{}^G \cap L_2{}^G$.

(iii) *Let N be any subgroup of G and let D be a full set of (H, N)-double coset representatives in G. Then*

$$L^G \cap (R*N) = \bigcap_{d \in D} (L^d \cap R*(H^d \cap N))^N.$$

We close this section by relating the results of § 1 to the more general considerations of [**5**, § 3]. Let $R*G$ be given with G finite and with R an arbitrary G-prime ring. Then, by [**5**, Lemma 3.1(i)], there exists a prime ideal Q of R with $\bigcap_{x \in G} Q^{\bar{x}} = 0$. As in § 1, we set $H = \{x \in G \mid Q^{\bar{x}} = Q\}$, $N = \operatorname{ann}_R Q$, and, in addition, we let $M = \sum_{x \in G} N^{\bar{x}}$ so that M is a non-zero G-invariant ideal of R. If L is a prime ideal of $R*H$ with $L \cap R = Q$, then it is shown in [**5**, Theorem 3.6] that

$$L^{\vee} = \{\alpha \in R*G \mid M\alpha \subset \bar{G}NL\bar{G}\}$$

is a prime ideal of $R*G$ with $L^{\vee} \cap R = 0$. Indeed, the map ${}^{\vee}$ yields a one-to-one correspondence between the set of prime ideals L of $R*H$ with $L \cap R = Q$ and the set of prime ideals P of $R*G$ with $P \cap R = 0$. The following proposition shows that this correspondence coincides with that of Theorem 1.7 (with $A = H$).

PROPOSITION 3.6. *Let $R*G$ be given with G finite and with R a G-prime ring. Let Q, H and ${}^{\vee}$ be as above. If L is a prime ideal of $R*H$ with $L \cap R = Q$, then $L^{\vee} = L^G$.*

Proof. Define $E = \{\gamma \in R*H \mid NM\gamma \subseteq L\}$. Since N and M are H-invariant, it follows easily that E is an ideal of $R*H$ and hence we have $(NM*H)E \subseteq L$. But L is prime and $L \cap R = Q$ implies that $NM*H \nsubseteq L$, so we conclude that $E \subseteq L$.

Let $\{x_1, x_2, \ldots, x_n\}$ be a right transversal for H in G, let $\alpha \in L^{\vee}$ and write $\alpha = \sum \alpha_i \bar{x}_i$ with $\alpha_i \in R*H$. Then, by definition, $M\alpha \subseteq \bar{G}NL\bar{G}$ so [**5**, Lemma 3.3(i)] yields

$$NM\alpha \subseteq N\bar{G}NL\bar{G} \subseteq L\bar{G} = \sum L\bar{x}_i.$$

Hence we see that $NM\alpha_i \subseteq L$ so $\alpha_i \in E \subseteq L$ and $\alpha \in L\bar{G}$. We have therefore shown that $L^{\vee} \subseteq L\bar{G}$ and thus $L^{\vee} \subseteq \mathrm{Id}_{R*G}(L\bar{G}) = L^G$.

Let $\tilde{\ }: R*H \to (R*H)/(Q*H) = (R/Q)*H$ denote the natural map. Then $\tilde{L} \cap \tilde{R} = 0$, by Lemma 1.3(iii), so [**5**, Theorem 3.6] asserts that $P = L^{\vee}$ is a prime ideal of $R*G$ with $P \cap R = 0$. Now let $I = L^G$. Then $I \subseteq L\bar{G}$ so we have $I \cap R \subseteq L \cap R = Q$ and hence

$$I \cap R \subseteq \bigcap_{x \in G} Q^{\bar{x}} = 0.$$

Finally, $P \subseteq I$ and $I \cap R = 0$ yields $L^{\vee} = P = I = L^G$ by Incomparability [**5**, Lemma 3.7], and the proposition is proved.

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